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Skew CR-warped products of Kaehler manifolds

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Abstract. Warped product CR-submanifolds of Kaehler manifolds were introduced by Chen in [9]. In this paper, we introduce a warped product skew-CR-submanifold, which is a generalization of warped product CR-submanifolds. We give a characterization supported by an example and obtain an inequality in terms of the length of the second fundamental form of such submanifolds. The equality case is also investigated.

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Key words: warped product manifold, CR-submanifold, slant submanifold, skew CR-submanifold.

1. Introduction

CR-submanifolds were defined by Bejancu [1] as a generalization complex and totally real submanifolds. A CR-submanifold is called proper if it is neither complex nor totally real submanifold. The geometry of CR-submanifolds has been studied in several papers since then.

Another generalization of complex (holomorphic) and totally real submanifolds is a slant submanifold. Slant submanifolds were defined by Chen [8]. Since then, such submanifolds have been investigated by many authors (see, [8] and references therein). A slant submanifold is called proper if it is neither holomorphic nor totally real submanifold and notice that a proper CR-submanifold is never a slant submanifold. In [21], Papaghiuc gave a generalization of this notion defining semi-slant submanifolds, obtaining slant and CR-submanifolds as particular cases. We note that slant and semi-slant submanifolds of Sasakian manifolds were studied in [4] and [5] by Cabrerizo, Carriazo, Fernandez and Fernandez. On the other hand, in [25] we studied hemi-slant submanifolds which were defined by Carriazo under the name of anti-slant submanifolds [6] and showed that such submanifolds also contain CR-submanifolds and slant submanifolds as particular subspaces. Moreover, we observe that there is no inclusion relation between semi-slant submanifolds and hemi-slant submanifolds.

In [22], Ronsse introduced skew CR-submanifolds of Kaehler manifolds. It is easy to observe that a skew CR-submanifold is a generalization of slant submanifolds as well as CR-submanifolds. In fact, semi-slant submanifolds [21] and hemi-slant submanifolds [25] are also particular classes of skew CR-submanifolds.

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Let (B, g_1) and (F, g_2) be two Riemannian manifolds, $f : B \to (0, \infty)$ and $\pi : B \times F \to B$, $\eta : B \times F \to F$ the projection maps given by $\pi(p, q) = p$ and $\eta(p, q) = q$ for every $(p, q) \in B \times F$. The warped product $M = B \times F$ is the manifold $B \times F$ equipped with the Riemannian structure such that

$$g(X, Y) = g_1(\pi_* X, \pi_* Y) + (f o \pi)^2 g_2(\eta_* X, \eta_* Y)$$

for every X and Y of M and $*$ is a symbol for the tangent map. The function f is called the warping function of the warped product manifold. In particular, if the warping function is constant, then the manifold M is said to be trivial. Let X, Y be vector fields on B and V, W vector fields on F, then from Lemma 7.3 of [2], we have

$$\nabla_X V = \nabla_V X = (\frac{Xf}{f})V \tag{1}$$

where ∇ is the Levi-Civita connection on M. We note that the concept of warped products was first introduced by Bishop and O'Neill [2] to construct examples of Riemannian manifolds with negative curvature. We also note that warped product manifolds have their applications in general relativity. Indeed, many spacetime models are examples of warped product manifolds. More precisely, Robertson-Walker spacetimes, asymptotically flat spacetimes, Schwarzschild spacetimes and Reissner-Nordström spacetimes are examples of warped product manifolds, for details [17].

It is known that there are two distributions on a CR-submanifold such that one of them is invariant and the other one is anti-invariant under the action of the complex structure of the ambient space. It is also known that a warped product manifold has fibres and leaves. Using this similarity, Chen considered warped product CR-submanifolds of Kaehler manifolds [9], [10] and showed that there do not exist warped product CR-submanifolds in the from $M_\perp \times_f M_T$ such that M_T is a holomorphic (complex) submanifold and $M_\perp$ is a totally real submanifold of a Kaehler manifold $\bar{M}$. Then he introduced the notion of CR-warped products of Kaehler manifolds as follows: A submanifold of a Kaehler manifold is called a CR-warped product if it is the warped product $M_T \times_f M_\perp$ of a holomorphic submanifold M_T and a totally real submanifold $M_\perp$. He also established a sharp relationship between the warping function f of a warped product CR-submanifold $M_T \times_f M_\perp$ of a Kaehler manifold $\bar{M}$ and the squared norm of the second fundamental form $\| h \|^2$. After Chen's papers, CR-warped product submanifolds have been studied in various manifolds [3, 19] and [24].

In [23], we proved that there are no warped product semi-slant (in the sense of Papaghiuc) submanifolds. In [25], we studied hemi-slant submanifolds and showed that there exists a class of warped product hemi-slant submanifolds.

In this paper, we introduce and study warped product skew CR-submanifolds of Kaehler manifolds. One can observe that such skew CR-submanifolds are generalization of CR-warped products and warped product hemi-slant submanifolds of Kaehler manifolds. Our construction of skew CR-warped submanifolds can be considered as a special multiply warped product manifolds. We note that multiply warped product manifolds were studied in [12] and they were applied to spacetime geometry. For example, in [12] and [13], the authors showed a representation of the interior Schwarzschild space-time as a multiply warped product space-time with certain warping functions. Also, Dobarro and Ünal obtained in [15] that Reissner-Nordström space-time and Kasner spacetime can be expressed as a multiply warped product space-time. In sections 3-4 of this paper, we give an example of warped product skew CR-submanifolds, obtain a characterization and establish an inequality which is a generalization of a sharp inequality obtained by Chen in [9].

2. Preliminaries

Let $(\bar{M}, g)$ be a Kaehler manifold . This means [26] that $\bar{M}$ admits a tensor field J of type (1,1) on $\bar{M}$ such that, $\forall X, Y \in \Gamma(T\bar{M})$, we have

$$J^2 = -I, \quad g(X,Y) = g(JX, JY), \quad (\bar{\nabla}_X J)Y = 0, \tag{2}$$

where g is the Riemannian metric and $\bar{\nabla}$ is the Levi-Civita connection on $\bar{M}$. Let M be a Riemannian manifold isometrically immersed in $\bar{M}$ and denote by the same symbol g for the induced Riemannian metric on M. Let $\Gamma(TM)$ be the Lie algebra of vector fields in M and $\Gamma(TM^\perp)$ the set of all vector fields normal to M. Denote by ∇ the Levi-Civita connection of M. Then the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{3}$$

and

$$\bar{\nabla}_X N = -A_N X + \nabla^\perp_X N \tag{4}$$

for any $X, Y \in \Gamma(TM)$ and any $N \in \Gamma(TM^\perp)$, where $\nabla^\perp$ is the connection in the normal bundle $TM^\perp$, h is the second fundamental form of M and A_N is the Weingarten endomorphism associated with N. The second fundamental form h and the shape operator A are related by

$$g(A_N X, Y) = g(h(X,Y), N). \tag{5}$$

For any $X \in \Gamma(TM)$ we write

$$JX = TX + FX, \tag{6}$$

where TX is the tangential component of JX and FX is the normal component of JX. Similarly, for any vector field normal to M, we put

$$JN = BN + CN, \tag{7}$$

where BN and CN are the tangential and the normal components of JN, respectively.

Let $\bar{M}$ be a Kaehler manifold with complex structure J and M is a Riemannian manifold isometrically immersed in $\bar{M}$. Then M is called holomorphic (complex) if $J(T_pM) \subset T_pM$ for every $p \in M$ where T_pM denotes the tangent space to M at the point p. M is called totally real if $J(T_pM) \subset T_pM^\perp$ for every $p \in M$, where $T_pM^\perp$ denotes the normal space to M at the point p. Besides holomorphic and totally real submanifolds, there are several other classes of submanifolds of a Kaehler manifold determined by the behavior of the tangent bundle of the submanifold under the action of the complex structure of the ambient manifold.

(1) The submanifold M is called a CR-submanifold [1] if there exists a differentiable distribution $\mathcal{D} : p \rightarrow \mathcal{D}_p \subset T_pM$ such that $\mathcal{D}$ is invariant with respect to J and the complementary distribution $\mathcal{D}^{\perp}$ is anti-invariant with respect to J.

(2) The submanifold M is called slant [8] if for all non-zero vector X tangent to M the angle $\theta(X)$ between JX and T_pM is a constant, i.e. it does not depend on the choice of $p \in M$ and $X \in T_pM$.

(3) The submanifold M is called semi-slant [21] if it is endowed with two orthogonal distributions $\mathcal{D}^T$ and $\mathcal{D}^{\theta}$, where $\mathcal{D}^T$ is invariant with respect to J and $\mathcal{D}^{\theta}$ is slant, i.e. $\theta(X)$ between JX and $\mathcal{D}^{\theta}_p$ is constant for $X \in \mathcal{D}^{\theta}_p$.

(4) The submanifold M is called a hemi-slant submanifold ([6] and[25]) if it is endowed with two orthogonal distributions $\mathcal{D}^{\theta}$ and $\mathcal{D}^{\perp}$, where $\mathcal{D}^{\theta}$ is slant and $\mathcal{D}^{\perp}$ is anti-invariant with respect to $\bar{J}$.

Finally, we recall the definition of skew CR-submanifolds from [22]. Let M be a submanifold of a Kaehler manifold $\bar{M}$. For any X_p and Y_p in T_pM we have $g(TX_p, Y_p) = -g(X_p, TY_p)$. Hence, it follows that T^2 is a symmetric operator on the tangent space T_pM, for all p. Therefore, its eigenvalues are real and diagonalizable. Moreover, its eigenvalues are bounded by -1 and 0. For each $p \in M$, we may set

$$\mathcal{D}^{\lambda}_p = Ker\{T^2 + \lambda^2(p)I\}_p,$$

where I is the identity transformation and $\lambda(p)$ belongs to closed real interval $[0, 1]$ such that $-\lambda^2(p)$ is an eigenvalue of $T^2(p)$. Notice that $\mathcal{D}^1_p = KerF$ and $\mathcal{D}^0_p = KerT$. $\mathcal{D}^1_p$ is the maximal $J-$ invariant subspace of T_pM and $\mathcal{D}^0_p$ is the maximal anti $J-$ invariant subspace of T_pM. From now on, we denote the distributions $\mathcal{D}^1$ and $\mathcal{D}^0$ by $\mathcal{D}^T$ and $\mathcal{D}^{\perp}$, respectively. Since T^2_p is symmetric and diagonalizable, if $-\lambda^2_1(p), ..., -\lambda^2_k(p)$ are the eigenvalues of T^2 at $p \in M$, then T_pM can be decomposed as a direct sum of mutually orthogonal eigenspaces, i.e.

$$T_pM = \mathcal{D}^{\lambda_1}_p \oplus ... \oplus \mathcal{D}^{\lambda_k}_p.$$

Each $\mathcal{D}^{\lambda_i}_p$, $1 \leq i \leq k$, is a $T-$ invariant subspace of T_pM. Moreover, if $\lambda_i \neq 0$, then $\mathcal{D}^{\lambda_i}_p$ is even dimensional. Let M be a submanifold of a Kaehler manifold $\bar{M}$. M is called a generic submanifold if there exists an integer k and functions λ_i, $1 \leq i \leq k$ defined on M with values in $(0, 1)$ such that

1. Each $-\lambda^2_i(p)$, $1 <\leq i \leq k$ is a distinct eigenvalue of T^2 with

$$T_pM = \mathcal{D}^T_p \oplus \mathcal{D}^{\perp}_p \oplus \mathcal{D}^{\lambda_1}_p \oplus ... \oplus \mathcal{D}^{\lambda_k}_p$$

for $p \in M$.

2. The dimensions of $\mathcal{D}^T_p$, $\mathcal{D}^{\perp}_p$ and $\mathcal{D}^{\lambda_i}$, $1 \leq i \leq k$ are independent of $p \in M$.

Moreover, if each λ_i is constant on M, then M is called a skew CR-submanifold. Thus, we observe that CR-submanifolds are a particular class of skew CR-submanifolds with $k = 0$, $\mathcal{D}^T \neq \{0\}$ and $\mathcal{D}^\perp \neq \{0\}$. And slant submanifolds are also a particular class of skew CR-submanifolds with $k = 1$, $\mathcal{D}^T = \{0\}$, $\mathcal{D}^\perp = \{0\}$ and λ_1 is constant. Moreover, if $\mathcal{D}^\perp = \{0\}$, $\mathcal{D}^T \neq \{0\}$ and $k = 1$, then M is a semi-slant submanifold. Furthermore, if $\mathcal{D}^T = \{0\}$, $\mathcal{D}^\perp \neq \{0\}$ and $k = 1$, then M is a hemi-slant submanifold.

Definition 1. *We say that a submanifold M is a skew CR-submanifold of order* 1 *if M is a skew CR-submanifold with $k = 1$ and λ_1 is constant.*

Thus a slant submanifold is a skew CR-submanifold of order 1 with $\mathcal{D}^T = \{0\}$ and $\mathcal{D}^\perp = \{0\}$. Moreover, a semi-slant submanifold is a skew CR-submanifold of order 1 with $\mathcal{D}^\perp = \{0\}$ and a hemi-slant submanifold is a skew CR-submanifold of order 1 with $\mathcal{D}^T = \{0\}$. We say that a skew CR-submanifold of order 1 is proper if $\mathcal{D}^T \neq \{0\}$ and $\mathcal{D}^\perp \neq \{0\}$.

Recall that slant submanifolds are characterized by

$$T^2X = \lambda X \tag{8}$$

such that $\lambda \in [-1, 0]$, for details see [8]). If M is slant, then $\lambda = -cos^2\theta$, where θ is the slant angle of M. Using (8), we also have another characterization for slant submanifolds. Namely, M is a slant submanifold if and only if there exists a constant $\kappa \in [-1, 0]$ such that

$$BFX = \kappa X \tag{9}$$

for $X \in \Gamma(TM)$. If M is a slant submanifold, then $BFX = -sin^2\theta\, X$ [25]. In this case, we also have

$$CFX = -FTX. \tag{10}$$

3. Warped product skew CR-submanifolds in Kaehler manifolds

Let M_1 be a semi-slant submanifold in the sense of Papaghiuc and $M_\perp$ a totally real submanifold of a Kaehler manifold. In this section we consider a warped product $M = M_1 \times_f M_\perp$ in a Kaehler manifold $\bar{M}$. Then, it is obvious that M is a proper skew CR-submanifold of order 1 of $\bar{M}$. So, we are entitled to call M a skew CR-warped product of $\bar{M}$. Then, from definition of a semi-slant submanifold and a skew CR-submanifold, we have

$$TM = \mathcal{D}^\theta \oplus \mathcal{D}^T \oplus \mathcal{D}^\perp, \tag{11}$$

where $D^\theta_p = \mathcal{D}^{\lambda_1}_p$. Thus, if $D^\theta = \{0\}$, then M is a CR-warped product [9]. If $D^T = \{0\}$, then M becomes a warped product hemi-slant submanifold [25].

Remark 1. *From [9, Theorem 3.1], we know that there are no proper warped product CR-submanifolds in the form $M_\perp \times_f M_T$ of a Kaehler manifold $\bar{M}$ such that $M_\perp$ is totally real and M_T is a holomorphic submanifold in $\bar{M}$. Also, from [25,* *Theorem 4.2], we know that there are no proper warped product submanifolds in the form* $M_\perp \times_f M_\theta$ *of a Kaehler manifold* $\bar{M}$ *such that* $M_\perp$ *is totally real and* M_θ *is a proper slant submanifold in* $\bar{M}$*. Thus, it follows that there are no warped product skew CR-submanifolds of order* 1 *in the form form* $M_\perp \times_f M_1$ *of a Kaehler manifold* $\bar{M}$ *such that* $M_\perp$ *is totally real and* M_1 *is a semi-slant submanifold (in the sense of Papaghiuc) in* $\bar{M}$*.*

We first give an example of a warped product skew CR-submanifold.

Example 1. *Consider a submanifold in* $\mathbb{R}^{12}$ *given by the following equations:*

$$x_1 = u_1 \cos\theta, x_2 = u_2\ , x_3 = u_1 \sin\theta, x_4 = 0,$$
$$x_5 = u_3 \cos u_5, x_6 = -u_4 \cos u_5\ , x_7 = u_3 \sin u_5, x_8 = -u_4 \sin u_5$$
$$x_9 = u_1 \sin u_5, x_{10} = 0\ , x_{11} = u_1 \cos u_5, x_{12} = 0.$$

Then, the tangent bundle of M *is spanned by*

$$\begin{aligned} Z_1 &= \cos\theta \frac{\partial}{\partial x_1} + \sin\theta \frac{\partial}{\partial x_3} + \sin u_5 \frac{\partial}{\partial x_9} + \cos u_5 \frac{\partial}{\partial x_{11}}, \ Z_2 = \frac{\partial}{\partial x_2} \\ Z_3 &= \cos u_5 \frac{\partial}{\partial x_5} + \sin u_5 \frac{\partial}{\partial x_7}, Z_4 = -\cos u_5 \frac{\partial}{\partial x_6} - \sin u_5 \frac{\partial}{\partial x_8} \\ Z_5 &= -u_3 \sin u_5 \frac{\partial}{\partial x_5} + u_4 \sin u_5 \frac{\partial}{\partial x_6} + u_3 \cos u_5 \frac{\partial}{\partial x_7} - u_4 \cos u_5 \frac{\partial}{\partial x_8} \\ &\quad + u_1 \cos u_5 \frac{\partial}{\partial x_9} - u_1 \sin u_5 \frac{\partial}{\partial x_{11}}. \end{aligned}$$

Then it is easy to see that $D^\theta = span\{Z_1, Z_2\}$ *is a slant distribution with slant angle* φ *such that* $\cos\varphi = \frac{\cos\theta}{\sqrt{2}}$ *and* $\mathcal{D}^T = span\{Z_3, Z_4\}$ *is a holomorphic distribution. Moreover, it can be easily seen that* $\bar{J}Z_5$ *is orthogonal to* M*. Thus* $\mathcal{D}^\perp = span\{Z_5\}$ *is an anti-invariant distribution. Hence, we conclude that* M *is a proper skew CR-submanifold of order* 1 *of* $\mathbb{R}^{12}$*. Moreover, it is easy to see that* $\mathcal{D}^T \oplus \mathcal{D}^\theta$ *and* $D^\perp$ *are integrable. Denoting the integral manifolds of* $\mathcal{D}^T$*,* $\mathcal{D}^\theta$ *and* $\mathcal{D}^\perp$ *by* M_T*,* M_θ *and* $M_\perp$*, respectively, then the induced metric tensor of* M *is*

$$\begin{aligned} ds^2 &= 2du_1^2 + du_2^2 + du_3^2 + du_4^2 + (u_3^2 + u_4^2 + u_1^2)du_5^2 \\ &= g_{M_\theta} + g_{M_T} + (u_3^2 + u_4^2 + u_1^2) g_{M_\perp}. \end{aligned}$$

Thus, it follows that M *is a warped product skew CR-submanifold of* $\mathbb{R}^{12}$ *with warping function* $f = \sqrt{u_3^2 + u_4^2 + u_1^2}$*.*

In the rest of this section, we are going to give a characterization of a skew CR-warped product of a Kaehler manifold $\bar{M}$.

Lemma 1. *Let* M *be a proper skew CR-submanifold of order* 1 *of a Kaehler manifold* $\bar{M}$*. Then, we have*

$$g(\nabla_{X_1} Y_1, Z) = g(A_{JZ} X_1, JY_1) \tag{12}$$
$$g(\nabla_{X_1} Y_2, Z) = \sec^2\theta\{-g(A_{FTY_2} X_1, Z) + g(A_{JZ} TY_2, X_1)\} \tag{13}$$
$$g(\nabla_{Y_2} X_1, Z) = g(A_{JZ} Y_2, JX_1) \tag{14}$$

for $X_1, Y_1 \in \Gamma(\mathcal{D}^T)$*,* $Y_2 \in \Gamma(\mathcal{D}^\theta)$ *and* $Z \in \Gamma(\mathcal{D}^\perp)$*.*

Proof. From (3) and (2), we obtain $g(\nabla_{X_1}Y_1, Z) = g(\bar{\nabla}_{X_1}JY_1, JZ)$ for $X_1, Y_1 \in \Gamma(\mathcal{D}^T)$ and $Z \in \Gamma(\mathcal{D}^\perp)$. Then using again (3) and (5), we have (12). With regards to statement (13), from (3) and (2), we get $g(\nabla_{X_1}Y_2, Z) = g(\bar{\nabla}_{X_1}JY_2, JZ)$. Using (6) we derive

$$g(\nabla_{X_1}Y_2, Z) = g(\bar{\nabla}_{X_1}TY_2, JZ) + g(\bar{\nabla}_{X_1}FY_2, JZ).$$

Thus, from (3) and (2) we have

$$g(\nabla_{X_1}Y_2, Z) = g(h(X_1, TY_2), JZ) + g(JFY_2, \bar{\nabla}_{X_1}Z).$$

Hence, using (7), we get

$$g(\nabla_{X_1}Y_2, Z) = g(h(X_1, TY_2), JZ) + g(BFY_2, \nabla_{X_1}Z) + g(CFY_2, h(X_1, Z)).$$

Then, from (9) and (10), we arrive at

$$g(\nabla_{X_1}Y_2, Z) = g(h(X_1, TY_2), JZ) - sin^2\theta\, g(Y_2, \nabla_{X_1}Z) - g(FTY_2, h(X_1, Z)).$$

Hence, we have

$$g(\nabla_{X_1}Y_2, Z) = g(h(X_1, TY_2), JZ) + sin^2\theta\, g(\nabla_{X_1}Y_2, Z) - g(FTY_2, h(X_1, Z)).$$

Thus, using (5) we obtain (13). In a similar way, one can obtain (14). $\square$

Lemma 2. *Let M be a proper skew CR-submanifold of order 1 of a Kaehler manifold $\bar{M}$. Then, we have*

$$g(\nabla_{X_2}Y_2, Z) = g(A_{JZ}TY_2 - A_{FTY_2}Z, X_2)sec^2\theta \tag{15}$$

$$g(\nabla_Z V, X_2) = -sec^2\theta\,\{g(A_{JV}Z, TX_2) - g(A_{FTX_2}Z, V)\} \tag{16}$$

and

$$g(\nabla_Z V, X_1) = -g(A_{JV}Z, JX_1) \tag{17}$$

for any $X_1 \in \Gamma(\mathcal{D}^T)$, $X_2, Y_2 \in \Gamma(\mathcal{D}^\theta)$ and $Z, V \in \Gamma(\mathcal{D}^\perp)$.

Proof. The proof of (17) is exactly the same as in (12). For the proof of (15), from (2), (3) and (6), we have $g(\nabla_{X_2}Y_2, Z) = g(\bar{\nabla}_{X_2}TY_2, JZ) + (\bar{\nabla}_{X_2}FY_2, JZ)$ for $X_2, Y_2 \in \Gamma(\mathcal{D}^\theta)$ and $Z, V \in \Gamma(\mathcal{D}^\perp)$. Then using (4) and (2) we get

$$g(\nabla_{X_2}Y_2, Z) = (h(X_2, TY_2), JZ) - g(\bar{\nabla}_{X_2}JFY_2, Z).$$

Thus, using (7) we derive

$$g(\nabla_{X_2}Y_2, Z) = (h(X_2, TY_2), JZ) - g(\bar{\nabla}_{X_2}BFY_2, Z) - g(\bar{\nabla}_{X_2}CFY_2, Z).$$

Hence, we obtain

$$g(\nabla_{X_2}Y_2, Z) = (h(X_2, TY_2), JZ) + sin^2\theta\, g(\nabla_{X_2}Y_2, Z) + g(\bar{\nabla}_{X_2}FTY_2, Z).$$

Then, from (4) we have

$$g(\nabla_{X_2}Y_2, Z) = (h(X_2, TY_2), JZ) + sin^2\theta\, g(\nabla_{X_2}Y_2, Z) - g(A_{FTY_2}X_2, Z).$$

Thus, using again (5) we obtain (15). In a similar way, one can obtain (16). $\square$

Lemma 3. *Let M be a proper skew CR-submanifold of order 1 of a Kaehler manifold $\bar{M}$. Then we have*

$$sin^2\theta g(\nabla_Z X_1, Y_2) = -g(A_{FTY_2}Z, X_1) + g(A_{FY_2}Z, JX_1)$$

for $X_1 \in \Gamma(\mathcal{D}^T)$, $Y_2 \in \Gamma(\mathcal{D}^\theta)$ and $Z \in \Gamma(\mathcal{D}^\perp)$.

Proof. From (3), (2) and (6) we obtain

$$g(\nabla_Z X_1, Y_2) = g(\bar{\nabla}_Z JX_1, TY_2) + g(\bar{\nabla}_Z JX_1, FY_2)$$

for $X_1 \in \Gamma(\mathcal{D}^T)$, $Y_2 \in \Gamma(\mathcal{D}^\theta)$ and $Z \in \Gamma(\mathcal{D}^\perp)$. Then, using (2), (6) and (3) we get

$$g(\nabla_Z X_1, Y_2) = -g(\bar{\nabla}_Z X_1, T^2 Y_2) - g(\bar{\nabla}_Z X_1, FTY_2) + g(h(Z, JX_1), FY_2).$$

Thus, from (8) we have

$$g(\nabla_Z X_1, Y_2) = cos^2\theta\, g(\nabla_Z X_1, Y_2) - g(h(Z, X_1), FTY_2) + g(h(Z, JX_1), FY_2).$$

Hence, using (5) we obtain the assertion of lemma. □

We now recall the following definition from [22]. For the distributions $\mathcal{D}^\theta$ and $\mathcal{D}^\perp$ on M, we say that M is $\mathcal{D}^\theta - \mathcal{D}^\perp$ mixed totally geodesic if for all $X \in \Gamma(\mathcal{D}^\theta)$ and $Y \in \Gamma(\mathcal{D}^\perp)$, we have $h(X, Y) = 0$.

We also recall the result of S. Hiepko [18], (cf. [14], Remark 2.1): Let D_1 be a vector subbundle in the tangent bundle of a Riemannian manifold M and let D_2 be its normal bundle. Suppose that the two distributions are involutive. We denote the integral manifolds of D_1 and D_2 by M_1 and M_2, respectively. Then M is locally isometric to warped product $M_1 \times_f M_2$ if the integral manifold M_1 is totally geodesic and the integral manifold M_2 is an extrinsic sphere, i.e. M_2 is a totally umbilical submanifold with a parallel mean curvature vector.

Now, we are ready to state and prove a characterization for a skew CR-warped product in a Kaehler manifold.

Theorem 1. *Let M be a $\mathcal{D}^\theta - \mathcal{D}^\perp$ mixed totally geodesic proper skew CR-submanifold of order 1 of a Kaehler manifold $\bar{M}$. Then M is a locally skew CR-warped product if and only if the following holds:*

(a) $A_{JZ}TY_2$ has no components in $\mathcal{D}^\theta$ for $Z \in \Gamma(\mathcal{D}^\perp)$ and $Y_2 \in \Gamma(\mathcal{D}^\theta)$.

(b) For $X_1 \in \Gamma(\mathcal{D}^T)$, $Z \in \Gamma(\mathcal{D}^\perp)$, $X_2 \in \Gamma(\mathcal{D}^\theta)$ and a function μ defined on M, we have

$$A_{JZ}JX_1 = X_1(\mu)Z \tag{18}$$
$$A_{FTX_2}Z = -X_2(\mu)cos^2\theta Z \tag{19}$$

such that $V(\mu) = 0$ for every $V \in \Gamma(\mathcal{D}^\perp)$.

Proof. Let M be a $\mathcal{D}^\theta - \mathcal{D}^\perp$ mixed totally geodesic skew CR warped product of a Kaehler manifold $\bar{M}$. Then M_1 is totally geodesic in M. Thus, $\nabla_X Y \in \Gamma(TM_1)$ for $X, Y \in \Gamma(TM_1)$. On the other hand, $\mathcal{D}^\theta - \mathcal{D}^\perp$ mixed totally geodesic M and (15) imply that

$$g(\nabla_{X_2} Y_2, Z) = sec^2\theta\, g(A_{JZ} TY_2, X_2)$$

for $X_2, Y_2 \in \Gamma(\mathcal{D}^\theta)$ and $Z \in \Gamma(\mathcal{D}^\perp)$. Then, since $\nabla_{X_2} Y_2 \in \Gamma(TM_1)$, we have

$$g(A_{JZ} TY_2, X_2) = 0$$

which implies (a). Moreover, from (12) and (14) we obtain

$$g(A_{JZ} X_1, JY_1) = 0 \quad \text{and} \quad g(A_{JZ} Y_2, JX_1) = 0$$

for $X_1, Y_1 \in \Gamma(\mathcal{D}^T)$, $Y_2 \in \Gamma(\mathcal{D}^\theta)$ and $Z \in \Gamma(\mathcal{D}^\perp)$. Since A is self-adjoint, we conclude that $A_{JZ} X_1$ has no components in TM_1. Then, $A_{JZ} JX_1 \in \Gamma(\mathcal{D}^\perp)$. Taking into account that M is $\mathcal{D}^\theta - \mathcal{D}^\perp$ mixed totally geodesic, from (1) and (17) we obtain

$$X_1(lnf) g(Z, V) = g(A_{JV} Z, JX_1)$$

for $V \in \Gamma(\mathcal{D}^\perp)$. Hence, we obtain (18). Since M is $\mathcal{D}^\theta - \mathcal{D}^\perp$ mixed totally geodesic, we have $g(A_{FTY_2} Z, Y_2) = 0$. Hence, $A_{FTY_2} Z$ has no components in $\mathcal{D}^\theta$. On the other hand, (18) implies that $g(A_{JV} X_1, Y_2) = -JX_1(\mu) g(V, Y_2) = 0$. Then, using (1) and (13), $g(A_{FTY_2} Z, X_1) = 0$ which implies that $A_{FTY_2} Z$ has also no components in $\mathcal{D}^T$. Then, from (11), we conclude that $A_{FTY_2} Z \in \Gamma(\mathcal{D}^\perp)$. Thus, mixed totally geodesic $\mathcal{D}^\theta - \mathcal{D}^\perp$ and (16) imply that

$$g(\nabla_Z V, Y_2) = g(A_{FTY_2} Z, V)\, sec^2\theta.$$

Then, using (1) we get

$$g(A_{FTY_2} Z, V) = -cos^2\theta Y_2(lnf) g(Z, V)$$

which is (19). Moreover, since $Z(lnf) = 0$ for a skew CR-product, we obtain $\mu = lnf$.

Let us to prove converse. Assume that M is $\mathcal{D}^\theta - \mathcal{D}^\perp$ mixed totally geodesic proper skew CR-submanifold of order 1 of a Kaehler manifold $\bar{M}$ such that (a) and (b) hold. Then, from (a), (b) and (12)-(15), it is easy to see that M_1 is totally geodesic in M. On the other hand, from [7], $\mathcal{D}^\perp$ is always integrable. We denote the integral manifold of $\mathcal{D}^\perp$ by $M_\perp$. Also let $h_\perp$ be the second fundamental form of $M_\perp$ in M. From (3) we have $g(h_\perp(Z, V), X_1) = g(\nabla_Z V, X_1)$ for $X_1, \in \Gamma(\mathcal{D}^T)$ and $Z, V \in \Gamma(\mathcal{D}^\perp)$. Then, (17) implies that $g(h_\perp(Z, V), X_1) = -g(A_{JV} Z, JX_1)$. Thus, from (18) we obtain

$$g(h_\perp(Z, V), X_1) = -X_1(\mu) g(V, Z). \tag{20}$$

In a similar way, from (3) we get $g(h_\perp(Z, V), X_2) = g(\nabla_Z V, X_2)$ for $X_2 \in \Gamma(\mathcal{D}^\theta)$. Then,since M is $D^\theta - D^\perp$ mixed totally geodesic, from (16) we obtain $g(h_\perp(Z, V), X_2) = g(A_{FTX_2} Z, V) sec^2\theta$. Using (19) we have

$$g(h_\perp(Z, V), X_2) = -X_2(\mu) g(Z, V). \tag{21}$$

Thus, for $X = X_1 + X_2 \in \Gamma(M_1)$, from (20) and (21) we obtain

$$\begin{aligned} g(h_\perp(Z,V),X) &= g(h_\perp(Z,V),X_1) + g(h_\perp(Z,V),X_2) \\ &= -[X_1(\mu) + X_2(\mu)]g(Z,V) \end{aligned} \tag{22}$$

which implies that $M_\perp$ is totally umbilical in M. Denoting the gradient of μ on $\mathcal{D}^T$ and $\mathcal{D}^\theta$ by $grad^T\mu$ and $grad^\theta\mu$, respectively, from (22) we can write

$$h_\perp(Z,V) = -[grad^T\mu + grad^\theta\,\mu]g(Z,V). \tag{23}$$

Thus, for $X = X_1 + X_2 \in \Gamma(TM_1)$, we have

$$\begin{aligned} g(\nabla_Z grad^T\mu + grad^\theta\,\mu, X) &= g(\nabla_Z grad^T\mu, X) + g(\nabla_Z grad^\theta\,\mu, X) \\ &= [Zg(grad^T\mu, X_2) - g(grad^T\mu, \nabla_Z X)] \\ &\quad + Zg(grad^\theta\,\mu, X_1) - g(grad^\theta\,\mu, \nabla_Z X) \\ &= [Z(X_2(\mu)) - g(grad^T\mu, \nabla_Z X)] \\ &\quad + Z(X_1(\mu)) - g(grad^\theta\,\mu, \nabla_Z X). \end{aligned}$$

Then, using (19) in Lemma 3.3, we get $g(\nabla_Z X_1, X_2) = -g(\nabla_Z X_2, X_1) = 0$. Hence, we arrive at

$$\begin{aligned} g(\nabla_Z grad^T\mu + grad^\theta\,\mu, X) &= [Z(X_2(\mu)) - g(grad^T\mu, \nabla_Z X_2)] \\ &\quad + Z(X_1(\mu)) - g(grad^\theta\,\mu, \nabla_Z X_1). \end{aligned}$$

Hence, by direct calculations, we get

$$\begin{aligned} g(\nabla_Z grad^T\mu + grad^\theta\,\mu, X) &= [Z(X_2(\mu)) - [Z,X_2](\mu) \\ &\quad + g(grad^T\mu, \nabla_{X_2} Z)] + Z(X_1(\mu)) \\ &\quad - [Z,X_1](\mu) + g(grad^\theta\,\mu, \nabla_{X_1} Z). \\ &= [X_2(Z(\mu)) + g(grad^T\mu, \nabla_{X_2} Z)] \\ &\quad - X_1(Z(\mu)) + g(grad^\theta\mu, \nabla_{X_1} Z). \end{aligned} \tag{24}$$

Since $Z(\mu) = 0$, we derive

$$g(\nabla_Z grad^T\mu + grad^\theta\,\mu, X) = -g(\nabla_{X_2} grad^T\mu, Z) - g(\nabla_{X_1} grad^\theta\mu, Z).$$

On the other hand, since M_1 is totally geodesic, we have $\nabla_{X_1} grad^\theta\mu, \nabla_{X_2} grad^T\mu \in \Gamma(TM_1)$. Hence, we obtain

$$g(\nabla_Z grad^T\mu + grad^\theta\,\mu, X) = 0.$$

Thus, we conclude that $grad^T\mu + grad^\theta\,\mu$ is parallel in M. This result and (23) imply that $M_\perp$ is an extrinsic sphere. Thus, the proof is complete. □

4. An inequality for skew CR-warped products

In this section, we obtain an inequality for the squared norm of the second fundamental form in terms of the warping function for skew CR-warped products in Kaehler manifolds. To do this, we need the following lemmas.

Lemma 4. *Let M be a skew CR-warped product of a Kaehler manifold $\bar{M}$. Then, we have*

$$g(h(X_1, Y_2), JZ) = 0 \tag{25}$$

and

$$g(h(X_1, Y_1), JZ) = 0 \tag{26}$$

for $X_1, Y_1 \in \Gamma(\mathcal{D}^T)$, $Y_2 \in \Gamma(D^\theta)$ and $Z \in \Gamma(\mathcal{D}^\perp)$.

Proof. From (2) and (3) we get

$$g(\nabla_{Y_2} JX_1, Z) = -g(\bar{\nabla}_{Y_2} X_1, JZ)$$

for $X_1 \in \Gamma(\mathcal{D}^T)$, $Y_2 \in \Gamma(D^\theta)$ and $Z \in \Gamma^{(}\mathcal{D}^\perp)$. Hence, we have $g(JX_1, \nabla_{Y_2} Z) = g(\bar{\nabla}_{Y_2} X_1, JZ)$. Then, using (1) and (3) we obtain

$$Y_2(lnf)g(JX_1, Z) = g(h(X_1, Y_2), JZ).$$

Hence, $g(h(X_1, Y_2), JZ) = 0$. In a similar way, from (3) and (2) we get $g(\nabla_{X_1} Z, Y_1) = g(\bar{\nabla}_{X_1} JZ, JY_1)$ for $X_1, Y_1 \in \Gamma(\mathcal{D}^T)$ and $Z \in \Gamma(\mathcal{D}^\perp)$. Then, using (1) and (4) we obtain $0 = g(A_{JZ} X_1, JY_1)$. Thus (3) implies (26). □

Lemma 5. *Let M be a skew CR-warped product of a Kaehler manifold $\bar{M}$. Then, we have*

$$X_1(lnf)g(Z, V) = g(h(Z, JX_1), JV) \tag{27}$$

and

$$g(h(Z, X_2), FY_2) = g(h(X_2, Y_2), JZ) \tag{28}$$

for $X_1 \in \Gamma(\mathcal{D}^T)$, $X_2, Y_2 \in \Gamma(\mathcal{D}^\theta)$ and $Z, V \in \Gamma(\mathcal{D}^\perp)$.

Proof. Equation (27) was obtained in [9], Lemma 4.1(3). For (28), from (3), we have $g(h(X_2, Z), FTY_2) = g(\bar{\nabla}_{X_2} Z, FTY_2)$ for $X_2, Y_2 \in \Gamma(\mathcal{D}^\theta)$ and $Z, V \in \Gamma(\mathcal{D}^\perp)$. Then, using (6) we get $g(h(X_2, Z), FTY_2) = g(\bar{\nabla}_{X_2} Z, JTY_2 - T^2 Y_2)$. Here, (8) and (2) imply that

$$g(h(X_2, Z), FTY_2) = cos^2\theta\, g(\nabla_{X_2} Z, Y_2) - g(\bar{\nabla}_{X_2} JZ, TY_2).$$

Thus, from (1) and (4), we have

$$g(h(X_2, Z), FTY_2) = g(A_{JZ} X_2, TY_2).$$

Then, using (5) we get

$$g(h(X_2, Z), FTY_2) = g(h(X_2, TY_2), JZ).$$

Thus, substituting TY_2 by Y_2 and using (8) we obtain (28). □

Lemma 6. *Let M be a skew CR-warped product of a Kaehler manifold $\bar{M}$. Then, we have*

$$TX_2(lnf)g(Z, V) = -g(h(Z, X_2), JV) + g(h(Z, V), FX_2) \tag{29}$$

for $X_2 \in \Gamma(\mathcal{D}^\theta)$, and $Z, V \in \Gamma(\mathcal{D}^\perp)$.

Proof. From (2), (3) and (6) we get

$$g(\nabla_{X_2} Z, V) = g(\bar{\nabla}_Z TX_2, JV) + g(\bar{\nabla}_Z FX_2, JV)$$

for $X_2 \in \Gamma(\mathcal{D}^\theta)$, and $Z, V \in \Gamma(\mathcal{D}^\perp)$. Hence, using (1) and (3), we obtain

$$X_2(lnf)g(Z, V) = g(h(Z, TX_2), JV) - g(FX_2, \bar{\nabla}_Z JV).$$

Using again (2) we arrive at

$$X_2(lnf)g(Z, V) = g(h(Z, TX_2), JV) + g(JFX_2, \bar{\nabla}_Z V).$$

Thus, from (7), we derive

$$X_2(lnf)g(Z, V) = g(h(Z, TX_2), JV) + g(BFX_2 + CFX_2, \bar{\nabla}_Z V).$$

Then, (3), (4), (9) and (10) imply that

$$X_2(lnf)g(Z, V) = g(h(Z, TX_2), JV) - sin^2\theta g(X_2, \nabla_Z V) - g(FTX_2, h(Z, V)).$$

Hence, we have

$$X_2(lnf)g(Z, V) = g(h(Z, TX_2), JV) + sin^2\theta g(\nabla_Z X_2, V) - g(FTX_2, h(Z, V)).$$

Thus, using(1) we obtain

$$X_2(lnf)g(Z, V) = g(h(Z, TX_2), JV) + sin^2\theta X_2(lnf)g(Z, V) - g(FTX_2, h(Z, V)).$$

Thus, we get

$$cos^2\theta\, X_2(lnf)g(Z, V) = g(h(Z, TX_2), JV) - g(FTX_2, h(Z, V)).$$

Now, substituting X_2 by TX_2 and using (8), we obtain(29). $\square$

Let M be an $(m_1 + m_2 + m_3)-$ dimensional proper skew CR-warped product of a Kaehler manifold $\bar{M}^{m_2+m_3}$. Then, we choose a canonical orthonormal frame $\{e_1, ..., e_{m_1},\ \bar{e}_1, ..., \bar{e}_{m_2},\ \tilde{e_1}, ..., \tilde{e}_{m_3},\ J\tilde{e_1}, ..., J\tilde{e}_{m_3},\ e_1^*, ..., e_{m_2}^*\}$ such that $\{e_1, ..., e_{m_1}\}$ is an orthonormal basis of $\mathcal{D}^T$, $\{\bar{e}_1, ..., \bar{e}_{m_2}\}$ is an orthonormal basis of $\mathcal{D}^\theta$ and $\{\tilde{e_1}, ..., \tilde{e}_{m_3}\}$ is an orthonormal basis of $\mathcal{D}^\perp$. It is known that the ranks of $\mathcal{D}^T$ and $\mathcal{D}^\theta$ are even. Hence, $m_1 = 2n_1$ and $m_2 = 2n_2$. Then, we can choose orthonormal frames $\{\bar{e}_1, ..., \bar{e}_{m_2}\}$ and$\{e_1^*, ..., e_{m_2}^*\}$ such that

$$\begin{aligned} \bar{e_2} &= sec\,\theta\, T\bar{e}_1, \ldots \bar{e}_{2n_2} = sec\,\theta\, T\bar{e}_{2n_2-1}, \\ e_1^* &= csc\,\theta\, F\bar{e}_1, \ldots e_{2n_2}^* = csc\,\theta\, F\bar{e}_{2n_2}, \end{aligned}$$

where θ is the slant angle of $\mathcal{D}^\theta$. Notice that it is called such an orthonormal frame an adapted frame [8].

We are now ready to state and prove the main theorem of this section.

Theorem 2. *Let M be an $(m_1 + m_2 + m_3)-$ dimensional proper $\mathcal{D}^\theta - \mathcal{D}^\perp$ mixed totally geodesic skew CR-warped product of a Kaehler manifold $\bar{M}^{m_2+m_3}$. Then we have*

(1) The squared norm of the second fundamental form of M satisfies

$$\| h \|^2 \geq m_3[2 \| \nabla^T(lnf) \|^2 + cot^2\theta \| \nabla^\theta(lnf) \|^2], \tag{30}$$

where $m_3 = dim(M_\perp)$, $\nabla^T(lnf)$ and $\nabla^\theta(lnf)$ are gradients of lnf on $\mathcal{D}^T$ and $\mathcal{D}^\theta$, respectively.

(2) If the equality sign of (30) holds identically, then M_1 is a totally geodesic submanifold and $M_\perp$ is a totally umbilical submanifold of $\bar{M}$.

Proof. Since

$$\| h \|^2 = \| h(\mathcal{D}^T,\mathcal{D}^T) \|^2 + \| h(\mathcal{D}^\theta,\mathcal{D}^\theta) \|^2 + \| h(\mathcal{D}^\perp,\mathcal{D}^\perp) \|^2 + 2 \| h(\mathcal{D}^T,\mathcal{D}^\perp) \|^2$$
$$+ 2 \| h(\mathcal{D}^T,\mathcal{D}^\theta) \|^2 + 2 \| h(\mathcal{D}^\perp,\mathcal{D}^\theta) \|^2$$

and M is $\mathcal{D}^\theta - \mathcal{D}^\perp$ mixed totally geodesic, we get

$$\| h \|^2 = \| h(\mathcal{D}^T,\mathcal{D}^T) \|^2 + \| h(\mathcal{D}^\theta,\mathcal{D}^\theta) \|^2 + \| h(\mathcal{D}^\perp,\mathcal{D}^\perp) \|^2 + 2 \| h(\mathcal{D}^T,\mathcal{D}^\perp) \|^2$$
$$+ 2 \| h(\mathcal{D}^T,\mathcal{D}^\theta) \|^2 .$$

Hence, we have

$$\begin{aligned}
\| h \|^2 = & \sum_{i,j=1}^{m_1}\sum_{a=1}^{m_3} g(h(e_i,e_j),J\tilde{e}_a)^2 + \sum_{i,j=1}^{m_1}\sum_{r=1}^{m_2} g(h(e_i,e_j),e_r^*)^2 \\
& + \sum_{a,b,c=1}^{m_3} g(h(\tilde{e}_a,\tilde{e}_b),J\tilde{e}_c)^2 + \sum_{a,b=1}^{m_3}\sum_{r=1}^{m_2} g(h(\tilde{e}_a,\tilde{e}_b),e_r^*)^2 \\
& + \sum_{p,q,r=1}^{m_2} g(h(\bar{e}_r,\bar{e}_s),e_p^*)^2 + \sum_{a=1}^{m_3}\sum_{r,s=1}^{m_2} g(h(\bar{e}_r,\bar{e}_s),J\tilde{e}_a)^2 \\
& + 2\sum_{i=1}^{m_1}\sum_{a=1}^{m_3}\sum_{r=1}^{m_2} g(h(e_i,\bar{e}_r),J\tilde{e}_a)^2 + 2\sum_{i=1}^{m_1}\sum_{r,s=1}^{m_2} g(h(e_i,\bar{e}_r),e_s^*)^2 \\
& + 2\sum_{i=1}^{m_1}\sum_{a,b=1}^{m_3} g(h(e_i,\tilde{e}_a),J\tilde{e}_b)^2 + 2\sum_{i=1}^{m_1}\sum_{a=1}^{m_3}\sum_{r=1}^{m_2} g(h(e_i,\tilde{e}_a),e_r^*)^2.
\end{aligned}$$

Then, using (25), (26), (28), (27) and considering the adapted frame, we obtain

$$\begin{aligned}
\| h \|^2 = & \sum_{i,j=1}^{m_1}\sum_{r=1}^{m_2} g(h(e_i,e_j),F\bar{e}_r)^2 csc^2\theta + \sum_{a,b,c=1}^{m_3} g(h(\tilde{e}_a,\tilde{e}_b),J\tilde{e}_c)^2 \\
& + \sum_{a,b=1}^{m_3}\sum_{r=1}^{m_2} g(h(\tilde{e}_a,\tilde{e}_b),F\bar{e}_r)^2 csc^2\theta + \sum_{p,q,r=1}^{m_2} g(h(\bar{e}_r,\bar{e}_s),F\bar{e}_p)^2 csc^2\theta \\
& + 2\sum_{i=1}^{m_1}\sum_{r,s=1}^{m_2} g(h(e_i,\bar{e}_r),F\bar{e}_s)^2 csc^2\theta + 2\sum_{i=1}^{m_1}\sum_{a,b=1}^{m_3} [Je_i(lnf)g(\tilde{e}_a,\tilde{e}_b)]^2 \\
& + 2\sum_{i=1}^{m_1}\sum_{a=1}^{m_3}\sum_{r=1}^{m_2} g(h(e_i,\tilde{e}_a),F\bar{e}_r)^2 csc^2\theta.
\end{aligned}$$

Thus, $\mathcal{D}^\theta - \mathcal{D}^\perp$ mixed totally geodesic M and (29) imply that

$$\begin{aligned}
\parallel h \parallel^2 = &\sum_{i,j=1}^{m_1}\sum_{r=1}^{m_2} g(h(e_i,e_j),F\bar{e}_r)^2 csc^2\theta + \sum_{a,b,c=1}^{m_3} g(h(\tilde{e}_a,\tilde{e}_b),J\tilde{e}_c)^2 \\
&+ \sum_{p,q,r=1}^{m_2} g(h(\bar{e}_r,\bar{e}_s),F\bar{e}_p)^2 csc^2\theta + \sum_{a,b=1}^{m_3}\sum_{r=1}^{m_2}[T\bar{e}_r(lnf)g(\tilde{e}_a,\tilde{e}_b)]^2 csc^2\theta \\
&+2\sum_{i=1}^{m_1}\sum_{r,s=1}^{m_2} g(h(e_i,\bar{e}_r),F\bar{e}_s)^2 csc^2\theta + 2\sum_{i=1}^{m_1}\sum_{a,b=1}^{m_3}[Je_i(lnf)g(\tilde{e}_a,\tilde{e}_b)]^2 \\
&+2\sum_{i=1}^{m_1}\sum_{a=1}^{m_3}\sum_{r=1}^{m_2} g(h(e_i,\tilde{e}_a),F\bar{e}_r)^2 csc^2\theta. \qquad (31)
\end{aligned}$$

On the other hand, by direct computations, we we have

$$\begin{aligned}
\sum_{p=1}^{m_2}[T\bar{e}_r(lnf)]^2 csc^2\theta = &[T\bar{e}_1(lnf)]^2 csc^2\theta + [sec\,\theta T^2\bar{e}_1(lnf)]^2 csc^2\theta \\
&+[T\bar{e}_2(lnf)]^2 csc^2\theta + [sec\,\theta T^2\bar{e}_2(lnf)]^2 csc^2\theta + \dots \\
&+[T\bar{e}_{2n_2-1}(lnf)]^2 csc^2\theta + [sec\,\theta T^2\bar{e}_{2n_2-1}(lnf)]^2 csc^2\theta
\end{aligned}$$

Then, using (8) we get

$$\begin{aligned}
\sum_{p=1}^{m_2}[T\bar{e}_r(lnf)]^2 csc^2\theta = &[T\bar{e}_1(lnf)sec\,\theta]^2 cos^2\theta csc^2\theta + [-cos\,\theta\bar{e}_1(lnf)]^2 csc^2\theta \\
&+[T\bar{e}_2(lnf)sec\,\theta]^2 cos^2\theta csc^2\theta + [-cos\,\theta\bar{e}_2(lnf)]^2 csc^2\theta \\
&+\dots + [T\bar{e}_{2n_2} sec\,\theta(lnf)]^2 cos^2\theta csc^2\theta \\
&+[-cos\,\theta\bar{e}_{2n_2-1}(lnf)]^2 csc^2\theta.
\end{aligned}$$

Hence, we arrive at

$$\sum_{p=1}^{m_2}[T\bar{e}_r(lnf)]^2 csc^2\theta = cot^2\theta(\nabla^\theta lnf)^2. \qquad (32)$$

Then, using (32) in (31) we obtain (30). If the equality sign of (30) holds, from (31) we have

$$h(\mathcal{D}^T,\mathcal{D}^T) = 0, \quad h(\mathcal{D}^\theta,\mathcal{D}^T) = 0, \quad h(\mathcal{D}^T,\mathcal{D}^\perp) \subset J(\mathcal{D}^\perp), \qquad (33)$$

$$h(\mathcal{D}^\perp,\mathcal{D}^\perp) \subset F(\mathcal{D}^\theta), \quad h(\mathcal{D}^\theta,\mathcal{D}^\theta) \subset J(\mathcal{D}^\perp). \qquad (34)$$

Since M is $\mathcal{D}^\theta - \mathcal{D}^\perp$ mixed totally geodesic, from (28) we have $g(h(X_2,Y_2),JZ) = 0$ for $X_2, Y_2 \in \Gamma(D^\theta)$ and $Z \in \Gamma(\mathcal{D}^\perp)$. Thus, using (34) we obtain $h(\mathcal{D}^\theta,\mathcal{D}^\theta) = 0$. Then (33) implies that M_1 being a totally geodesic in $\bar{M}$ due to M_1 is totally geodesic submanifold of M. Furthermore, the first equation of (34), (29) and $\mathcal{D}^\theta - \mathcal{D}^\perp$ mixed totally geodesic M imply that

$$g(h(Z,V), FX_2) = TX_2(lnf)g(Z,V)$$

for $X_2 \in \Gamma(D^\theta)$ and $Z, V \in \Gamma(\mathcal{D}^\perp)$. Hence, since $M_\perp$ is totally umbilical in M, it follows that $M_\perp$ is a totally umbilical submanifold of $\bar{M}$. Thus, the proof is complete. $\square$

Remark 2. *In case $\mathcal{D}^\theta = \{0\}$, Theorem 2 coincides with Theorem 5.1 of [9]. Thus, Theorem 2 is a generalization of Theorem 5.1 of [9].*

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