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Linear operators that strongly preserve regularity of fuzzy matrices

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Abstract. An $n \times n$ fuzzy matrix A is called regular if there is an $n \times n$ fuzzy matrix G such that $AGA = A$. We study the problem of characterizing those linear operators T on the fuzzy matrices such that $T(X)$ is regular if and only if X is. Consequently, we obtain that T strongly preserves regularity of fuzzy matrices if and only if there are permutation matrices P and Q such that it has the form $T(X) = PXQ$ or $T(X) = PX^tQ$ for all fuzzy matrices X.
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1. Introduction

Let $\mathcal{F} = [0, 1]$ be a set of reals between 0 and 1 with addition $(+)$, multiplication $(\cdot)$ and the ordinary order $\leq$ such that

$$x + y = \max\{x, y\} \text{ and } x \cdot y = \min\{x, y\}$$

for all $x, y \in \mathcal{F}$. We call $\mathcal{F}$ a fuzzy semiring. For all $x, y \in \mathcal{F}$, we suppress the dot of $x \cdot y$ and simply write xy. Let $\mathcal{M}_n(\mathcal{F})$ denote a set of all $n \times n$ fuzzy matrices with entries in $\mathcal{F}$. Then addition, multiplication by scalars, and the product of fuzzy matrices on $\mathcal{M}_n(\mathcal{F})$ are defined as if $\mathcal{F}$ were a field.

Regular matrices play a central role in the theory of matrices, and they have many applications in network and switching theory and information theory [4, 5, 7]. Recently, a number of authors have studied characterizations of regular matrices over various semirings [1, 3, 4, 5, 7, 8].

In matrix theory, one of the most active and fertile subjects is the study of those linear operators on matrices that leave certain properties or relations of matrices invariant. Although the linear preservers concerned are mostly linear operators on matrix spaces over some fields or rings, the same problem has been found on matrix spaces over various semirings [2, 9].

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In this paper, we study certain properties of regular fuzzy matrices. We also characterize linear operators on $\mathcal{M}_n(\mathcal{F})$ that strongly preserve regular fuzzy matrices.

2. Preliminaries

The matrix I_n is an $n \times n$ identity matrix, J_n is an $n \times n$ matrix all of whose entries are 1, and O_n is an $n \times n$ zero matrix. We will suppress the subscripts on these matrices when the orders are evident from the context and we write I, J and O, respectively. For any matrix A, A^t is denoted by the transpose of A. A zero-one matrix in $\mathcal{M}_n(\mathcal{F})$ with only one entry equal to 1 is called a cell. If the nonzero entry occurs in the ith row and the jth column, we denote the cell by E_{ij}.

A matrix A in $\mathcal{M}_n(\mathcal{F})$ is said to be invertible if there is a matrix B in $\mathcal{M}_n(\mathcal{F})$ such that $AB = BA = I$. It is well known [9] that $n \times n$ permutation matrices are the only invertible matrices in $\mathcal{M}_n(\mathcal{F})$.

The notion of the generalized inverse of an arbitrary matrix apparently originated from the work of Moore [6].

Let A be a matrix in $\mathcal{M}_n(\mathcal{F})$. Consider a matrix $X \in \mathcal{M}_n(\mathcal{F})$ in equation

$$AXA = A. \tag{1}$$

If (1) has a solution X, then X is called a generalized inverse of A. Furthermore, A is called regular if there is a solution of (1).

Clearly, J and O are fuzzy regular in $\mathcal{M}_n(\mathcal{F})$ because $JEJ = J$ and $OEO = O$, where E is any cell in $\mathcal{M}_n(\mathcal{F})$. Thus in general, a solution of (1), although it exists, is not necessarily unique.

Let $\mathcal{R}_n(\mathcal{F})$ be a set of all regular matrices in $\mathcal{M}_n(\mathcal{F})$; that is

$$\mathcal{R}_n(\mathcal{F}) = \{X \in \mathcal{M}_n(\mathcal{F}) |\ X \text{ is regular}\}.$$

The following Proposition is an immediate consequence of the definitions of a fuzzy regular matrix and a permutation matrix.

Proposition 1. *Let A be a matrix in $\mathcal{M}_n(\mathcal{F})$. If P and Q are permutation matrices of degree n, then the following are equivalent:*

(i) $A \in \mathcal{R}_n(\mathcal{F})$,

(ii) $\alpha A \in \mathcal{R}_n(\mathcal{F})$ *for all nonzero* $\alpha \in \mathcal{F}$,

(iii) $PAQ \in \mathcal{R}_n(\mathcal{F})$,

(iv) $A^t \in \mathcal{R}_n(\mathcal{F})$.

Also we can easily show that for any $A \in \mathcal{M}_n(\mathcal{F})$ and for all $B \in \mathcal{M}_p(\mathcal{F})$,

$$\begin{bmatrix} A & O \\ O & B \end{bmatrix} \in \mathcal{R}_{n+p}(\mathcal{F}) \text{ if and only if } A \in \mathcal{R}_n(\mathcal{F}) \text{ and } B \in \mathcal{R}_p(\mathcal{F}). \tag{2}$$

In particular, all idempotent matrices in $\mathcal{M}_n(\mathcal{F})$ are regular.

The (factor) rank, $r(A)$, of a nonzero $A \in \mathcal{M}_n(\mathcal{F})$ is defined as the least integer k such that $A = BC$, where B and C are $n \times k$ and $k \times n$ fuzzy matrices, respectively. The rank of a zero matrix is zero.

Proposition 2. *If $A \in \mathcal{M}_n(\mathcal{F})$ with $r(A) = 1$, then $A \in \mathcal{R}_n(\mathcal{F})$. And if A is any matrix in $\mathcal{M}_n(\mathcal{F})$ whose row rank or column rank is 1, then $A \in \mathcal{R}_n(\mathcal{F})$.*

Proof. If $r(A) = 1$ or if the row rank or the column rank of A is 1, then we can easily show that A is of the form

$$A = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} [b_1 \ \cdots \ b_n],$$

where $a_i b_j \neq 0$ for some i and j. Let $\alpha = \max\{a_1, \ldots, a_m, b_1, \ldots, b_n\}$. Then we have $A(\alpha J)A = A$. Hence $A \in \mathcal{R}_n(\mathcal{F})$. □

The number of nonzero entries of a matrix is denoted by $|A|$. For a matrix $A = [a_{ij}] \in \mathcal{M}_n(\mathcal{F})$, an entry a_{rs} in A is called maximal if $a_{ij} \leq a_{rs}$ for all i and j. Since $|A|$ is finite, A has at least one maximal entry.

Lemma 1. *Let $A \in \mathcal{M}_2(\mathcal{F})$. Then $A \in \mathcal{R}_2(\mathcal{F})$ if and only if* (1) *A has at least two maximal entries, or* (2) *A has only one maximal entry and there are permutation matrices P and Q such that*

$$PAQ = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with $a > \max\{b, c, d\}$ and $d \geq bc$.

Proof. First, suppose that A has at least two maximal entries. By Proposition 1, we may write

$$A = \begin{bmatrix} p & p \\ q & r \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} p & q \\ r & p \end{bmatrix},$$

where $p \geq \max\{q, r\}$. If $A = \begin{bmatrix} p & p \\ q & r \end{bmatrix}$, we have

$$\begin{bmatrix} p & p \\ q & r \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p & p \\ q & r \end{bmatrix} = \begin{bmatrix} p & p \\ q & r \end{bmatrix} \text{ and } \begin{bmatrix} p & p \\ q & r \end{bmatrix} \begin{bmatrix} p & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} p & p \\ q & r \end{bmatrix} = \begin{bmatrix} p & p \\ q & r \end{bmatrix}$$

for $q \geq r$ and $r \geq q$, respectively. Hence $A \in \mathcal{R}_2(\mathcal{F})$. If

$$A = \begin{bmatrix} p & q \\ r & p \end{bmatrix},$$

then clearly A is idempotent and hence regular. Next, assume that there are permutation matrices P and Q such that

$$PAQ = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with $a > \max\{b, c, d\}$ and $d \geq bc$. Clearly, PAQ is idempotent and hence regular. By Proposition 1, $A \in \mathcal{R}_2(\mathcal{F})$.

Conversely, suppose that $A \in \mathcal{R}_2(\mathcal{F})$. Now, assume that A has only one maximal entry. Without loss of generality, we may write

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with $a > \max\{b, c, d\}$. Since $A \in \mathcal{R}_2(\mathcal{F})$, there is a nonzero

$$G = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in \mathcal{M}_2(\mathcal{F})$$

such that $AGA = A$, equivalently

$$\begin{bmatrix} ax + bz + cy + bcw & bx + bz + dy + bdw \\ cx + dz + cy + cdw & bcx + bdz + cdy + dw \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

From $(1, 1)$th entries of AGA and A, we have $x \geq a$ because $a > \max\{b, c, d\}$. Again from $(2, 2)$th entries, we obtain $d \leq w$ and $bc \leq d$. Hence the result follows. □

Corollary 1. *Let $n \geq 2$. For every cell E in $\mathcal{M}_n(\mathcal{F})$, there is a matrix $A \in \mathcal{M}_n(\mathcal{F})$ such that $A \in \mathcal{R}_n(\mathcal{F})$, but $E + A \notin \mathcal{R}_n(\mathcal{F})$.*

Proof. Without loss of generality, we may assume $E = E_{11}$. Consider a matrix

$$A = \begin{bmatrix} B & O \\ O & O \end{bmatrix} \in \mathcal{M}_n(\mathcal{F}),$$

where $B = \begin{bmatrix} 0 & p \\ p & 0 \end{bmatrix}$ with $0 < p < 1$. Then B is regular, while $C = \begin{bmatrix} 1 & p \\ p & 0 \end{bmatrix}$ is not regular by Lemma 1. It follows from (2) that A is regular, while

$$E + A = \begin{bmatrix} C & O \\ O & O \end{bmatrix}$$

is not regular. □

Lemma 2. *Suppose that $A \in \mathcal{M}_n(\mathcal{F})$ has only one maximal entry $a_{rs}(\neq 0)$. If there is a certain 2×2 submatrix B of A and permutation matrices P and Q such that*

$$PBQ = \begin{bmatrix} a_{rs} & \alpha \\ \beta & 0 \end{bmatrix}$$

with $\alpha \neq 0$ and $\beta \neq 0$, then A is not regular.

Proof. Without loss of generality (if necessary, permute rows and columns of A), we assume that

$$B = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{bmatrix}$$

with $a_{12} \neq 0$ and $a_{21} \neq 0$, where a_{11} is the only one maximal entry in A. If A is regular, then there is a nonzero $X \in \mathcal{M}_n(\mathcal{F})$ such that $AXA = A$. Equating the $(1,1)$th entries of AXA and A, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{1i}x_{ij}a_{j1} = a_{11}$$

so that $a_{1i}x_{ij}a_{j1} = a_{11}$ for some i and j. If $i \neq 1$ or $j \neq 1$, then $a_{1i}x_{ij}a_{j1} < a_{11}$ because a_{11} is the only one maximal entry in A. Hence $i = j = 1$ and so $a_{11}x_{11}a_{11} = a_{11}$, equivalently $x_{11} \geq a_{11}$. And so, the $(2,2)$th entry of AXA is

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{2i}x_{ij}a_{j2} \geq a_{21}x_{11}a_{12} > 0.$$

But the $(2,2)$th entry of A is zero, which is a contradiction. Thus the result follows. □

An operator T on $\mathcal{M}_n(\mathcal{F})$ is said to be linear if $T(\alpha A + \beta B) = \alpha T(A) + \beta T(B)$ for all $\alpha, \beta \in \mathcal{F}$ and for all $A, B \in \mathcal{M}_n(\mathcal{F})$. An operator T on $\mathcal{M}_n(\mathcal{F})$ is said to be singular if $T(A) = O$ for some nonzero $A \in \mathcal{M}_n(\mathcal{F})$. A linear operator on $\mathcal{M}_n(\mathcal{F})$ is completely determined by its behavior on the set of cells in $\mathcal{M}_n(\mathcal{F})$.

Let T be a linear operator on $\mathcal{M}_n(\mathcal{F})$. We say that

(i) T preserves regularity if $T(A) \in \mathcal{R}_n(\mathcal{F})$ whenever $A \in \mathcal{R}_n(\mathcal{F})$;

(ii) T strongly preserves regularity when $T(A) \in \mathcal{R}_n(\mathcal{F})$ if and only if $A \in \mathcal{R}_n(\mathcal{F})$.

Example 1. *Let A be any nonzero regular matrix in $\mathcal{M}_n(\mathcal{F})$. Define a linear operator T on $\mathcal{M}_n(\mathcal{F})$ by*

$$T(X) = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} x_{i,j}\right)A$$

for all $X \in \mathcal{M}_n(\mathcal{F})$. Then we can easily show that T is nonsingular and preserves regularity. But T does not preserve fuzzy nonregular matrices. Hence T does not strongly preserve regularity of fuzzy matrices.

For $A, B \in \mathcal{M}_n(\mathcal{F})$, we say A dominates B (written $A \geq B$ or $B \leq A$) if $a_{ij} = 0$ implies $b_{ij} = 0$ for all i and j.

Lemma 3. *Let $n \geq 2$. If T strongly preserves regularity on $\mathcal{M}_n(\mathcal{F})$, then T is nonsingular.*

Proof. If $T(X) = O$ for some nonzero $X \in \mathcal{M}_n(\mathcal{F})$, then $T(E) = O$ for all cells $E \leq X$. For such E, there is a matrix A such that $A \in \mathcal{R}_n(\mathcal{F})$ and $E + A \notin \mathcal{R}_n(\mathcal{F})$ by Corollary 1. Nevertheless, $T(E + A) = T(A)$, a contradiction to the fact that T strongly preserves regularity. Hence $T(X) \neq O$ for all nonzero $X \in \mathcal{M}_n(\mathcal{F})$. Therefore T is nonsingular. □

Let A and B be matrices in $\mathcal{M}_n(\mathcal{F})$. Then the matrix $A \circ B$ denotes the Hadamard product or Schur product. That is, the (i, j)th entry of $A \circ B$ is $a_{ij}b_{ij}$.

Lemma 4. *Let B be a matrix in $\mathcal{M}_n(\mathcal{F})$ with $J \leq B$ and $n \geq 2$. Let T be a linear operator on $\mathcal{M}_n(\mathcal{F})$ defined by $T(X) = X \circ B$ for all $X \in \mathcal{M}_n(\mathcal{F})$. If T strongly preserves regularity on $\mathcal{M}_n(\mathcal{F})$, then $B = J$.*

Proof. Since $J \leq B$, all entries of B are nonzero and $T(J) = B$. We will now show that $B = J$. Or, equivalently $b_{ij} = 1$ for all i and j. It is sufficient to consider b_{11}; for b_{ij} is any entry of $T(J)$, let P' be the transposition matrix that exchanges 1st and ith rows from identity matrix I_n, and Q' the transposition matrix that exchanges 1st and jth columns from identity matrix I_n. Define a linear operator L on $\mathcal{M}_n(\mathcal{F})$ by $L(X) = P'T(X)Q'$ for all X. Since T strongly preserves regularity, so does L. Furthermore the $(1, 1)$th entry of $L(J)$ is b_{ij}.

If $b_{11} \neq 1$, let $\alpha = \min\{b_{11}, b_{12}, b_{21}\}$. Then $\alpha \neq 1$. Let $A = E_{11} + \alpha(E_{12} + E_{21})$. By Lemma 1, A is not regular, and hence, $T(A) = b_{11}E_{11} + \alpha(E_{12} + E_{21})$ is not regular so that $b_{11} \neq \alpha$. Thus $\alpha \in \{b_{12}, b_{21}\}$. If $\alpha = b_{12}$, consider the matrix $A_1 = b_{11}(E_{11}+E_{12})+\alpha E_{21}$. Then A_1 is fuzzy regular, but $T(A_1) = b_{11}E_{11}+\alpha(E_{12}+E_{21})$ is not fuzzy regular by Lemma 1, a contradiction. For the case $\alpha = b_{21}$, if we consider the matrix $A_2 = b_{11}(E_{11} + E_{21}) + \alpha E_{12}$, then A_2 is fuzzy regular while $T(A_2)$ is not fuzzy regular, a contradiction. Therefore $b_{11} = 1$. Hence $B = J$. $\square$

3. Characterizations of fuzzy regularity preservers

In this section we obtain characterizations of the linear operators that strongly preserve fuzzy regular matrices.

Let $\mathcal{M}_n(\{0, 1\})$ denote the set of all zero-one matrices in $\mathcal{M}_n(\mathcal{F})$. For matrices $A, B \in \mathcal{M}_n(\{0, 1\})$ with $A \geq B$, we define $A \setminus B$ to the matrix C where

$$c_{ij} = \begin{cases} 0, & \text{if } b_{ij} = 1 \\ a_{ij}, & \text{if } b_{ij} = 0. \end{cases}$$

Proposition 3. *Let $A \in \mathcal{M}_n(\mathcal{F})$ be a sum of k cells with $r(A) = k$, where $n \geq 3$. Then $J \setminus A \in \mathcal{R}_n(\mathcal{F})$ if and only if $k \leq 2$.*

Proof. Without loss of generality, we assume that

$$A = \sum_{t=1}^{k} E_{tt}.$$

If $k \leq 2$, then $(J \setminus A)(E_{12} + E_{21})(J \setminus A) = J \setminus A$ and so $J \setminus A \in \mathcal{R}_n(\mathcal{F})$.

Let $k \geq 3$. Now we will show that $Y = J \setminus A \notin \mathcal{R}_n(\mathcal{F})$. If not, there is a nonzero $B \in \mathcal{M}_n(\mathcal{F})$ such that $Y = YBY$. Then the (t, t)th entry of YBY becomes

$$\sum_{i,j \in I} b_{ij} \tag{3}$$

for all $t = 1, \ldots, k$, where $I = \{1, \ldots, n\} \setminus \{t\}$. From $y_{11} = 0$ and (3), we have

$$b_{ij} = 0 \quad \text{for all} \quad i, j = 2, \ldots, n. \tag{4}$$

Consider the first row and the first column of B. It follows from $y_{22} = 0$ and (3) that

$$b_{i1} = 0 = b_{1j} \quad \text{for all} \quad i, j = 1, 3, 4, \ldots, n. \tag{5}$$

Also, from $y_{33} = 0$, we obtain $b_{12} = b_{21} = 0$, and hence $B = O$ by (4) and (5), a contradiction. Thus $J \setminus A \notin \mathcal{R}_n(\mathcal{F})$. □

The pattern, A^*, of a matrix $A \in \mathcal{M}_n(\mathcal{F})$ is the matrix in $\mathcal{M}_n(\{0,1\})$ whose (i,j)th entry is 0 if and only if $a_{ij} = 0$. By the definition, we obtain

$$(AB)^* = A^*B^* \quad \text{and} \quad (A+B)^* = A^* + B^*$$

for all $A, B \in \mathcal{M}_n(\mathcal{F})$. It follows that if $A \in \mathcal{R}_n(\mathcal{F})$, then $A^* \in \mathcal{R}_n(\mathcal{F})$, but not conversely. For example, let

$$A = \begin{bmatrix} p & q \\ r & 0 \end{bmatrix}$$

with $0 < p < q < r \leq 1$. Then $A^* \in \mathcal{R}_2(\mathcal{F})$, while $A \notin \mathcal{R}_2(\mathcal{F})$ by Lemma 1.

For a linear operator T on $\mathcal{M}_n(\mathcal{F})$, define its pattern, T^* by $T^*(A) = [T(A)]^*$ for all $A \in \mathcal{M}_n(\mathcal{F})$.

Lemma 5. *Let T strongly preserve regularity on $\mathcal{M}_2(\mathcal{F})$. For any cell E, $T^*(E)$ is a cell. Furthermore T^* is bijective on the set of cells.*

Proof. By Lemma 3, $|T^*(E)| \geq 1$ for every cell E. Suppose $|T^*(E_1)| \geq 2$ for some cell E_1. Then there are permutation matrices P and Q such that $PE_1Q = E_{11}$. Let E_2 and E_3 be cells with $PE_2Q = E_{12}$ and $PE_3Q = E_{21}$, respectively. Choose p, q and $r \in \mathcal{F}$ such that

$$0 < p < q < r < \min\{a_i : a_i \text{ are all nozero entries of } T(E_i)\}.$$

Let $A = P(rE_1 + pE_2 + qE_3)Q$. Then

$$A = \begin{bmatrix} r & p \\ q & 0 \end{bmatrix}$$

and $T(A)$ has at least two maximal entries. It follows from Lemma 1 that A is not fuzzy regular, while $T(A)$ is fuzzy regular, a contradiction. Thus $|T^*(E)| = 1$ for all cell E and hence the first assertion follows.

Next suppose $T^*(E) = T^*(F)$ for some distinct cells E and F. Let G be a cell different from E and F. Then there are permutation matrices P_1 and Q_1 such that

$$P_1(E + F + G)Q_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

If we take

$$B = \begin{bmatrix} 1 & \alpha \\ \alpha & 0 \end{bmatrix}$$

with $0 < \alpha < 1$, then B is not fuzzy regular by Lemma 1. But $|T(B)| = 1$ or 2. Hence $T(B)$ is fuzzy regular by Lemma 1, a contradiction. Hence the last assertion follows. $\square$

Hereafter, unless otherwise specified, we assume that $n \geq 3$ and T strongly preserves regularity on $\mathcal{M}_n(\mathcal{F})$. Then we can easily show that T^* strongly preserves regularity on $\mathcal{M}_n(\{0,1\})$. Also by Proposition 3,

$$n^2 - 3 = \max\{|N| : N \in \mathcal{M}_n(\{0,1\}) \text{ and } N \notin \mathcal{R}_n(\mathcal{F})\}.$$

Let

$$\Phi = \{N \in \mathcal{M}_n(\{0,1\}) |\ N \notin \mathcal{R}_n(\mathcal{F}) \text{ and } |N| = n^2 - 3\}.$$

Proposition 4. *For distinct cells E and F, $T^*(E) \not\leq T^*(F)$. In particular, if $|T^*(E)| = |T^*(F)| = 1$, then $T^*(E) \neq T^*(F)$.*

Proof. Suppose $T^*(E) \leq T^*(F)$ for some distinct cells E and F. Then there are cells E_1 and E_2 different from F such that $r(E + E_1 + E_2) = 3$. Since $F \leq J \setminus (E + E_1 + E_2)$, we have $T^*(E) \leq T^*(F) \leq T^*(J \setminus (E + E_1 + E_2))$ so that

$$T^*(J \setminus (E_1 + E_2)) = T^*(E) + T^*(J \setminus (E + E_1 + E_2)) = T^*(J \setminus (E + E_1 + E_2)).$$

But this is impossible as $J \setminus (E + E_1 + E_2) \notin \mathcal{R}_n(\mathcal{F})$ while $J \setminus (E_1 + E_2) \in \mathcal{R}_n(\mathcal{F})$ by Proposition 3. $\square$

Lemma 6. *If $n = 3$, then $T^*(\Phi) \subseteq \Phi$.*

Proof. Now, $\Phi = \{N \in \mathcal{M}_3(\{0,1\}) |\ N \notin \mathcal{R}_3(\mathcal{F}) \text{ and } |N| = 6\}$. Let $N \in \Phi$ be arbitrary. It suffices to show $|T^*(N)| = 6$. Since $T^*(N) \notin \mathcal{R}_3(\mathcal{F})$, we have $|T^*(N)| \leq 6$. Suppose $|T^*(N)| \leq 5$. Write

$$N = \sum_{i=1}^{6} E_i \quad \text{and} \quad J = \sum_{i=1}^{9} E_i$$

for cells $E_1, \ldots, E_9$. By Lemma 3, $|T^(E_i)| \geq 1$ for all i. Now we will claim that there are distinct cells E_i, E_j, E_k in $\{E_1, \ldots, E_6\}$ such that $T^*(E_i + E_j + E_k) = T^*(N)$. If the claim is true, then $N \notin \mathcal{R}_3(\mathcal{F})$, while $E_i + E_j + E_k \in \mathcal{R}_3(\mathcal{F})$, a contradiction to the fact that T^* strongly preserves regularity on $\mathcal{M}_3(\{0,1\})$. Hence $|T^*(N)| = 6$.

Clearly the claim holds if there is a cell $E_i \in \{E_1, \ldots, E_6\}$ such that $|T^*(E_i)| \geq 3$. Suppose $|T^*(E_i)| \leq 2$ for $i = 1, \ldots, 6$. Without loss of generality, we may assume that

$$|T^*(E_1)| = \cdots = |T^*(E_r)| = 2 \quad \text{and} \quad |T^*(E_{r+1})| = \cdots = |T^*(E_6)| = 1$$

for some r. If $r = 0$ or 1, then by Proposition 4, $T^*(E_1), \ldots, T^*(E_6)$ are all disjoint and hence $|T^*(N)| = |T^*(E_1) + \cdots + T^*(E_6)| = |T^*(E_1)| + \cdots + |T^*(E_6)| \geq 6$, which is impossible. Thus $r \geq 2$. Now suppose $|T^*(E_i + E_j)| \leq 3$ for all $1 \leq i < j \leq r$. Then there is a cell F such that $F \leq T^*(E_i)$ for all $i = 1, \dots, r$. By Proposition 4, the six cells $T^*(E_1) \setminus F, \dots, T^*(E_r) \setminus F, T^*(E_{r+1}), \dots, T^*(E_6)$ are disjoint and so

$$\begin{aligned}|T^*(N)| &= |T^*(E_1) + \cdots + T^*(E_6)| \\ &\geq |T^*(E_1) \setminus F| + \cdots + |T^*(E_r) \setminus F| + |T^*(E_{r+1})| + \cdots + |T^*(E_6)| = 6,\end{aligned}$$

which is impossible. Thus $|T^*(E_i + E_j)| = 4$ for some $1 \leq i < j \leq r$. Furthermore, we can find a cell E_k with $k \in \{1, \dots, 6\} \setminus \{i, j\}$ such that

$$T^*(E_i + E_j) + T^*(E_k) = T^*(E_i + E_j + E_k) = T^*(N).$$

So our claim holds. □

Lemma 7. *Let $n \geq 4$. For any matrix $A \in \mathcal{M}_n(\{0, 1\})$ with $|A| \leq n^2 - 3$, we have $|T^*(A)| \geq |A|$. Consequently, $T^*(\Phi) \subseteq \Phi$.*

Proof. We will prove the first part of the result by induction on $m = |A|$ with $A \in \mathcal{M}_n(\{0, 1\})$, where $|A| \leq n^2 - 3$. If $m = 1$, then the result holds by Lemma 3. Assume that the result holds for all $B \in \mathcal{M}_n(\{0, 1\})$ with $|B| < m$. Let $A \in \mathcal{M}_n(\{0, 1\})$ be arbitrary with $|A| = m$.

Suppose that at least two rows and two columns of A contain zero entries. Since $n \geq 4$, there are cells $E_1, E_2 \not\leq A$ and $E_3 \leq A$ such that $r(E_1 + E_2 + E_3) = 3$. If $|T^*(A)| = |T^*(A \setminus E_3)|$, then $T^*(A) = T^*(A \setminus E_3)$ and so

$$\begin{aligned}T^*(J \setminus (E_1 + E_2)) &= T^*(A) + T^*(J \setminus (A + E_1 + E_2)) \\ &= T^*(A \setminus E_3) + T^*(J \setminus (A + E_1 + E_2)) \\ &= T^*(J \setminus (E_1 + E_2 + E_3)),\end{aligned}$$

a contradiction by Proposition 3. Hence we have $|T^*(A)| > |T^*(A \setminus E_3)| \geq |A \setminus E_3| = m - 1$ by assumption. Thus the result follows.

Now suppose that A has zero entries in exactly one row (or one column). Since $|A| \leq n^2 - 3$, without loss of generality, we may assume that A has zero entries in the first row with $a_{11} = a_{12} = a_{13} = 0$. By assumption,

$$|T^*(A)| \geq |T^*(A \setminus E_{21})| \geq |A \setminus E_{21}| = m - 1.$$

Suppose $|T^*(A)| = m - 1$. Take

$$G_1 = E_{21} + E_{33}, \quad G_2 = E_{21} + E_{34}, \quad G_3 = E_{22} + E_{33} \quad \text{and} \quad G_4 = E_{22} + E_{34}.$$

We claim that there is an index $i \in \{1, 2, 3, 4\}$ such that $T^*(A \setminus G_i) = T^*(A)$. If the claim holds, then we can take a cell $F \in \{E_{11}, E_{12}, E_{13}\}$ such that $r(F + G_i) = 3$ and

$$\begin{aligned}T^*(J \setminus F) &= T^*(A) + T^*(J \setminus (A + F)) = T^*(A \setminus G_i) + T^*(J \setminus (A + F)) \\ &= T^*(J \setminus (F + G_i)),\end{aligned}$$

a contradiction by Proposition 3. Thus, $|T^*(A)| \geq m$ and the result follows.

Suppose the claim does not hold. Then for each $i \in \{1,2,3,4\}$, $|T^*(A\backslash G_i)| = m-2$ and so $T^*(A \setminus G_i) = T^*(A) \setminus H_i$ for some cell $H_i \leq T^*(A)$. Notice that for any $i \neq j$, $(A \setminus G_i) + (A \setminus G_j)$ equals either A or $A \setminus E$ for some $E \in \{E_{21}, E_{22}, E_{33}, E_{34}\}$. Since $m-1 = |T^*(A)| \geq |T^*(A \setminus E)| \geq m-1$, we have $T^*(A \setminus E) = T^*(A)$ and so $T^*((A \setminus G_i) + (A \setminus G_j)) = T^*(A)$ in both cases. Then

$$\begin{aligned} T^*(A) &= T^*((A \setminus G_i) + (A \setminus G_j)) = T^*(A \setminus G_i) + T^*(A \setminus G_j) \\ &= (T^*(A) \setminus H_i) + (T^*(A) \setminus H_j). \end{aligned}$$

Thus we must have $H_i \neq H_j$ and hence $|H_1 + H_2 + H_3 + H_4| = 4$.

Now let $B = A \setminus (E_{21} + E_{22} + E_{33} + E_{34})$. Since $T^*(B) \leq T^*(A \setminus G_i)$ and $H_i \not\leq T^*(A \setminus G_i)$, we have $H_i \not\leq T^*(B)$ for all $i = 1,2,3,4$ and hence $H_1 + H_2 + H_3 + H_4 \not\leq T^*(B)$. Then $T^*(B) \leq T^*(A) \setminus (H_1 + H_2 + H_3 + H_4)$ and hence by assumption,

$$m - 4 = |B| \leq |T^*(B)| \leq |T^*(A) \setminus (H_1 + H_2 + H_3 + H_4)| = (m-1) - 4 = m - 5$$

which is impossible. Thus our claim holds. □

Corollary 2. *The map $T^*|_\Phi$ is bijective from Φ onto Φ.*

Proof. By Lemmas 6 and 7, $T^*(\Phi) \subseteq \Phi$. Suppose $T^*(A) = T^*(B)$ for some distinct $A, B \in \Phi$. Then

$$T^*(A + B) = T^*(A) + T^*(B) = T^*(A).$$

But $A + B \in \mathcal{R}_n(\mathcal{F})$ and $A \notin \mathcal{R}_n(\mathcal{F})$, a contradiction to the fact that T^* strongly preserves regularity on $\mathcal{M}_n(\{0,1\})$. Thus T^* is injective in Φ. Since Φ is finite, $T^*(\Phi) = \Phi$. Thus the result follows. □

Lemma 8. *For any cell E, $T^*(E)$ is a cell. Furthermore, T^* is bijective on the set of cells.*

Proof. Let E be an arbitrary cell. Notice that for any $A \in \Phi$, as $T^*(\Phi) = \Phi$,

$$E \leq A \in \Phi \quad \Leftrightarrow \quad E + A \in \Phi \quad \Leftrightarrow \quad T^*(E) + T^*(A) \in \Phi \quad \Leftrightarrow \quad T^*(E) \leq T^*(A) \in \Phi.$$

Hence $T^*(\{A \in \Phi : E \leq A\}) = \{A \in \Phi : T^*(E) \leq A\}$. It follows from Corollary 2 that the two sets $T^*(\{A \in \Phi : E \leq A\})$ and $\{A \in \Phi : T^*(E) \leq A\}$ have the same number of elements. This is possible only if $T^*(E)$ is a cell. Thus the first assertion follows. The last assertion follows by Proposition 4. □

A matrix L is called a line matrix if

$$L = \sum_{k=1}^{n} E_{ik} \quad \text{or} \quad \sum_{l=1}^{n} E_{lj}$$

for some $i, j \in \{1, \ldots, n\}$; $R_i = \sum\limits_{k=1}^{n} E_{ik}$ is the ith row matrix and $C_j = \sum\limits_{l=1}^{n} E_{lj}$ is the jth column matrix.

Corollary 3. *Let $n \geq 2$. If T strongly preserves regularity on $\mathcal{M}_n(\mathcal{F})$, then T^* preserves all line matrices.*

Proof. By Lemmas 5 and 8, T^* is bijective on the set of cells. Suppose that T^* does not map some line matrix into a line matrix. Without loss of generality, we assume that $T(E_{11}) = aE_{11}$ and $T(E_{12}) = bE_{22}$ with $ab \neq 0$. Furthermore there is a cell E_{ij} different from E_{11} and E_{12} such that $T(E_{ij}) = cE_{12}$ with $c \neq 0$. Suppose $i = 1$ or $j \geq 3$. Choose $p \in \mathcal{F}$ such that $0 < p < \min\{a, b, c\}$. Then

$$T[p(E_{11} + E_{12}) + E_{ij}] = \begin{bmatrix} p & c \\ 0 & p \end{bmatrix} \oplus O_{n-2} \notin \mathcal{R}_n(\mathcal{F})$$

by Lemma 1, while $p(E_{11} + E_{12}) + E_{ij} \in \mathcal{R}_n(\mathcal{F})$ by (2) and Proposition 2, a contradiction. Thus we may assume that $i \geq 2$ and $j \leq 2$. Without loss of generality, we assume $i = 2$. Furthermore there is a cell E_{rs} different from E_{11}, E_{12} and E_{2j} such that $T(E_{rs}) = dE_{21}$ with $d \neq 0$. By the same manner in the previous argument, we may assume $r \geq 2$ and $s \leq 2$. Suppose $r \geq 3$. Choose $p \in \mathcal{F}$ such that $0 < p < \min\{a, b, c, d\}$. Let

$$A = E_{11} + p(E_{12} + E_{2j} + E_{rs}) \quad \text{and} \quad B = E_{12} + p(E_{11} + E_{2j} + E_{rs})$$

according as $j = 1$ and $j = 2$. Then A and B are not fuzzy regular by Lemma 2. But

$$T(A) = \begin{bmatrix} a & p \\ p & p \end{bmatrix} \oplus O_{n-2} \quad \text{and} \quad T(B) = \begin{bmatrix} p & p \\ p & b \end{bmatrix} \oplus O_{n-2}$$

are fuzzy regular by Lemma 1 and (2), a contradiction. Thus, we may assume $i = r = 2$ with $j = 1$ and $s = 2$. Choose $q_i \in \mathcal{F}$ such that $0 < q_4 < q_3 < q_2 < q_1 < \min\{a, b, c, d\}$ and let

$$C = \begin{bmatrix} q_1 & q_2 \\ q_3 & q_4 \end{bmatrix} \oplus O_{n-2}.$$

Then

$$T(C) = \begin{bmatrix} q_1 & q_3 \\ q_4 & q_2 \end{bmatrix} \oplus O_{n-2}.$$

It follows from Lemma 1 and (2) that $T(C) \in \mathcal{R}_n(\mathcal{F})$, while $C \notin \mathcal{R}_n(\mathcal{F})$, a contradiction. Thus the result follows. □

Now, we are ready to prove the main theorem.

Theorem 1. *Let $n \geq 2$ and let T be a linear operator on $\mathcal{M}_n(\mathcal{F})$. Then T strongly preserves regularity on $\mathcal{M}_n(\mathcal{F})$ if and only if there are permutation matrices P and Q such that $T(X) = PXQ$ or $T(X) = PX^tQ$ for all $X \in \mathcal{M}_n(\mathcal{F})$.*

Proof. By Proposition 1, the sufficiency is obvious. To prove the necessity, assume that T strongly preserves regularity on $\mathcal{M}_n(\mathcal{F})$. Then T^* is bijective on the set of cells by Lemmas 5 and 8. Also by Corollary 3, T^* preserves all line matrices. Since no combination of s row matrices and t column matrices can dominate J where $s + t = n$ unless $s = 0$ or $t = 0$, we have that either

(1) the image of T^* of each row matrix is a row matrix and the image of T^* of each column matrix is a column matrix, or

(2) the image of T^* of each row matrix is a column matrix and the image of T^* of each column matrix is a row matrix.

If (1) holds, then there are permutations σ and τ of $\{1,\dots,n\}$ such that $T^*(R_i) = R_{\sigma(i)}$ and $T^*(C_j) = C_{\tau(j)}$ for all $i, j = 1,\dots,n$. Let P and Q be permutation matrices corresponding to σ and τ, respectively. Then we have

$$T(E_{ij}) = b_{ij}E_{\sigma(i)\tau(j)} = P(b_{ij}E_{ij})Q,$$

where $b_{ij} \neq 0$ for all cells E_{ij}. By the action of T on the cells, we have $T(X) = P(X \circ B)Q$. Define a linear operator L on $\mathcal{M}_n(\mathcal{F})$ by $L(X) = P^tT(X)Q^t = X \circ B$. Since T strongly preserves regularity on $\mathcal{M}_n(\mathcal{F})$, so does L. By Lemma 4, $B = J$. Thus we have $T(X) = PXQ$ for all $X \in \mathcal{M}_n(\mathcal{F})$.

If (2) holds, then a parallel argument shows that there are permutation matrices P and Q such that $T(X) = PX^tQ$ for all $X \in \mathcal{M}_n(\mathcal{F})$. $\square$

Thus we obtain the characterizations of the linear operators that strongly preserve the regularity of fuzzy matrices.

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