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Convergence of Ishikawa iterative sequence for strongly pseudocontractive operators in arbitrary Banach spaces

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Abstract. Under the condition of removing the restriction any bounded, we give the convergence of the Ishikawa iteration process to a unique fixed point of a strongly pseudocontractive operator in arbitrary real Banach space. Furthermore, general convergence rate estimate is given in our results, which extend the recent results of Ciric [3] and Soltuz [12].

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1. Introduction

Let X be an arbitrary real Banach space with norm $\|\cdot\|$ and dual X^* , and J denote by the normalized duality mapping from X into 2^{X^*} given by

$$J(x) = \left\{ f \in X^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \right\}, \forall x \in X,$$

where $\langle \cdot, \cdot \rangle$ is a generalized duality pairing. In the sequel, $D(T)$ and $R(T)$ denote the domain and the range of T, respectively. The Hahn-Banach theorem assures that $J(x) \neq \emptyset$ for each $x \in X$. It is easy to see (c.f. [12]) that

$$\langle x, j(y) \rangle \leq \|x\| \cdot \|y\| \tag{1}$$

for all $x, y \in X$ and each $j(y) \in J(y)$.

An operator $T : D(T) \subset X \to X$ is called a strongly pseudocontractive operator, if for all $x, y \in D(T)$, there exist $j(x-y) \in J(x-y)$ and a constant $k \in (0,1)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \leq k \|x-y\|^2 .$$

Let $T : X \to X$ be a mapping on X, if for all $x, y \in X$, there exist $j(x-y) \in J(x-y)$ and a constant $k \in (0,1)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \geq k \|x-y\|^2 ,$$

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then T is called a strongly accretive operator.

For arbitrary $x_0 \in X$ define the sequence $\{x_n\}$ by

$$x_{n+1} = (1-\alpha_n)\,x_n + \alpha_n T y_n,$$
$$y_n = (1-\beta_n)\,x_n + \beta_n T x_n, \quad n \geq 0,$$

where $\alpha_n, \beta_n \in [0,1]$ satisfy suitable conditions (see e.g. [1–10, 12–16]). If $\beta_n = 0$ for each $n \geq 0$, then Ishikawa iterations reduce to the Mann iterations [7].

Zhou [16] considered the Ishikawa iteration process with parameters $\alpha_n \geq a > 0$. Osilike in [9] have proved that two assumptions of the main theorem in [16] are contradictory. Soltuz [12] presented a correction for the result of Zhou [16] for strongly pseudocontractive operators with bounded range $R(T)$ and $k < \frac{1}{2}$. Recently Ciric [3] extended the result of Soltuz [12] to all strongly pseudocontractive operators which satisfy $\{Tx_n\}$ and $\{Ty_n\}$ are bounded and $k < 1$.

The purpose of this paper is to study convergence of Ishikawa iterative sequences for strongly pseudocontractive operators with $k < 1$ in arbitrary real Banach spaces under the condition of removing the restriction $\{Tx_n\}$ and $\{Ty_n\}$ being bounded, and to give that general convergence rate estimate in our results, which largely unify and extend the corresponding results obtained by Ciric [3] and Soltuz[12].

The following results will be needed in the sequel.

Lemma 1 (see [12, 3])**.** *Let X be a real Banach space and let $J : X \to 2^{X^*}$ be a normalized duality mapping. Then*

$$\|x+y\|^2 \leq \|x\|^2 + 2\,\langle y, j\,(x+y)\rangle$$

for all $x, y \in X$ and each $j\,(x+y) \in J\,(x+y)$.

Lemma 2 (see [10, 11])**.** *Let $\{\rho_n\}$ be a sequence of non-negative real numbers which satisfy*

$$\rho_{n+1} \leq (1-\omega)\rho_n + \delta_n, \quad n \geq 0,$$

where $\omega \in (0,1)$ is a fixed number and $\delta_n \geq 0$ is such that $\delta_n \to 0$ as $n \to \infty$. Then $\rho_n \to 0$ as $n \to \infty$.

Lemma 3 (see [1])**.** *If $T : X \to X$ is a strongly accretive operator, then for any $f \in X$, mapping $S : X \to X$, defined by $Sx = f - Tx + x$ is a strongly pseudocontractive operator, i.e. for any $x, y \in X$:*

$$\langle Sx - Sy, j(x-y)\rangle \leq (1-k)\,\|x-y\|^2\,,$$

where $k \in (0,1)$ is the strongly accretive constant of T.

2. Main results

We now state the main results of this section.

Theorem 1. *Let X be a real Banach space, D a non-empty, convex subset of X and let $T : D \to D$ be a continuous and strongly pseudocontractive mapping with a* *pseudocontractive parameter* $k \in (0,1)$*. For arbitrary* $x_0 \in D$*, let Ishikawa iteration sequence* $\{x_n\}$ *be defined by*

$$\begin{cases} x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n T y_n \\ y_n = (1-\beta_n)\, x_n + \beta_n T x_n, \quad n \geq 0. \end{cases} \tag{2}$$

where $\alpha_n, \beta_n \in [0,1]$*, and constants* $a, \tau \in (0, 1-k)$ *are such that*

$$0 < a \leq \alpha_n < 1 - k - \tau, \quad n \geq 0. \tag{3}$$

If

$$\| \, T x_{n+1} - T y_n \, \| \to 0, \tag{4}$$

as $n \to \infty$*, then the sequence* $\{x_n\}$ *converges strongly to a unique fixed point of* T *in* D*; moreover,*

$$\|x_n - x^*\| \leq \sqrt{(1-a\tau)^n \|x_0 - x^*\|^2 + \frac{(1-(1-a\tau)^n)\, M}{a\tau}} \quad , \quad n \geq 0,$$

where $M = \sup \left\{ \frac{1}{k^2} \|T y_n - T x_{n+1}\|^2, \, n \geq 0 \right\}$.

Proof. The existence of a fixed point follows from the result of Deimling [4], and the uniqueness from the strongly pseudocontractivity of T. Let x^* be such that $Tx^* = x^*$. From Lemma 1, we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|(1-\alpha_n)(x_n - x^*) + \alpha_n (T y_n - T x^*)\|^2 \\ &\leq (1-\alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \langle T y_n - T x_{n+1}, j(x_{n+1} - x^*)\rangle \\ &\quad + 2\alpha_n \langle T x_{n+1} - x^*, j(x_{n+1} - x^*)\rangle . \end{aligned} \tag{5}$$

Now we consider the first and second term on the right–hand side of (5). By strongly pseudocontractivity of T, we get

$$2\alpha_n \langle T x_{n+1} - T x^*, j(x_{n+1} - x^*)\rangle \leq 2\alpha_n k \|x_{n+1} - x^*\|^2 , \tag{6}$$

for each $j(x_{n+1} - x^*) \in J(x_{n+1} - x^*)$, and a constant $k \in (0,1)$.

From (1) and inequality $ab \leq \frac{a^2+b^2}{2}$, we obtain that

$$\begin{aligned} 2\alpha_n \langle T y_n - T x_{n+1}, j(x_{n+1} - x^*)\rangle &\leq 2\alpha_n \|T y_n - T x_{n+1}\| \|x_{n+1} - x^*\| \\ &\leq \|T y_n - T x_{n+1}\|^2 + \alpha_n^2 \|x_{n+1} - x^*\|^2, \end{aligned} \tag{7}$$

Substituting (6) and (7) into (5), we infer that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq (1-\alpha_n)^2 \|x_n - x^*\|^2 + \|T y_n - T x_{n+1}\|^2 + \alpha_n^2 \|x_{n+1} - x^*\|^2 \\ &\quad + 2\alpha_n k \|x_{n+1} - x^*\|^2 , \end{aligned}$$

which means that

$$\|x_{n+1} - x^*\|^2 \leq \frac{(1-\alpha_n)^2}{1 - 2k\alpha_n - \alpha_n^2} \|x_n - x^*\|^2 + \frac{\|T y_n - T x_{n+1}\|^2}{1 - 2k\alpha_n - \alpha_n^2} \tag{8}$$

for all $n \geq 0$.

From (3) it follows that

$$\begin{aligned} 1 - 2k\alpha_n - \alpha_n^2 &\geq 1 - 2k(1-k-\tau) - (1-k-\tau)^2 \\ &= k^2 + \tau(2-\tau) \\ &> k^2 > 0. \end{aligned} \tag{9}$$

By (8), (9) and (3) and note that $0 < 1 - 2k\alpha_n - \alpha_n^2 < 1$, we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq \frac{(1-\alpha_n)^2}{1 - 2k\alpha_n - \alpha_n^2} \|x_n - x^*\|^2 + \frac{\|Ty_n - Tx_{n+1}\|^2}{1 - 2k\alpha_n - \alpha_n^2} \\ &= \left(1 - \frac{2\alpha_n(1-k-\alpha_n)}{1 - 2k\alpha_n - \alpha_n^2}\right) \|x_n - x^*\|^2 + \frac{\|Ty_n - Tx_{n+1}\|^2}{1 - 2k\alpha_n - \alpha_n^2} \\ &\leq \left(1 - \frac{2\alpha_n(1-k-(1-k-\tau))}{1 - 2k\alpha_n - \alpha_n^2}\right) \|x_n - x^*\|^2 + \frac{1}{k^2}\|Ty_n - Tx_{n+1}\|^2 \\ &= \left(1 - \frac{2\alpha_n\tau}{1 - 2k\alpha_n - \alpha_n^2}\right) \|x_n - x^*\|^2 + \frac{1}{k^2}\|Ty_n - Tx_{n+1}\|^2 \\ &\leq (1 - 2\alpha_n\tau) \|x_n - x^*\|^2 + \frac{1}{k^2}\|Ty_n - Tx_{n+1}\|^2 \\ &\leq (1 - a\tau) \|x_n - x^*\|^2 + \frac{1}{k^2}\|Ty_n - Tx_{n+1}\|^2 \end{aligned} \tag{10}$$

for all $n \geq 0$. Set $\omega = a\tau$, $\rho_n = \|x_n - x^*\|^2$, $\delta_n = \frac{1}{k^2}\|Ty_n - Tx_{n+1}\|^2, n \geq 0$. By Lemma 2 ensures that $x_n \to x^*$ as $n \to \infty$, that is, $\{x_n\}$ converges strongly to the unique fixed point x^* of the T. Furthermore, using (10) we get

$$\begin{aligned} \|x_n - x^*\|^2 &\leq (1 - a\tau) \|x_{n-1} - x^*\|^2 + M \\ &\leq (1 - a\tau)^n \|x_0 - x^*\|^2 + \frac{(1 - (1 - a\tau)^n) M}{a\tau} \end{aligned}$$

for all $n \geq 0$, which implies that

$$\|x_n - x^*\| \leq \sqrt{(1 - a\tau)^n \|x_0 - x^*\|^2 + \frac{(1 - (1 - a\tau)^n) M}{a\tau}}\ , n \geq 0.$$

This completes the proof. □

Remark 1. *Theorem 1 improves and extends Theorem 1 of Soltuz [12] in its three aspects:*

(i) It abolishes the condition that the range of T is bounded.

(ii) It extends $0 < k < \frac{1}{2}$ to $k \in (0, 1)$.

(iii) General convergence rate estimate is given in our result.

Remark 2. *Theorem 1 improves and extends Theorem 1 of Ciric [3] in the following ways:*

(i) It abolishes the condition that $\{Tx_n\}$ and $\{Ty_n\}$ are bounded.

(ii) General convergence rate estimate is given in our result.

Theorem 2. *Let X be a real Banach space and let $S : X \to X$ be a continuous strongly accretive operator with a strongly accretive constant $k \in (0,1)$. For any given $f \in X$, define a mapping $T : X \to X$ by*

$$Tx = f - Sx + x$$

for all $x \in X$, where $\alpha_n, \beta_n \in [0,1]$, and constants $a, \tau \in (0, 1-k)$ are such that

$$0 < a \le \alpha_n < 1 - k - \tau, n \ge 0,$$

then for arbitrary $x_0 \in X$ the sequence $\{x_n\}$, defined by (2) and satisfying (4) in Theorem 1, converges strongly to a unique solution of the equation $Sx = f$. Moreover,

$$\|x_n - x^*\| \le \sqrt{(1-a\tau)^n \|x_0 - x^*\|^2 + \frac{(1-(1-a\tau)^n)\,M}{a\tau}} \quad , \quad n \ge 0,$$

where $M = \sup\left\{\frac{1}{k^2}\|Ty_n - Tx_{n+1}\|^2, n \ge 0\right\}$.

Proof. Obviously, if $x^* \in X$ is a solution of the equation $Sx = f$, then x^* is a fixed point of T. Also it is easy to prove that T is continuous and strongly pseudocontractive with the strongly pseudocontractivity constant $(1-k)$. Thus, Theorem 2 follows from Theorem 1. □

Remark 3. *Theorem 2 improves and extends Theorem 2 of Ciric [3] in the following ways:*

(i) It abolishes the condition that the range of $(I - S)$ is bounded.

(ii) General convergence rate estimate is given in Theorem 2.

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