

## Common fixed points of noncommuting almost contractions in cone metric spaces

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**Abstract.** In this paper we prove the existence of coincidence points and common fixed points for a large class of almost contractions in cone metric spaces. These results generalize, extend and unify several well-known recent related results in literature.

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**Key words:** metric space, weakly compatible mapping, contractive type mappings, coincidence point, common fixed point, iterative method, convergence theorem, error estimate

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### 1. Introduction

In a very recent paper [19], the authors established some fixed point theorems in cone metric spaces, an ambient space which is obtained by replacing the real axis in the definition of the distance, by an ordered real Banach space whose order is induced by a normal cone  $P$ .

Thereafter, Abbas and Jungck [1] used this setting as ambient space in order to formulate and prove several common fixed point theorems that extend well-known fixed point theorems for contractive type mappings from the case of usual metric spaces. In direct relation to these results, in [26] the authors pointed out that all the fixed point theorems, established in [19] for the case of a cone metric space ordered by a normal cone  $P$  with a normal constant  $K$ , could be formulated and proved in a more general case of a cone metric space. Moreover, they presented several interesting and useful facts about normal and regular cones, illustrated with appropriate examples.

On the other hand, the author [13] obtained coincidence and common fixed point theorems, similar to the ones in [1], but for a more general class of almost contractions, by restricting the ambient space to the case of usual metric spaces.

Although in view of the very recent paper by Du [16], the category of cone metric spaces and that of metric spaces are the same, reviewing some results in cone metric spaces is interesting yet, especially the existence of maximum and minimum.

It is therefore the main aim of the present paper to extend and unify all the results in [19, 1, 13, 14, 26], in view of the important considerations from [26]. To this end, we present in the next two sections some definitions and basic results that will be needed to state and prove our main results.

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For other very recent fixed point and common fixed point results in cone metric spaces, we also refer to [2, 3, 17, 18, 22, 25, 27, 28, 36].

## 2. Preliminaries

**Definition 1.** Let  $E$  be a real Banach space and  $P$  a subset of  $E$ .  $P$  is called a cone if

- (i)  $P$  is closed, nonempty and  $P \neq \{0\}$ ;
- (ii)  $ax + by \in P$  for all  $x, y \in P$  and nonnegative real numbers  $a, b$ ;
- (iii)  $P \cap (-P) = \{0\}$ .

Note also that the relations  $intP + intP \subseteq intP$  and  $\lambda intP \subseteq intP$  ( $\lambda > 0$ ) hold.

For a given cone  $P \subseteq E$ , we can define on  $E$  a partial ordering  $\leq$  with respect to  $P$  by putting  $x \leq y$  if and only if  $y - x \in P$ . Further,  $x < y$  stands for  $x \leq y$  and  $x \neq y$ , while  $x \ll y$  stands for  $y - x \in intP$ , where as usually  $intP$  denotes the interior of  $P$ .

**Definition 2.** Let  $X$  be a non-empty set. A mapping  $d : X \times X \rightarrow E$  satisfying

- (d1)  $0 \leq d(x, y)$  for all  $x, y \in X$  and  $d(x, y) = 0$  if and only if  $x = y$ ;
- (d2)  $d(x, y) = d(y, x)$  for all  $x, y \in X$ ;
- (d3)  $d(x, y) \leq d(x, z) + d(z, y)$  for all  $x, y, z \in X$ ,

is called a cone metric on  $X$ , while  $(X, d)$  is called a cone metric space.

Note that in papers [5] and [6] these notions were termed as “generalized metric” and “generalized metric space”, respectively.

**Definition 3.** Let  $(X, d)$  be a cone metric space,  $x \in X$  and  $\{x_n\}_{n \geq 1}$  a sequence in  $X$ . Then

- (i)  $\{x_n\}_{n \geq 1}$  converges to  $x$  whenever for every  $\varepsilon \in E$  with  $0 \ll \varepsilon$ , there is a natural number  $N$  such that  $d(x_n, x) \ll \varepsilon$  for all  $n \geq N$ . We denote this by  $\lim_{n \rightarrow \infty} x_n = x$  or  $x_n \rightarrow x$ , as in the usual case.
- (ii)  $\{x_n\}_{n \geq 1}$  is a Cauchy sequence whenever for every  $\varepsilon \in E$  with  $0 \ll \varepsilon$  there is a natural number  $N$  such that  $d(x_{n+p}, x_n) \ll \varepsilon$  for all  $n \geq N$  and all  $p$ ;
- (iii)  $(X, d)$  is a complete cone metric space if every Cauchy sequence is convergent.

**Definition 4** (see [1]). Let  $S$  and  $T$  be selfmaps of a nonempty set  $X$ . If there exists  $x \in X$  such that  $Sx = Tx$  then  $x$  is called a coincidence point of  $S$  and  $T$ , while  $y = Sx = Tx$  is called a point of coincidence of  $S$  and  $T$ . If  $Sx = Tx = x$ , then  $x$  is a common fixed point of  $S$  and  $T$ .

**Definition 5** (see [21]). Let  $S$  and  $T$  be selfmaps of a nonempty set  $X$ . The pair of mappings  $S$  and  $T$  is said to be weakly compatible if they commute at their coincidence points.

The next Proposition, which is given in [1] as Proposition 1.4, will be needed to prove the last part of our main result.

**Proposition 1.** *Let  $S$  and  $T$  be weakly compatible selfmaps of a nonempty set  $X$ . If  $S$  and  $T$  have a unique point of coincidence  $y = Sx = Tx$ , then  $y$  is the unique common fixed point of  $S$  and  $T$ .*

### 3. Some classical fixed point theorems

The classical Banach's contraction principle is one of the most useful results in nonlinear analysis. In a metric space setting its statement is given by the next theorem.

**Theorem B.** *Let  $(X, d)$  be a complete metric space and  $T : X \rightarrow X$  a map satisfying*

$$d(Tx, Ty) \leq a d(x, y), \quad \text{for all } x, y \in X, \quad (1)$$

where  $0 \leq a < 1$  is constant. Then:

(p1)  $T$  has a unique fixed point  $x^*$  in  $X$ ;

(p2) The Picard iteration  $\{x_n\}_{n=0}^{\infty}$  defined by

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, \dots \quad (2)$$

converges to  $x^*$ , for any  $x_0 \in X$ .

(p3) The following estimate holds:

$$d(x_{n+i-1}, x^*) \leq \frac{a^i}{1-a} d(x_n, x_{n-1}), \quad n = 0, 1, 2, \dots, \quad i = 1, 2, \dots \quad (3)$$

(p4) The rate of convergence of Picard iteration is given by

$$d(x_n, x^*) \leq a d(x_{n-1}, x^*), \quad n = 1, 2, \dots \quad (4)$$

#### 3.1. Remarks

Theorem B has many applications in solving nonlinear equations. Its merit is not only to state the existence and uniqueness of the fixed point of the strict contraction  $T$  but also to show that the fixed point can be approximated by means of Picard iteration (2). Moreover, for this iterative method both a priori

$$d(x_n, x^*) \leq \frac{a^n}{1-a} d(x_0, x_1), \quad n = 0, 1, 2, \dots$$

and a posteriori

$$d(x_n, x^*) \leq \frac{a}{1-a} d(x_n, x_{n-1}), \quad n = 0, 1, 2, \dots, \quad i = 1, 2, \dots$$

error estimates are available, which are both contained in (3).

Despite these important features, Theorem B suffers from one drawback - the contractive condition (1) forces  $T$  be continuous on  $X$ .

It was then natural to ask if there exist or not weaker contractive conditions which do not imply the continuity of  $T$ . This was answered in the affirmative by R. Kannan [23] in 1968, who proved a fixed point theorem which extends Theorem B to mappings that need not be continuous on  $X$  (but are continuous at their fixed point, (see [31]), by considering instead of (1) the next condition: there exists  $b \in [0, 1/2)$  such that

$$d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)], \quad \text{for all } x, y \in X. \quad (5)$$

Following the Kannan's theorem, a lot of papers were devoted to obtaining fixed point theorems or common fixed point theorems for various classes of contractive type conditions that do not require the continuity of  $T$ , see for example, [32, 33, 10] and the references therein.

One of them, actually a sort of dual of the Kannan fixed point theorem, due to Chatterjea [15], is based on a condition similar to (5): there exists  $c \in [0, 1/2)$  such that

$$d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)], \quad \text{for all } x, y \in X. \quad (6)$$

For a presentation and comparison of such kind of fixed point theorems, see [29, 30, 24, 7].

These fixed point results were then complemented by corresponding results regarding the existence of common fixed points of such contractive type mappings. So, Jungck [20] proved in 1976 a common fixed point theorem for commuting maps, thus generalizing Theorem B. In the same spirit, very recently M. Abbas and G. Jungck [1], obtained coincidence and common fixed point theorems for the class of Banach contractions, Kannan contractions and Chatterjea contractions in cone metric spaces with a normal cone, without making use of the commutative property, but based on the so called concept of weakly compatible mappings, introduced by Jungck [21].

On the other hand, in 1972, Zamfirescu [35] obtained a very interesting fixed point theorem, by combining the contractive conditions (1) of Banach, (5) Kannan and (6) Chatterjea.

Note that, as shown later by Rhoades [29], the contractive conditions (1), (5) and (6) are independent.

The Zamfirescu fixed point theorem has been further extended to almost contractions [8], a class of contractive type mappings which exhibits totally different features than the ones of the particular results incorporated, i.e., an almost contraction generally does not have a unique fixed point, see Example 1 in [8].

We give here the full statement of the main result from [8] in view of its extension to coincidence and common fixed point theorems.

**Theorem 1.** *Let  $(X, d)$  be a complete metric space and  $T : X \rightarrow X$  an almost contraction, that is, a mapping for which there exist a constant  $\delta \in (0, 1)$  and some  $L \geq 0$  such that*

$$d(Tx, Ty) \leq \delta \cdot d(x, y) + Ld(y, Tx), \quad \text{for all } x, y \in X. \quad (7)$$

*Then*

- 1)  $F(T) = \{x \in X : Tx = x\} \neq \emptyset$ ;
- 2) For any  $x_0 \in X$ , the Picard iteration  $\{x_n\}_{n=0}^\infty$  given by (1.2) converges to some  $x^* \in F(T)$ ;
- 3) The following estimate holds

$$d(x_{n+i-1}, x^*) \leq \frac{\delta^i}{1-\delta} d(x_n, x_{n-1}), \quad n = 0, 1, 2, \dots, \quad i = 1, 2, \dots \quad (8)$$

Starting from this background, it is the main aim of the next section to extend and unify all the results in [19, 1, 13, 14, 26], in view of the important considerations from [26].

#### 4. Main results

In [1], the authors obtained three coincidence and common fixed point theorems, corresponding to Banach contraction condition (Theorem 2.1), Kannan's contractive condition (Theorem 2.3) and Chatterjea's contractive condition (Theorem 2.4), respectively, in cone metric spaces with a normal cone  $P$ .

Then, the author [14] extended all the coincidence and common fixed point theorems in [1] to a more general class of discontinuous noncommuting mappings, and in and [13] to almost contractions, by restricting the ambient space to the case of usual metric spaces. We now establish the corresponding results from [1, 19, 13, 26], in an arbitrary cone metric space. Note that our technique of proof, adapted from [13], is significantly different from the one used in [1].

We start this section by presenting a coincidence point theorem.

**Theorem 2.** *Let  $(X, d)$  be a cone metric space and let  $T, S : X \rightarrow X$  be two mappings for which there exist a constant  $\delta \in (0, 1)$  and some  $L \geq 0$  such that*

$$d(Tx, Ty) \leq \delta \cdot d(Sx, Sy) + Ld(Sy, Tx), \quad \text{for all } x, y \in X. \quad (9)$$

*If the range of  $S$  contains the range of  $T$  and  $S(X)$  is a complete subspace of  $X$ , then  $T$  and  $S$  have a coincidence point in  $X$ . Moreover, for any  $x_0 \in X$ , the iteration  $\{Sx_n\}$  defined by (10) converges to some coincidence point  $x^*$  of  $T$  and  $S$ .*

**Proof.** Let  $x_0$  be an arbitrary point in  $X$ . Since  $T(X) \subset S(X)$ , we can choose a point  $x_1$  in  $X$  such that  $Tx_0 = Sx_1$ . Continuing in this way, for a  $x_n$  in  $X$ , we can find  $x_{n+1} \in X$  such that

$$Sx_{n+1} = Tx_n, \quad n = 0, 1, \dots \quad (10)$$

If  $x := x_n$ ,  $y := x_{n-1}$  are two successive terms of the sequence defined by (10), then by (9) we have

$$d(Sx_n, Sx_{n+1}) = d(Tx_{n-1}, Tx_n) \leq L \cdot d(Sx_n, Tx_{n-1}) + \delta \cdot d(Sx_{n-1}, Sx_n),$$

which in view of (10) yields  $d(Sx_n, Tx_{n-1}) = 0$  and hence

$$d(Sx_{n+1}, Sx_n) \leq \delta \cdot d(Sx_n, Sx_{n-1}), \quad n = 0, 1, 2, \dots \quad (11)$$

Now by induction, from (11) we obtain

$$d(Sx_{n+k}, Sx_{n+k-1}) \leq \delta^k \cdot d(Sx_n, Sx_{n-1}), \quad n, k = 0, 1, \dots \quad (k \neq 0), \quad (12)$$

and then, for  $p > i$ , we get after straightforward calculations

$$d(Sx_{n+p}, Sx_{n+i-1}) \leq \frac{\delta^i(1 - \delta^{p-i+1})}{1 - \delta} \cdot d(Sx_n, Sx_{n-1}), \quad n \geq 0, i \geq 1. \quad (13)$$

Take  $i = 1$  (13). Then, by an inductive process, we get

$$d(Sx_{n+p}, Sx_n) \leq \frac{\delta}{1 - \delta} \cdot d(Sx_n, Sx_{n-1}) \leq \frac{\delta^n}{1 - \delta} \cdot d(Sx_1, Sx_0), \quad n \geq 0.$$

Let now  $0 \ll \varepsilon$  be given. Choose  $\delta > 0$  such that  $\varepsilon + N_\delta(0) \subset P$ , where  $N_\delta(0) = \{y \in E : \|y\| \leq \delta\}$ . Also choose a natural number  $N_1$  such that

$$\frac{\delta^n}{1 - \delta} \cdot d(Sx_1, Sx_0) \in N_\delta(0), \quad \text{for all } n \geq N_1.$$

Then

$$\frac{\delta^n}{1 - \delta} \cdot d(Sx_1, Sx_0) \ll \varepsilon, \quad \text{for all } n \geq N_1$$

and hence

$$d(Sx_{n+p}, Sx_n) \leq \frac{\delta}{1 - \delta} \cdot d(Sx_n, Sx_{n-1}) \ll \varepsilon, \quad \text{for all } n \geq N_1 \text{ and all } p,$$

which shows that  $\{Sx_n\}$  is a Cauchy sequence.

Since  $S(X)$  is complete, there exists a  $x^*$  in  $S(X)$  such that

$$\lim_{n \rightarrow \infty} Sx_{n+1} = x^*. \quad (14)$$

We can find  $p \in X$  such that  $Sp = x^*$ . By (11) and (12) we further have

$$d(Sx_n, Tp) = d(Tx_{n-1}, Tp) \leq \delta d(Sx_{n-1}, Sp) \leq \delta^{n-1} d(Sx_1, Sp),$$

which shows that we also have

$$\lim_{n \rightarrow \infty} Sx_n = Tp. \quad (15)$$

By (14) and (15) it results now that  $Tp = Sp$ , that is,  $p$  is a coincidence point of  $T$  and  $S$  (or  $x^*$  is a point of coincidence of  $T$  and  $S$ ).  $\square$

**Remark 1.** *Let us note that the coincidence point ensured by Theorem 2 is not generally unique, see Example 1 in [8].*

*In order to obtain a common fixed point theorem from the coincidence Theorem 2, we need the uniqueness of the coincidence point, which could be obtained by imposing an additional contractive condition, similarly to (9).*

**Theorem 3.** *Let  $(X, d)$  be a cone metric space and let  $T, S : X \rightarrow X$  be two mappings satisfying (9) for which there exist a constant  $\theta \in (0, 1)$  and some  $L_1 \geq 0$  such that*

$$d(Tx, Ty) \leq \theta \cdot d(Sx, Sy) + L_1 d(Sx, Tx), \quad \text{for all } x, y \in X. \quad (16)$$

*If the range of  $S$  contains the range of  $T$  and  $S(X)$  is a complete subspace of  $X$ , then  $T$  and  $S$  have a unique coincidence point in  $X$ . Moreover, if  $T$  and  $S$  are weakly compatible, then  $T$  and  $S$  have a unique common fixed point in  $X$ . In both cases, for any  $x_0 \in X$ , the iteration  $\{Sx_n\}$  defined by (10) converges to the unique common fixed point (coincidence point)  $x^*$  of  $S$  and  $T$ .*

**Proof.** By the proof of Theorem 2, we have that  $T$  and  $S$  have at least a point of coincidence, say  $x^* = Tp = Sp$ ,  $p \in X$ . Now let us show that  $T$  and  $S$  actually have a unique point of coincidence. Assume there exists  $q \in X$  such that  $Tq = Sq$ . Then, by (16) we get

$$d(Sq, Sp) = d(Tq, Tp) \leq 2\delta d(Sq, Tq) + \delta d(Sq, Tp) = \delta d(Sq, Sp)$$

which yields

$$(1 - \delta)d(Sq, Sp) \leq 0.$$

As by definition,  $0 \leq d(Sq, Sp)$ , that is,  $d(Sq, Sp) \in P$ , by the previous inequality we obtain  $-d(Sq, Sp) \in P$ , since  $\frac{1}{1-\delta} > 0$ . This means that  $d(Sq, Sp) = 0$ , which shows that  $Sq = Sp = x^*$ , that is  $T$  and  $S$  have a unique point of coincidence,  $x^*$ .

Now if  $T$  and  $S$  have are weakly compatible, by Proposition 1 it follows that  $x^*$  is their unique common fixed point.  $\square$

A stronger but simpler contractive condition that ensures the uniqueness of the coincidence point and which actually unifies (9) and (16), has been very recently obtained by Babu et al. [4]. We state in the following the common fixed point theorem corresponding to the fixed point result in [4].

**Theorem 4.** *Let  $(X, d)$  be a cone metric space and let  $T, S : X \rightarrow X$  be two mappings for which there exist a constant  $\delta \in (0, 1)$  and some  $L \geq 0$  such that*

$$d(Tx, Ty) \leq \delta \cdot d(Sx, Sy) + L \min\{d(Sx, Tx), d(Sy, Ty), d(Sx, Ty), d(Sy, Tx)\}, \quad (17)$$

*for all  $x, y \in X$ . If the range of  $S$  contains the range of  $T$  and  $S(X)$  is a complete subspace of  $X$ , then  $T$  and  $S$  have a unique coincidence point in  $X$ . Moreover, if  $T$  and  $S$  are weakly compatible, then  $T$  and  $S$  have a unique common fixed point in  $X$ . In both cases, for any  $x_0 \in X$ , the iteration  $\{Sx_n\}$  defined by (10) converges to the unique common fixed point (coincidence point)  $x^*$  of  $S$  and  $T$ .*

**Proof.** If  $x := x_n$ ,  $y := x_{n-1}$  are two successive terms of the sequence defined by (10), then by (17) we have

$$d(Sx_n, Sx_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \delta \cdot d(Sx_{n-1}, Sx_n) + L \cdot M,$$

where

$$M = \min\{d(Sx_n, Tx_n), d(Sx_{n-1}, Tx_{n-1}), d(Sx_n, Tx_{n-1}),$$

$d(Sx_{n-1}, Tx_n)\} = 0$ , since  $d(Sx_n, Tx_{n-1}) = 0$ . The rest of the proof follows as in the case of Theorem 3.  $\square$

## 5. Particular cases

- 1) If in (9) we have  $L \equiv 0$ , then by Theorem 2 we obtain a generalization of Theorem 2.1 in [1]. If the cone metric space reduces to a usual metric space, then by Theorem 2 we obtain Theorem 2 in [13] which, in turn, generalizes the Jungck common fixed point [20];
- 2) If in Theorem 2, the cone  $P = \mathbb{R}_+$ , the nonnegative real semi-axis, then by Theorem 2 we obtain the main result (Theorem 3) in [13];
- 3) Also note that by Theorem 2 we obtain a significant generalization of Theorem 2.8 in [26], which has been obtained there by imposing for the contractive inequality (9) the very restrictive condition  $\delta + L < 1$ . The following corollaries are also obtained by our main results

**Corollary 1.** *Let  $(X, d)$  be a cone metric space and let  $T, S : X \rightarrow X$  be two mappings for which there exist  $b \in [0, \frac{1}{2})$  such that for all  $x, y \in X$ ,*

$$d(Tx, Ty) \leq b[d(Sx, Tx) + d(Sy, Ty)].$$

*If the range of  $S$  contains the range of  $T$  and  $S(X)$  is a complete subspace of  $X$ , then  $T$  and  $S$  have a unique coincidence point in  $X$ . Moreover, if  $T$  and  $S$  are weakly compatible, then  $T$  and  $S$  have a unique common fixed point in  $X$ .*

*In both cases, the iteration  $\{Sx_n\}$  defined by (10) converges to the unique (coincidence) common fixed point  $x^*$  of  $S$  and  $T$ , for any  $x_0 \in X$ .*

**Corollary 2.** *Let  $(X, d)$  be a cone metric space and let  $T, S : X \rightarrow X$  be two mappings for which there exist  $c \in [0, \frac{1}{2})$  such that for all  $x, y \in X$ ,*

$$d(Tx, Ty) \leq c[d(Sx, Ty) + d(Sy, Tx)].$$

*If the range of  $S$  contains the range of  $T$  and  $S(X)$  is a complete subspace of  $X$ , then  $T$  and  $S$  have a unique coincidence point in  $X$ . Moreover, if  $T$  and  $S$  are weakly compatible, then  $T$  and  $S$  have a unique common fixed point in  $X$ .*

*In both cases, the iteration  $\{Sx_n\}$  defined by (10) converges to the unique (coincidence) common fixed point  $x^*$  of  $S$  and  $T$ , for any  $x_0 \in X$ .*

By noting that Banach contraction condition does imply (9) (with  $L=0$ ), by Corollary 1, Corollary 2 and the similar result corresponding to condition (1), we obtain the main result in [14].

**Corollary 3.** *Let  $(X, d)$  be a cone metric space and let  $T, S : X \rightarrow X$  be two mappings for which there exist  $a \in [0, 1)$ ,  $b, c \in [0, \frac{1}{2})$  such that for all  $x, y \in X$ , at least one of the following conditions is true:*

$$(z_1) \quad d(Tx, Ty) \leq a d(Sx, Sy),$$

$$(z_2) \quad d(Tx, Ty) \leq b[d(Sx, Tx) + d(Sy, Ty)],$$

$$(z_3) \quad d(Tx, Ty) \leq c[d(Sx, Ty) + d(Sy, Tx)].$$



If the range of  $S$  contains the range of  $T$  and  $S(X)$  is a complete subspace of  $X$ , then  $T$  and  $S$  have a unique coincidence point in  $X$ . Moreover, if  $T$  and  $S$  are weakly compatible, then  $T$  and  $S$  have a unique common fixed point in  $X$ .

In both cases, the iteration  $\{Sx_n\}$  defined by (10) converges to the unique (coincidence) common fixed point  $x^*$  of  $S$  and  $T$ .

Several other results can be obtained as particular cases of our main results, see [7, 9, 15, 23] and references therein.

## 6. An example

We present now a non-trivial example that illustrates how general and important the result given by Theorem 2 in this paper is.

Note that all results established in this paper remain valid if we replace the assumption “ $S(X)$  is a complete subspace of  $X$ ” by “ $(X, d)$  is a complete metric space”, which will be used in the following.

**Example 1.** Let  $E = \mathbb{R}^2$  be a Euclidean plane, and

$$P = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$$

its positive cone.

If we consider  $X = \{(x, 0) \in \mathbb{R}^2 : 0 \leq x \leq 1\}$  and define  $d : X \times X \rightarrow P$  by

$$d((x, 0), (y, 0)) = \left( |x - y|, \frac{1}{2} |x - y| \right), \quad \forall (x, 0), (y, 0) \in X, \quad (18)$$

then  $(X, d)$  is a complete cone metric space. Let  $T, S : X \rightarrow X$  be defined by

$$T(x, 0) = \begin{cases} \left( \frac{x}{4}, 0 \right), & 0 \leq x < \frac{2}{3}, \\ \left( \frac{2}{3}, 0 \right), & \frac{2}{3} \leq x \leq 1 \end{cases}$$

and

$$S(x, 0) = \begin{cases} (x, 0), & 0 \leq x \leq \frac{2}{3}, \\ (1, 0), & \frac{2}{3} < x \leq 1, \end{cases}$$

respectively.

We have  $T(X) = \{(x, 0) : 0 \leq x < 1/6\} \cup \{(2/3, 0)\} \subset \{(x, 0) : 0 \leq x \leq 2/3\} \cup \{(1, 0)\} = S(X)$ . In order to show that  $S$  and  $T$  do satisfy the contractive condition (9) in Theorem 2, let us denote

$$\begin{aligned} M_1 &= [0, 2/3) \times [0, 2/3), & M_2 &= [0, 2/3) \times (2/3, 1], \\ M_3 &= [0, 2/3) \times \{2/3\}, & M_4 &= [2/3, 1] \times [2/3, 1]. \end{aligned}$$

Clearly,  $[0, 1] \times [0, 1] = M_1 \cup M_2 \cup M_3 \cup M_4$ .

Case 1.  $(x, y) \in M_1$ . In this case  $S$  and  $T$  satisfy (9). Indeed, by (9) we get

$$\left( \left| \frac{x}{4} - \frac{y}{4} \right|, \frac{1}{2} \left| \frac{x}{4} - \frac{y}{4} \right| \right) \leq a \cdot \left( |x - y|, \frac{1}{2} |x - y| \right) + L \cdot \left( \left| y - \frac{x}{4} \right|, \frac{1}{2} \left| y - \frac{x}{4} \right| \right)$$

and both components reduce to the inequality

$$\left| \frac{x}{4} - \frac{y}{4} \right| \leq \delta |x - y| + L \cdot \left| y - \frac{x}{4} \right|,$$

which holds for all  $x, y \in [0, 2/3)$  and any constant  $L \geq 0$  if we simply take  $\delta$  such that  $\delta \geq 1/4$ .

Case 2.  $(x, y) \in M_2$ . As  $T(x, 0) = \left(\frac{x}{4}, 0\right)$ ,  $T(y, 0) = \left(\frac{2}{3}, 0\right)$  and  $S(x, 0) = (x, 0)$ ,  $S(y, 0) = (1, 0)$ , in view of (18), condition (9) reduces to show that there exist the constants  $\delta$  and  $L$ ,  $0 \leq \delta < 1$ , and  $L \geq 0$ , such that

$$\left| \frac{x}{4} - \frac{2}{3} \right| \leq \delta |x - 1| + L \cdot \left| 1 - \frac{x}{4} \right|, \quad \forall x \in [0, \frac{2}{3}]. \quad (19)$$

As for  $x \in [0, \frac{2}{3})$  we have  $\left| \frac{x}{4} - \frac{2}{3} \right| \in \left( \frac{1}{2}, \frac{2}{3} \right]$  and  $\left| 1 - \frac{x}{4} \right| \in \left( \frac{5}{6}, 1 \right]$ , in order to have

(19) fulfilled, it suffices to take  $L \geq \frac{4}{5}$  and allow  $0 \leq \delta < 1$  be arbitrary.

Case 3.  $(x, y) \in M_3$ . In this case, (9) reduces to show that there exist the constants  $\delta$  and  $L$ ,  $0 \leq \delta < 1$ , and  $L \geq 0$  such that

$$\left| \frac{x}{4} - \frac{2}{3} \right| \leq \delta \left| x - \frac{2}{3} \right| + L \cdot \left| 1 - \frac{x}{4} \right|, \quad \forall x \in [0, \frac{2}{3}), \quad (20)$$

which, by the previous case, is indeed satisfied for any  $0 \leq \delta < 1$  if we similarly take  $L \geq \frac{4}{5}$ .

Case 4.  $(x, y) \in M_4$ . In this case (9) holds for any constants  $\delta$  and  $L$ ,  $0 \leq \delta < 1$ , and  $L \geq 0$ , since its left hand-side is always equal to  $(0, 0)$ .

By summarizing, we conclude that  $S$  and  $T$  satisfy the contractive condition (9) in Theorem 2 with  $\delta = \frac{1}{4}$  and  $L = \frac{4}{5}$ .

Hence, Theorem 2 applies and  $T$  and  $S$  have two common fixed points, namely  $(0, 0)$  and  $(2/3, 0)$ .

**Remark 2.** Note that  $T$  and  $S$  in Example 1 satisfy neither conditions (16) in Theorem 3 and (17) in Theorem 3, nor the contractive conditions in Corollaries 1-3.

Let us check first for condition (16). In view of (18), for  $x = 0$  and  $y = \frac{2}{3}$ , condition (16) would require that there exist the constants  $\theta$  and  $L_1$ , with  $0 < \theta < 1$  and  $L_1 \geq 0$  such that:

$$\left| 0 - \frac{2}{3} \right| \leq \theta \left| 0 - \frac{2}{3} \right| + L_1 |0 - 0|,$$

which yields the contradiction  $\theta \geq 1$ . Thus Theorem 2.1 in [1] does not apply to Example 1.

$T$  and  $S$  in Example 1 do not satisfy the contractive condition in Corollary 1 either. Indeed, for  $x = 0$  and  $y = \frac{2}{3}$  this condition would require the existence of a constant  $b$ ,  $0 \leq b < 1/2$ , such that

$$\left|0 - \frac{2}{3}\right| \leq b \left[|0 - 0| + \left|1 - \frac{2}{3}\right|\right],$$

which obviously yields the contradiction  $2 \leq b < 1/2$ . Thus Corollary 1 and Theorem 2.3 in [1] do not apply to Example 1.

Moreover,  $T$  and  $S$  in Example 1 do not satisfy the contractive condition in Corollary 2. Indeed, for  $x = \frac{2}{3} - \epsilon$ ,  $\epsilon > 0$  and  $y = \frac{2}{3}$  this condition would require the existence of a constant  $c$ ,  $0 \leq c < 1/2$ , such that

$$\left|\frac{\frac{2}{3} - \epsilon}{4} - \frac{2}{3}\right| \leq c \left[\left|\frac{2}{3} - \epsilon - \frac{2}{3}\right| + \left|\frac{2}{3} - \frac{\frac{2}{3} - \epsilon}{4}\right|\right],$$

which by letting  $\epsilon \rightarrow 0$  obviously yields the contradiction  $1 \leq c < 1/2$ . Thus Corollary 2 and Theorem 2.4 in [1] do not apply to Example 1. As a direct consequence of the arguments above, Corollary 3 as well as Theorems 2 and 3 in [14] do not apply to Example 1 either.

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