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Tricyclic biregular graphs whose energy exceeds the number of vertices

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Abstract. The eigenvalues of a graph are the eigenvalues of its adjacency matrix. The energy $E(G)$ of the graph G is the sum of the absolute values of the eigenvalues of G. A graph is said to be (a, b)-biregular if its vertex degrees assume exactly two different values: a and b. A connected graph with n vertices and m edges is tricyclic if $m = n + 2$. The inequality $E(G) \geq n$ is studied for connected tricyclic biregular graphs, and conditions for its validity are established.
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1. Introduction

In this paper we are concerned with simple graphs, i.e. graphs without multiple and directed edges and without loops. Let G be such a graph, and let n and m be, the number of its vertices and edges, respectively. Eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the adjacency matrix of G are called the eigenvalues of G and form the spectrum of G [2]. A spectrum–based graph invariant that recently attracted much attention of mathematicians is the energy defined as

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

Details of the theory of graph energy can be found in the reviews [3, 8], recent papers [1, 4, 6, 7, 9, 11, 12, 13, 17], and references cited therein.

One of the problems in the theory of graph energy is the characterization of graphs whose energy exceeds the number of vertices, i.e. of graphs satisfying the inequality

$$E(G) \geq n \ . \tag{1}$$

The first results along these lines were communicated in [10] and a systematic study was initiated in [4]. In particular, it was shown that (1) is satisfied by (i) regular graphs [10], (ii) graphs whose all eigenvalues are non-zero [4], and (iii) graphs having

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a large number of edges, $m \geq n^2/4$ [4]. In view of these results, it was purposeful to examine the validity of (1) for biregular graphs (whose definition is given in the subsequent section), especially those possessing a small number of edges. Acyclic, unicyclic, and bicyclic biregular graphs satisfying (1) were studied in [1, 6, 11, 12]. In the present paper we extend these researches to the (much more complicated) case of tricyclic biregular graphs.

Graphs violating condition (1), i.e. graphs whose energy is less than the number of vertices, are referred to as hypoenergetic graphs [9]. Some results on hypoenergetic graphs were recently obtained for trees [7, 13] as well as unicyclic and bicyclic graphs [17].

2. Preliminaries

All graphs considered in this paper are assumed to be connected.

Let a and b be integers, $1 \leq a < b$. A graph G is said to be (a, b)-biregular if its vertex degrees assume exactly two different values: a and b.

Let n be the number of vertices in the graph G and m the number of its edges. The (connected) graph G is said to be tricyclic if $m = n + 2$.

In this paper we are interested in (connected) biregular tricyclic graphs whose energy exceeds the number of vertices, i. e. which obey inequality (1).

It is known [14, 16] that the energy of any graph satisfies the inequality

$$E(G) \geq \sqrt{\frac{(M_2)^3}{M_4}}, \tag{2}$$

where M_2 and M_4 are the second and fourth spectral moments, respectively [2]. These moments can be easily calculated from simple structural details of the underlying graph:

$$M_2 = 2m,$$
$$M_4 = 2\sum_{i=1}^{n}(d_i)^2 - 2m + 8q,$$

where q is the number of quadrangles and d_i the degree of the i-th vertex, $i = 1, \ldots, n$.

From (2) it is evident that whenever condition (3)

$$\sqrt{\frac{(M_2)^3}{M_4}} \geq n \tag{3}$$

is satisfied, then inequality (1) will also be satisfied. In what follows we establish necessary and sufficient conditions under which (3) holds for tricyclic biregular graphs. By this we establish sufficient (but not necessary) conditions for the validity of inequality (1).

We begin with the equalities

$$n_a + n_b = n \tag{4}$$

and

$$a \cdot n_a + b \cdot n_b = 2m \tag{5}$$

where n_a and n_b are the numbers of vertices of G of degree a and b, respectively. Bearing in mind that for any tricyclic graph $m = n + 2$, we obtain

$$n_a = \frac{n(b-2)-4}{b-a}, \quad n_b = \frac{n(2-a)+4}{b-a}.$$

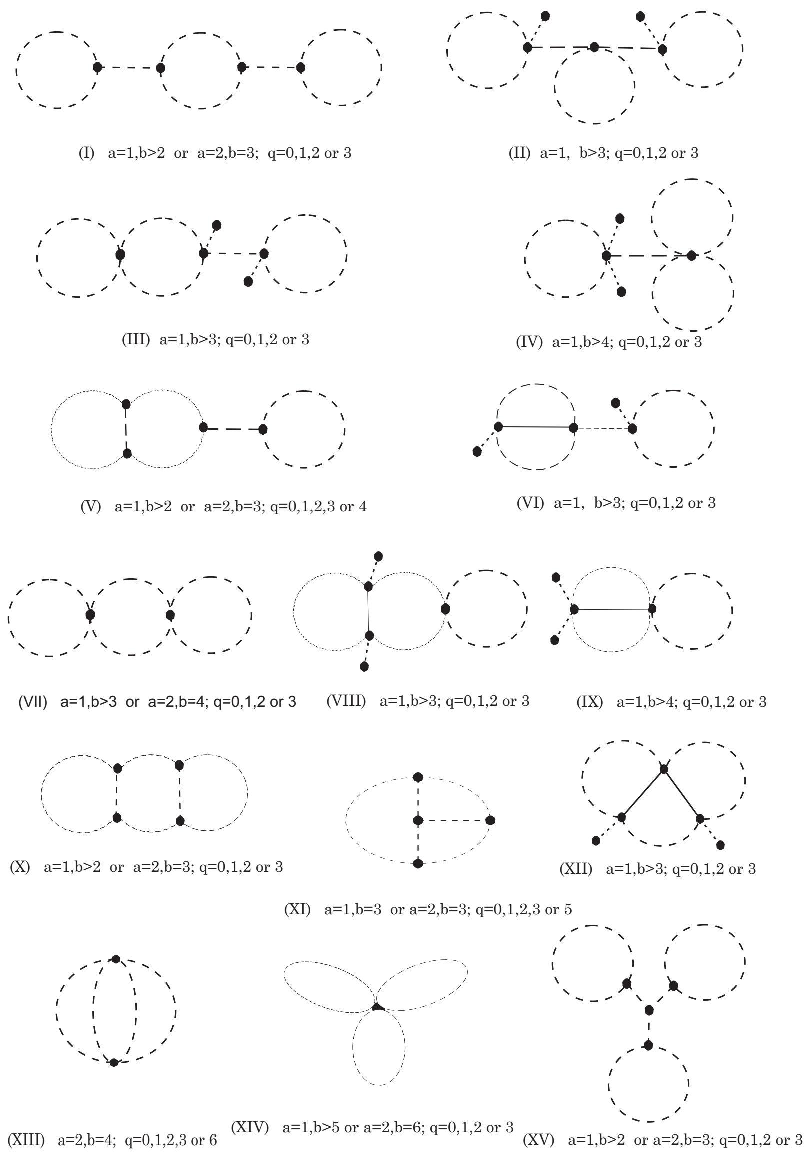

Figure 1. *Classes of tricyclic graphs. If these graphs are (a, b)-biregular, then their vertex degrees a and b and the number of quadrangles q may assume the indicated values*

Next, we have

$$\sum_{i=1}^{n}(d_i)^2 = a^2 \cdot n_a + b^2 \cdot n_b = (4+2n)(b+a) - abn.$$

By this, we arrive at expressions for the second and fourth spectral moments:

$$\begin{aligned} M_2 &= 2(n+2), \\ M_4 &= 2(2a+2b-1)(n+2) - 2abn + 8q \end{aligned}$$

by means of which inequality (3) becomes

$$\sqrt{\frac{4(n+2)^3}{(2a+2b-1)(n+2) - abn + 4q}} \geq n. \tag{6}$$

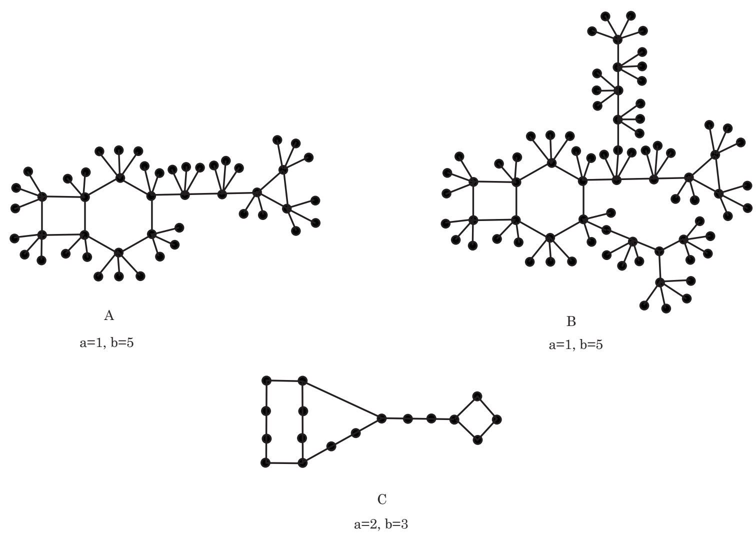

Figure 2. *Explaining the diagrams depicted in Figure 1 by examples of tricyclic biregular graphs of class V. In graphs A and B vertex degrees are $a = 1$ and $b = 5$. Graph B differs from graph A in two (1,5)-biregular trees attached. Evidently, if the smaller vertex degree (a) is equal to one, then the greater vertex degree (b) may assume any value greater than two. In graph C vertex degrees are $a = 2$ and $b = 3$. If $a = 2$, then it must be $b = 3$. Further, it cannot be $a > 2$. In all three graphs A, B, and C, the number of quadrangles q is equal to one*

There are 15 different classes of biregular tricyclic graphs [5, 15]. Each of these is illustrated in Figure 1 and under each diagram all possible values for a, b, and q are given. Dotted lines indicate that an arbitrary number of vertices can be put on them. If $a = 2$, these are vertices of degree 2. If $a = 1$, we can attach entire $(1, b)$-biregular trees. Additional explanations are given in Figure 2.

3. Main results

From Figures 1 and 2 it should be evident that $a \in \{1,2\}$, i.e. that the smallest vertex degree of a biregular tricyclic graph cannot be greater than 2. Bearing this in mind, we divide all classes of tricyclic (a,b)-biregular graphs into two groups and examine each group separately.

Theorem 1. *Let G be a connected tricyclic $(1,b)$-biregular graph with n vertices. Then, inequality (3) holds if and only if either $b=3$ and $q=0,1,2$, or $b=3$, $q=3$, and $n \leq 24$.*

Theorem 2. *For every connected tricyclic $(1,b)$-biregular graph with $b \geq 4$, inequality (3) is not satisfied.*

Theorem 3. *For every connected tricyclic $(2,3)$-biregular graph inequality (3) holds.*

Theorem 4. *Let G be a connected tricyclic $(2,4)$-biregular graph. Then inequality (3) holds if and only if $q \neq 6$.*

Theorem 5. *Let G be a connected tricyclic $(2,6)$-biregular graph. Then inequality (3) holds if and only if $q \neq 3$.*

Evidently, the immediate consequence of Theorem 1 is that relation (1) holds if either $b=3$ and $q=0,1,2$, or $b=3$, $q=3$, and $n \leq 24$. Analogous consequences are also deduced from Theorems 2–5.

4. Proofs

Proof. (of Theorem 1)
We need to consider each class except XIII and $q \in \{0,1,2,3\}$.

Let $a=1$. Then inequality (6) becomes

$$\sqrt{\frac{4(n+2)^3}{n(b+1)+2(1+2b+2q)}} \geq n, \tag{7}$$

and from this we obtain

$$b \leq \frac{3n^3+n^2(22-4q)+48n+32}{n^2(n+4)}. \tag{8}$$

For $q=0,1,2$, and 3 we have

$$b \leq \frac{3n^3+22n^2+48n+32}{n^2(n+4)} \tag{9}$$

$$b \leq \frac{3n^3+18n^2+48n+32}{n^2(n+4)} \tag{10}$$

$$b \leq \frac{3n^3+14n^2+48n+32}{n^2(n+4)} \tag{11}$$

$$b \leq \frac{3n^3+10n^2+48n+32}{n^2(n+4)}, \tag{12}$$

respectively.

We may substitute n on the right–hand sides of the inequalities (9)–(12) by $x \in \mathbb{R}$ and then examine the respective functions of the variable x. Calculating the first derivatives of the first three functions, we conclude that for every $x \geq 1$ these monotonically decrease and their lower bound is 3. On the other hand, the function corresponding to (12), namely

$$f(x) = \frac{3x^3 + 10x^2 + 48x + 32}{x^2(x+4)}, \quad x \geq 1, x \in \mathbb{R},$$

has a stationary point $x = 50.8797$ at which it reaches its minimal value 2.98097. Thus, for $x \in (1, 50.8797)$, the function f monotonically decreases, for $x \in (50.8797, +\infty)$ it monotonically increases, and its upper bound is 3. Since b is never less than 3, we are interested only in the first interval. There, the function f has values greater than or equal to 3 if $x \in [1, 24]$.

We start with graphs for which $b > 2$. These pertain to the classes I, V, X, XI, and XV. With the condition $b > 2$, the expressions on the right–hand sides of (9)–(12) must be at least 3. For Eqs. (9)–(11) this is true for every $n \in \mathbb{N}$, whereas for (12) we have the condition $n \leq 24$. With these conditions for n, we conclude that one possible value for b is 3. Now, we will see that b cannot have any other value.

For example, if we take into consideration class I, then the smallest such graph with $q = 0$ has 14 vertices. With $n = 14$ the value of the expression on the right–hand side of (9) is equal to 3.75 and it decreases with increasing n. Therefore it must be $b = 3$.

If $q = 1$ the smallest graph has 16 vertices and from (10) we get $b \leq 3.45$ and again $b = 3$.

For $q = 2$ we have $n = 18$ and from (11) we obtain $b \leq 3.21$, implying $b = 3$.

If $q = 3$ we have $n = 20$ and from (12) we get $b \leq 3.02$. Here $n \leq 24$, and therefore (12) holds for $n = 20, 22, 24$.

In a similar way, classes V, X, and XV are analyzed: For class V and $q = 0, 1, 2$ we have $n \geq 14, 10, 12$, respectively, and the inequalities (9), (10), and (11) are satisfied only for $b = 3$. For $q = 3$ we have $n \geq 16$ and we conclude that inequality (12) holds only for $b = 3$ and $n = 16, 18, 20, 22, 24$.

In the same way we conclude that for class X the corresponding inequalities hold only for $b = 3$. Specially, for $q = 3$ there is a limited number of graphs for which (12) holds. These are the ones with 12, 14, 16, 18, 20, 22, and 24 vertices.

For class XV we get $b = 3$ as well, and for $q = 3$ it must be $n \in \{22, 44\}$.

For class XI we know that $b = 3$, so the inequalities (9)–(11) are true for every n. For $q = 3$ the smallest graph has 6 vertices, and thus (12) is true for $n =$ 6, 8, 10, 12, 14, 16, 18, 20, 22, and 24.

By taking into account the smallest possible number of vertices, we conclude that for graphs with $b > 3$ inequalities (9)–(12) are not satisfied. □

Proof. (of Theorem 2)
The proof follows from Theorem 1 and from the information on the smallest number of vertices in such graphs. □

In agreement with Theorem 2, for $b \geq 4, 5, 6$ (classes II, III, IV, VI, VII, VIII, IX, XII, and XIV), the inequalities (9)–(12) are not satisfied. Some of the smallest such graphs with different values for q are depicted in Figure 3. By an easy graph–theoretical reasoning, it can be seen that the graphs (2), (3), and (4) are unique. Namely, if we want to construct such (connected, tricyclic, biregular) graphs belonging to the prescribed class, with the required values for the parameters a, b, and q, then we realize that this can be done in just a single way. (The same holds for the "unique" graphs mentioned later in this paper.)

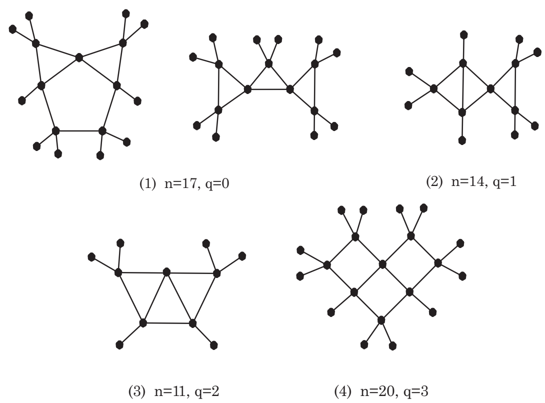

Figure 3. *Tricyclic* $(1,4)$*-biregular graphs with the minimum number of vertices. The graphs (1) belong to classes XII and VII, respectively. Graph (2) belongs to class VIII. Graphs (3) and (4) belong to class XII*

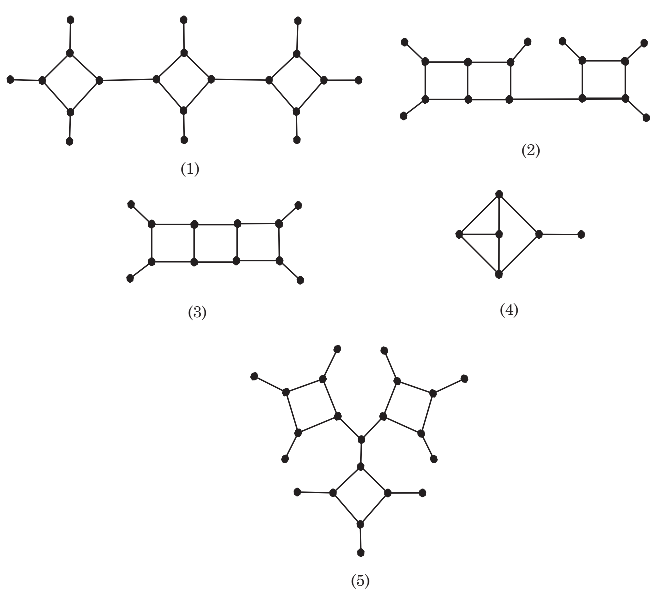

Figure 4. *Tricyclic* $(1,3)$*-biregular graphs with* $q = 3$ *and the minimum number of vertices. Graphs (1), (2), (3), (4), and (5) belong to classes I, V, X, XI, and XV, respectively*

Figure 4 shows examples of tricyclic $(1,3)$-biregular graphs with $q = 3$ and the minimum number of vertices. Graphs (1), (2), (3), (4), and (5) belong to classes I, V, X, XI, and XV, respectively. Inequality (3) holds for these graphs because they have $n \leq 24$ vertices.

Proof. (of Theorem 3)

For $a = 2$ and $b = 3$ the respective graphs belong to classes I, V, X, XI, and XV. Then inequality (6) becomes

$$\sqrt{\frac{4(n+2)^3}{3n+18+4q}} \geq n,$$

and we obtain the inequality

$$n^3 + (6-4q)n^2 + 48n + 32 \geq 0.$$

Possible values for q are 0, 1, 2, 3, 4, and 5. With each of these values the upper inequality holds for arbitrary $n \in \mathbb{N}$. $\square$

Proof. (of Theorem 4)

Graphs with $a = 2$ and $b = 4$ pertain to classes VII and XIII, and the number of quadrangles q can be 0, 1, 2, 3, and 6. From (6) we obtain

$$\sqrt{\frac{4(n+2)^3}{3n+22+4q}} \geq n$$

and thus

$$n^3 + (2-4q)n^2 + 48n + 32 \geq 0.$$

For $q = 0, 1, 2, 3,$ and 6, the latter inequality becomes:

$$n^3 + 2n^2 + 48n + 32 \geq 0 \tag{13}$$
$$n^3 - 2n^2 + 48n + 32 \geq 0 \tag{14}$$
$$n^3 - 6n^2 + 48n + 32 \geq 0 \tag{15}$$
$$n^3 - 10n^2 + 48n + 32 \geq 0 \tag{16}$$
$$n^3 - 22n^2 + 48n + 32 \geq 0, \tag{17}$$

respectively.

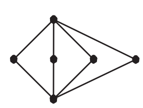

Figure 5. *The unique tricyclic* $(2,4)$*-biregular graph possessing six quadrangles*

Inequalities (13)–(16) hold for arbitrary $n \in \mathbb{N}$, while (17) holds only for $n \leq 3$ and $n \geq 20$. Bearing in mind that the tricyclic $(2,4)$-biregular graph with $q = 6$ is unique and has 6 vertices, see Figure 5, it is the only example of such graph for which inequality (3) is not fulfilled. $\square$

Proof. (of Theorem 5)

If $a = 2$ and $b = 6$, then the graph belongs to class XIV. From (6) we obtain

$$\sqrt{\frac{4(n+2)^3}{3n+30+4q}} \geq n$$

and

$$n^3 - (6+4q)n^2 + 48n + 32 \geq 0.$$

For $q = 0, 1, 2, 3$ the latter inequality becomes

$$n^3 - 6n^2 + 48n + 32 \geq 0 \tag{18}$$

$$n^3 - 10n^2 + 48n + 32 \geq 0 \tag{19}$$

$$n^3 - 14n^2 + 48n + 32 \geq 0 \tag{20}$$

$$n^3 - 18n^2 + 48n + 32 \geq 0, \tag{21}$$

respectively.

Again, we have the exception (21) which holds only for $n \leq 4$ and $n \geq 15$. Since there exists a unique tricyclic $(2, 6)$-biregular graph with $q = 3$ and it has 10 vertices, see Figure 6, it is the only example of such graph for which inequality (3) is not fulfilled. □

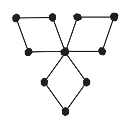

Figure 6. *The unique tricyclic* $(2, 6)$*-biregular graph with three quadrangles*

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