Math. Commun., Vol. **15**, No. 1, pp. 205-212 (2010)

Parametric general quasi variational inequalities

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Received June 27, 2009; accepted January 4, 2010

Abstract. In this paper, we prove that the general quasi variational inequalities are equivalent to the fixed point problem using the projection technique. We use this equivalent formulation to study the sensitivity analysis of the general quasi variational inequality. Our approach is very simple. Several special cases are also discussed.
AMS subject classifications: 49J40, 90C33

Key words: variational inequalities, sensitivity analysis, fixed-point problem

1. Introduction

The ideas and techniques of variational inequalities are being applied in a variety of diverse areas of pure and applied sciences, and proved to be innovative and productive. It has been shown that variational inequalities provide a novel, natural, simple, unified and efficient framework for a general treatment of a wide class of unrelated linear and nonlinear problems. This theory combines theoretical and algorithmic advances with a new and novel domain of applications. Analysis of these problems requires a blend of techniques from convex analysis, functional analysis and numerical analysis, see [1-35] and the references therein.

In recent years, variational inequalities have been generalized and extended in several directions using new and novel techniques. Noor [25] has introduced and considered a new class of variational inequalities which is called the general quasi variational inequality. In this paper, we study sensitivity analysis of general quasi variational inequalities, that is, examining how the solutions of such problems change when the data of the problems are changed. This is an important problem for several reasons. We would like to mention that sensitivity analysis provides useful information for designing or planning various equilibrium systems. Sensitivity analysis can provide a new insight and can stimulate new ideas and techniques for problem solving. Dafermos [4] studied sensitivity analysis of variational inequalities using the

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fixed point technique. This technique has been modified and extended by many authors, see Noor-Noor and Rassias [27]. This approach has strong geometrical flavour. We extend this for general quasi variational inequalities. We show that general quasi variational inequalities are equivalent to the fixed point problem. This alternative equivalent form is used to consider sensitivity analysis of general quasi variational inequalities without assuming the differentiability of the given data. Our analysis is in the spirit of Dafermos [5] and Noor [18,19]. Since general quasi variational inequalities include general variational inequalities, quasi variational inequalities and complementarity problems as special cases, our result continues to hold for these problems.

2. Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|.\|$, respectively. Let $K : H \longrightarrow H$ be a point to set mapping, which is closed and convex valued. In other words, for every $u \in H$, the set $K(u)$ is closed and convex.

For given nonlinear operators $T, g : H \rightarrow H$, consider the problem of finding $u \in H : g(u) \in K(u)$ such that

$$\langle \rho Tu + u - g(u), g(v) - u \rangle \geq 0, \quad \forall v \in H : g(v) \in K(u), \tag{1}$$

where $\rho > 0$ is a constant. Inequality of type (1) is called a general quasi variational inequality involving two operators, which was introduced and studied by Noor [25]. This class of quasi variational inequalities is a quite general and unified one.

If $K(u) \equiv K$, that is, the convex set $K(u)$ is independent of the solution u, then general quasi variational inequalities (1) are equivalent to finding $u \in H : g(u) \in K(u)$ such that

$$\langle Tu + u - g(u), g(v) - u \rangle \geq 0, \quad \forall v \in H : g(v) \in K, \tag{2}$$

which is called a general variational inequality involving two operators and was introduced and studied by Noor [22]. It has been shown [22] that the minimum of a nonconvex differentiable function on a nonconvex set K in H can be characterized by a general variational inequality (2).

Noor [14] considered and studied the following quasi variational inequality of finding $u \in H : g(u) \in K(u)$ such that

$$\langle \rho Tu + g(u) - g(u), v - g(u) \rangle \geq 0, \quad \forall v \in K(u). \tag{3}$$

Obviously problems (1) and (3) are quite different and have applications in pure and applied sciences.

For $g = I$, the identity operator, the general quasi variational inequality (1) is equivalent to finding $u \in K(u)$ such that

$$\langle Tu, v - u \rangle \geq 0, \quad \forall v \in K(u), \tag{4}$$

which is known as a classical quasi variational inequality, introduced and studied by Bensoussan and Lions [3] in the study of impulse control theory. See also [1-32].

If $K(u) \equiv K$, and $g = I$, the identity operator, then problem (1) is equivalent to finding $u \in K$ such that

$$\langle Tu, v - u \rangle \geq 0, \quad \forall v \in K, \tag{5}$$

which is known as a classical variational inequality which was introduced in 1964 by Stampacchia [32]. For recent applications, numerical methods, sensitivity analysis, dynamical systems and formulation of variational inequalities, see [1-35] and the references therein. In brief, we conclude that the general quasi variational inequality (1) is quite general and includes several classes of variational inequalities and related optimization problems as special cases.

We also need the following standard and classical result.

Lemma 1 (see [13]). *Let $K(u)$ be a closed and convex set in H. Then, for a given $z \in H$, $u \in K(u)$ satisfies the inequality*

$$\langle u - z, v - u \rangle \geq 0, \quad \forall v \in K(u),$$

if and only if

$$u = P_{K(u)} z,$$

where $P_{K(u)}$ is the projection of H onto the closed convex set $K(u)$ in H.

We would like to point out that the implicit projection operator $P_{K(u)}$ is not nonexpansive. We shall assume that the implicit projection operator $P_{K(u)}$ satisfies the Lipschitz type continuity, which plays an important and fundamental role in the existence theory and in developing numerical methods for solving quasi variational inequalities.

Assumption 1. *The implicit projection operator $P_{K(u)}$ satisfies the condition*

$$\|P_{K(u)}w - P_{K(v)}w\|| \leq \nu\|u - v||, \quad \forall u, v, w \in H,$$

where $\nu > 0$ is a positive constant.

Assumption 1 has been used to prove the existence of a solution of quasi variational inequalities as well as in analyzing convergence of the iterative methods, see [18, 19, 20].

In many important applications [2, 3, 19] the convex-valued set $K(u)$ can be written as

$$K(u) = m(u) + K, \tag{6}$$

where $m(u)$ is a point-point mapping and K is a convex set. In this case, we have

$$P_{K(u)}w = P_{m(u)+K}(w) = m(u) + P_K[w - m(u)], \quad \forall u, v \in H.$$

We note that if $K(u)$ is defined by (6) and $m(u)$ is a Lipschitz continuous mapping with constant $\gamma > 0$, then

$$\begin{aligned} \|P_{K(u)}w - P_{K(v)}w\| &= \|m(u) - m(v) + P_K[w - m(u)] - P_K[w - m(v)\| \\ &\leq 2\|m(u) - m(v)\| \leq 2\gamma\|u - v\|, \quad \forall u, v, w \in H, \end{aligned}$$

which shows that Assumption 1 holds with $\nu = 2\gamma$.

We now consider parametric versions of problem (1). To formulate the problem, let M be an open subset of H in which the parameter λ takes values. Let $T(u, \lambda)$ be the given operator defined on $H \times H \times M$ and take value in $H \times H$. From now onward, we denote $T_\lambda(.) \equiv T(., \lambda)$ unless otherwise specified.

The parametric general variational inequality problem is to find $(u, \lambda) \in H \times M$ such that

$$\langle \rho T_\lambda u + u - g(u), g(v) - u \rangle \geq 0, \quad \forall v \in H : g(v) \in K. \tag{7}$$

We also assume that for some $\overline{\lambda} \in M$, problem (7) has a unique solution $\overline{u}$.

From Lemma 1, we see that parametric general quasi variational inequalities are equivalent to the fixed point problem:

$$F_\lambda(u) \equiv u = P_K[g(u) - \rho T_\lambda u], \quad \forall (u, \lambda) \in X \times M. \tag{8}$$

We use this equivalence to study sensitivity analysis of general quasi variational inequalities. We assume that for some $\overline{\lambda} \in M$, problem (7) has a solution $\overline{u}$ and X is a closure of a ball in H centered at $\overline{u}$. We want to investigate those conditions under which, for each λ in a neighborhood of $\overline{\lambda}$, problem (7) has a unique solution $z(\lambda)$ near $\overline{u}$ and the function $u(\lambda)$ is (Lipschitz) continuous and differentiable.

Definition 1. *Let $T_\lambda(.)$ be an operator on $X \times M$. Then, the operator $T_\lambda(.)$ is said to be:*

(a) Locally strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle T_\lambda(u) - T_\lambda(v), u - v \rangle \geq \alpha \|u - v\|^2, \quad \forall \lambda \in M, u, v \in X.$$

(b) Locally Lipschitz continuous, if there exists a constant $\beta > 0$ such that

$$\|T_\lambda(u) - T_\lambda(v)\| \leq \beta \|u - v\|, \quad \forall \lambda \in M, u, v \in X.$$

3. Main results

We consider the case when the solutions of the parametric general quasi variational inequality (8) lie in the interior of X. Following the ideas of Dafermos [5] and Noor [18,19], we consider the map $F_\lambda(u)$ as defined by (8). We have to show that the map $F_\lambda(u)$ has a fixed point, which is a solution of the parametric general quasi variational inequality (7). First of all, we prove that the map $F_\lambda(u)$, defined by (8), is a contraction map with respect to z uniformly in $\lambda \in M$.

Lemma 2. *Let $T_\lambda(.)$ be a locally strongly monotone with constant $\alpha > 0$ and locally Lipschitz continuous with constant $\beta > 0$. Let the operator g be strongly monotone with constants $\sigma > 0$ and Lipschitz continuous with constants $\delta > 0$, respectively. If Assumption 1 holds and for all $u_1, u_2 \in X$ and $\lambda \in M$, we have*

$$\|F_\lambda(u_1) - F_\lambda(u_2)\| \leq \theta \|u_1 - u_2\|,$$

where

$$\theta = \left\{ \sqrt{1 - 2\sigma + \delta^2} + \nu + \sqrt{1 - 2\alpha\rho + \beta^2\rho^2} \right\} \tag{9}$$

for

$$|\rho - \frac{\alpha}{\beta^2}| < \frac{\sqrt{\alpha^2 - \beta^2 k(2-k)}}{\beta^2}, \quad \alpha > \beta\sqrt{k(2-k)}, \quad k < 1, \tag{10}$$

where

$$k = \sqrt{1 - 2\sigma + \delta^2} + \nu. \tag{11}$$

Proof. In order to prove the existence of a solution of (7), it is enough to show that the mapping $F_\lambda(u)$, defined by (8), is a contraction mapping.

For $u_1 \neq u_2 \in H$, and using Assumption 1, we have

$$\begin{aligned} \|F_\lambda(u_1) - F_\lambda(u_2)\| &= \|P_{K(u_1)}[g(u_1) - \rho T_\lambda u_1] - P_{K(u_2)}[g(u_2) - \rho T_\lambda u_2]\| \\ &\leq \|P_{K(u_1)}[g(u_1) - \rho T_\lambda u_1] - P_{K(u_2)}[g(u_1) - \rho T_\lambda u_1]\| \\ &\quad + \|P_{K(u_2)}[g(u_1) - \rho T_\lambda u_1] - P_{K(u_2)}[g(u_2) - \rho T_\lambda u_2]\| \\ &\leq \nu\|u_1 - u_2\| + \|g(u_1) - g(u_2) - \rho(T_\lambda u_1 - T_\lambda u_2)\| \\ &= \nu\|u_1 - u_2\| \\ &\quad + \| - (u_1 - u_2 - (g(u_1) - g(u_2))) + u_1 - u_2 - \rho(T_\lambda u_1 - T_\lambda u_2)\| \\ &\leq \nu\|u_1 - u_2\| + \|u_1 - u_2 - (g(u_1) - g(u_2))\| \\ &\quad + \|u_1 - u_2 - \rho(T_\lambda u_1 - T_\lambda u_2)\|. \end{aligned} \tag{12}$$

Since the operator T_λ is a locally strongly monotone with constant $\alpha > 0$ and locally Lipschitz continuous with constant $\beta > 0$, it follows that

$$\begin{aligned} ||u_1 - u_2 - \rho(T_\lambda u_1 - T_\lambda u_2)||^2 &\leq ||u_1 - u_2||^2 - 2\rho\langle T_\lambda u_1 - T_\lambda u_2, u_1 - u_2\rangle \\ &\quad + \rho^2||T_\lambda u_1 - T_\lambda u_2||^2 \\ &\leq (1 - 2\rho\alpha + \rho^2\beta^2)||u_1 - u_2||^2. \end{aligned} \tag{13}$$

In a similar way, we have

$$||u_1 - u_2 - (g(u_1) - g(u_2))||^2 \leq (1 - 2\sigma + \delta^2)||u_1 - u_2||^2, \tag{14}$$

where we have used the fact that g is strongly monotone with constant $\sigma > 0$ and Lipschitz continuous with constant $\delta > 0$.

From (9), (11), (13) and (14), we have

$$\begin{aligned} \|F_\lambda(u_1) - F_\lambda(u_2)\| &\leq \left\{ \nu + \sqrt{(1 - 2\sigma + \delta^2)} + \sqrt{(1 - 2\rho\alpha + \rho^2\beta^2)} \right\} \|u_1 - u_2\| \\ &= \left\{ k + \sqrt{(1 - 2\rho\alpha + \rho^2\beta^2)} \right\} \|u_1 - u_2\| \\ &= \theta\|u_1 - u_2\|, \end{aligned}$$

where

$$\theta = k + \sqrt{(1 - 2\rho\alpha + \rho^2\beta^2)}. \tag{15}$$

From (10), it follows that $\theta < 1$. Thus it follows that the mapping $F_\lambda(u)$, defined by (8), is a contraction mapping and consequently it has a fixed point, which belongs to $K(u)$ satisfying the general quasi variational inequality (7), the required result. □

Remark 1. *From Lemma 2, we see that the map $F_\lambda(z)$ defined by (8) has a unique fixed point $u(\lambda)$, that is, $u(\lambda) = F_\lambda(u)$. Also, by assumption, the function $\overline{u}$, for $\lambda = \overline{\lambda}$ is a solution of the parametric general quasi variational inequality (7). Again using Lemma 2, we see that $\overline{u}$, for $\lambda = \overline{\lambda}$, is a fixed point of $F_\lambda(u)$ and it is also a fixed point of $F_{\overline{\lambda}}(u)$. Consequently, we conclude that*

$$u(\overline{\lambda}) = \overline{u} = F_{\overline{\lambda}}(u(\overline{\lambda})).$$

Using Lemma 2, we can prove the continuity of the solution $u(\lambda)$ of the parametric general quasi variational inequality (7) using the technique of Noor [13,14]. However, for the sake of completeness and to convey an idea of the techniques involved, we give its proof.

Lemma 3. *Assume that the operator $T_\lambda(.)$ is locally Lipschitz continuous with respect to the parameter λ. If the operator $T_\lambda(.)$ is locally Lipschitz continuous and the map $\lambda \to P_{K_\lambda} u$ is continuous (or Lipschitz continuous), then the function $u(\lambda)$ satisfying (8) is (Lipschitz) continuous at $\lambda = \overline{\lambda}$.*

Proof. For all $\lambda \in M$, invoking Lemma 2 and the triangle inequality, we have

$$\begin{aligned} \|u(\lambda) - u(\bar{\lambda})\| &\leq \|F_\lambda(u(\lambda)) - F_{\bar{\lambda}}(u(\bar{\lambda})\| + \|F_\lambda(u(\bar{\lambda})) - F_{\bar{\lambda}}(u(\bar{\lambda}))\| \\ &\leq \theta\|u(\lambda) - u(\bar{\lambda})\| + \|F_\lambda(u(\bar{\lambda})) - F_{\bar{\lambda}}(u(\bar{\lambda}))\|. \end{aligned} \tag{16}$$

From (8) and the fact that the operator T_λ is Lipschitz continuous with respect to the parameter λ, we have

$$\begin{aligned} \|F_\lambda(u(\bar{\lambda})) - F_{\bar{\lambda}}(u(\bar{\lambda}))\| &= \|u(\bar{\lambda}) - u(\bar{\lambda}) + \rho(T_\lambda(u(\bar{\lambda}), u(\bar{\lambda})) - T_{\bar{\lambda}}(u(\bar{\lambda}), u(\bar{\lambda})))\| \\ &\leq \rho\mu\|\lambda - \bar{\lambda}\|. \end{aligned} \tag{17}$$

Combining (16) and (17), we obtain

$$\|u(\lambda) - u(\bar{\lambda})\| \leq \frac{\rho\mu}{1-\theta}\|\lambda - \bar{\lambda}\|, \quad \text{for all } \lambda, \bar{\lambda} \in M,$$

from which the required result follows. □

We now state and prove the main result of this paper which is the motivation of our next result.

Theorem 1. *Let $\overline{u}$ be the solution of the parametric general quasi variational inequality (7) for $\lambda = \overline{\lambda}$. Let $T_\lambda(u)$ be a locally strongly monotone Lipschitz continuous operator for all $u, v \in X$. If the map $\lambda \to P_K$ is (Lipschitz) continuous at $\lambda = \overline{\lambda}$, then there exists a neighborhood $N \subset M$ of $\overline{\lambda}$ such that for $\lambda \in N$, the parametric general quasi variational inequality (7) has a unique solution $u(\lambda)$ in the interior of X, $u(\overline{\lambda}) = \overline{u}$ and $u(\lambda)$ is (Lipschitz) continuous at $\lambda = \overline{\lambda}$.*

Proof. Its proof follows from Lemmas 2, 3 and Remark 1. □

Acknowledgement

The authors would like to thank Dr. S. M. Junaid Zaidi, Rector, CIIT, for providing excellent research facilities.

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