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On the stability of solutions of nonlinear differential equations of fifth order with delay

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Abstract. Criteria for global asymptotic stability of a null solution of a nonlinear differential equation of fifth order with delay

$$\begin{aligned}&x^{(5)}(t)+\psi(x(t-r),x'(t-r),x''(t-r),x'''(t-r),x^{(4)}(t-r))x^{(4)}(t)\\&+f(x''(t-r),x'''(t-r))+\alpha_3x''(t)+\alpha_4x'(t)+\alpha_5x(t)=0\end{aligned}$$

are obtained by using Lyapunov's second method. By defining a Lyapunov functional, sufficient conditions are established, which guarantee the null solution of this equation is globally asymptotically stable. Our result consists of a new theorem on the subject.
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Key words: stability, Lyapunov functional, differential equation, fifth order, delay

1. Introduction

This paper is concerned with the problem of the stability of the null solution of the nonlinear differential equation of fifth order with constant delay

$$\begin{aligned}&x^{(5)}(t)+\psi(x(t-r),x'(t-r),x''(t-r),x'''(t-r),x^{(4)}(t-r))x^{(4)}(t)\\&+f(x''(t-r),x'''(t-r))+\alpha_3x''(t)+\alpha_4x'(t)+\alpha_5x(t)=0\end{aligned}\tag{1}$$

or its equivalent system

$$\begin{aligned}x'(t)&=y(t)\\y'(t)&=z(t)\\z'(t)&=w(t)\\w'(t)&=u(t)\\u'(t)&=-\psi(x(t-r),y(t-r),z(t-r),w(t-r),u(t-r))u(t)\\&\quad-f(z(t),w(t))-\alpha_3z(t)-\alpha_4y(t)-\alpha_5x(t)\\&\quad+\int_{t-r}^{t}f_z(z(s),w(s))w(s)ds+\int_{t-r}^{t}f_w(z(s),w(s))u(s)ds,\end{aligned}\tag{2}$$

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where ψ and f are continuous functions for the arguments displayed explicitly in Eq. (1); r is a positive constant, that is, r is fixed constant delay; α_3, α_4 and α_5 are some positive constants. It is assumed that $f(z,0) = 0$ and the derivatives $f_z(z,w) \equiv \frac{\partial}{\partial z} f(z,w)$ and $f_w(z,w) \equiv \frac{\partial}{\partial w} f(z,w)$ exist and are continuous for all z, w. All solutions considered are also assumed to be real valued.

To the best of our knowledge from literature, in the last three decades, much attention has been paid to investigation of stability of solutions of nonlinear differential equations of fifth order without delay:

$$x^{(5)}(t) + A_1 x^{(4)}(t) + A_2 x'''(t) + A_3 x''(t) + A_4 x'(t) + A_5 x(t) = 0,$$

in which $x \in \Re$, $t \in [0,\infty)$, A_1, A_2, A_3, A_4 and A_5 are not necessarily constants. For a comprehensive treatment of the subject we refer the reader to the papers of Abou-El Ela and Sadek [1], Burganskaja [5], Chukwu [7], Sinha [17], Tejumola and Afuwape [18], Tunç [19], Yu [21] and the references thereof for some works performed on the topic, which include some nonlinear differential equations of fifth order without delay. Throughout all aforementioned papers, Lyapunov's second (or direct) method [14] has been used as a basic tool to prove the problems established in these papers. Meanwhile, for some recent works, it is worth mentioning that in 2007 Adesina and Ukpera [2] investigated the convergence of solutions for the following nonlinear differential equation of fifth order without delay

$$x^{(5)} + ax^{(4)} + bx''' + f(x'') + g(x') + h(x) = p(t, x, x', x'', x''', x^{(4)}),$$

by using Lyapunov's second method, where a, b are positive constants, the functions f, g, h and p are real valued and continuous in their respective arguments. The authors presented some sufficient conditions for all solutions of the above equation to be convergence. Namely, two solutions convergent to each other if their difference and those of their derivatives up to order four approach zero as time approaches infinity. In this work, the authors showed that the nonlinear functions involved in the above equation are not necessarily differentiable, but satisfy certain increment ratios that lie in the closed sub-interval of the Routh-Hurwitz interval. Later, in 2008, Adesina and Ukpera [3] discussed boundedness, global exponential stability and periodicity of solutions of a class of nonlinear differential equations of fifth order with constant delay. Finally, in 2009, Adesina and Ukpera [4] also established some sufficient conditions in order for all solutions of a certain nonlinear differential equation of fifth order without delay to converge to a limiting regime under some boundedness restrictions and proved that this limiting regime is periodic or almost periodic. It should be noted that our equation, Eq. (1) and the assumptions that will be established here are different from those in the papers of Adesian and Ukpera [2-4].

To the best of our knowledge, so far, nonlinear delay differential Eq. (1) has not been the subject of investigation for stability of solutions. Our motivation comes from the papers mentioned above, which were carried out on the nonlinear differential equations of fifth order without delay and with delay. Throughout this paper, we also use Lyapunov's second (or direct) method [14] to investigate the stability of the null solution of delay differential Eq. (1) by defining a Lyapunov functional.

2. Preliminaries

In order to reach the main result of this paper, we will give some important basic information for the general autonomous delay differential system (see also Èl'sgol'ts [8], Èl'sgol'ts and Norkin [9], Hale [11], Kolmanovskii and Myshkis [12], Krasovskii [13], Razumikhin ([15, 16]) and Yoshizawa [20]).

We consider a general autonomous delay differential system

$$x' = f(x_t),\ x_t(\theta) = x(t+\theta), \quad -r \leq \theta \leq 0,\ t \geq 0, \tag{3}$$

where $f : C_H \to \Re^n$ is a continuous mapping, $f(0) = 0$, and we suppose that f takes closed bounded sets into bounded sets of $\Re^n$. Here $(C, \ \|.\|)$ is the Banach space of continuous function $\phi : [-r,\ 0] \to \Re^n$ with supremum norm, $r > 0$; C_H is the open H-ball in C; $C_H := \{\phi \in (C\,[-r, 0]\,,\ \ \Re^n) : \|\phi\| < H\}$. Standard existence theory, see Burton [6], shows that if $\phi \in C_H$ and $t \geq 0$, then there is at least one continuous solution $x(t, t_0, \phi)$ such that on $[t_0,\ t_0 + \alpha)$ satisfying Eq. (3) for $t > t_0$, $x_t(t, \phi) = \phi$ and α is a positive constant. If there is a closed subset $B \subset C_H$ such that the solution remains in B, then $\alpha = \infty$. Further, the symbol $|.\,|$ will denote the norm in $\Re^n$ with $|x| = \max_{1 \leq i \leq n} |x_i|$.

Definition 1 (see [6]). *Let $f(0) = 0$. The zero solution of Eq. (3) is:*

(i) stable if for each $t_1 \geq t_0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $[\ \|\phi\| \leq \delta,\ t \geq t_1]$ imply that $|x(t, t_1, \phi)| < \varepsilon$.

(ii) asymptotically stable if it is stable and if for each $t_1 \geq t_0$ there is an η such that $\|\phi\| \leq \eta$ implies that $x(t, t_0, \phi) \to 0$ as $t \to \infty$.

Definition 2 (see [6]). *A continuous positive definite function $W : \Re^n \to [0,\ \infty)$ is called a wedge.*

Definition 3 (see [6]). *A continuous function $W : [0,\ \infty) \to [0,\ \infty)$ with $W(0) = 0$, $W(s) > 0$ if $s > 0$, and W strictly increasing is a wedge. (We denote wedges by W or W_i, where i is an integer).*

Definition 4 (see [6]). *Let D be an open set in $\Re^n$ with $0 \in D$. A function $V : D \to [0, \infty)$ is called*

(i) positive definite if $V(0) = 0$ and if there is a wedge W_1 with $V(x) \geq W_1(|x|)$,

(ii) decresent if there is a wedge W_2 with $V(x) \leq W_2(|x|)$.

Definition 5 (see [17]). *If V is a continuous scalar function in C_H, we define the derivative of V along the solutions of Eq. (3) by the following relation*

$$\dot{V}(\phi) = \limsup_{h \to 0^+} \frac{V(x_h(\phi)) - V(\phi))}{h}. \tag{4}$$

Proposition 1 (see [10]). *Suppose $f(0) = 0$. Let V be a continuous functional defined on $C_H = C$ with $V(0) = 0$, and let $u(s)$ be a function, non-negative and continuous for $0 \leq s < \infty$, $u(s) \to \infty$ as $u \to \infty$ such that for all $\phi \in C$*

(i) $u(|\phi(0)|) \le V(\phi)$, $V(\phi) \ge 0$,

(ii) $\dot{V}(\phi) < 0$ *for* $\phi \ne 0$.

Then all solutions of Eq. (3) approach zero as $t \to \infty$ *and the origin is globally asymptotically stable.*

Note that $C_H = C$ when $H = \infty$; and that the set R of ϕ in C for which $\dot{V}_{(3)}(\phi) = 0$ has the largest invariant set $M = \{0\}$ by the condition $\dot{V}(\phi) < 0$ for $\phi \ne 0$.

3. Main result

Our main result is the following theorem.

Theorem 1. *In addition to basic assumptions imposed on the functions* ψ *and* f *appeared in Eq. (1), we assume that there are positive constants* α_1, α_2, α_3, α_4, α_5, ε, ε_0, δ, λ, ρ, M *and* L *such that the following conditions hold for every* x, y, z, w *and* u *:*

(i)

$$\begin{aligned}
\alpha_1 &> 0, \alpha_1\alpha_2 - \alpha_3 > 0, (\alpha_1\alpha_2 - \alpha_3)\alpha_3 - (\alpha_1\alpha_4 - \alpha_5)\alpha_1 > 0,\\
\delta_0 &:= (\alpha_3\alpha_4 - \alpha_2\alpha_5)(\alpha_1\alpha_2 - \alpha_3) - (\alpha_1\alpha_4 - \alpha_5)^2 > 0, \alpha_5 > 0,\\
\Delta_1 &:= \frac{(\alpha_3\alpha_4 - \alpha_2\alpha_5)(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - (\alpha_1\alpha_4 - \alpha_5) > 2\varepsilon\alpha_2,\\
\Delta_2 &:= \frac{\alpha_3\alpha_4 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5} - \frac{\alpha_1\alpha_4 - \alpha_5}{\alpha_1\alpha_2 - \alpha_3} - \frac{\varepsilon}{\alpha_1} > 0.
\end{aligned}$$

(ii)

$$\begin{aligned}
2\varepsilon_0 &\le \psi(x(t-r), y(t-r), z(t-r), w(t-r), u(t-r)) - \alpha_1\\
&\le \min\left\{\frac{\varepsilon\alpha_2(\alpha_1\alpha_4 - \alpha_5)^2}{4\alpha_4^2(\alpha_1\alpha_2 - \alpha_3)^2}, \frac{\varepsilon}{4\alpha_1^2}, \frac{\varepsilon\alpha_4}{4\delta^2}\right\}.
\end{aligned}$$

(iii)

$$\frac{f(z,w)}{w} \ge \alpha_2, \quad w \ne 0$$

and

$$\left(\frac{f(z,w)}{w} - \alpha_2\right)^2 \le \min\left[\frac{\varepsilon^2\alpha_2(\alpha_1\alpha_4 - \alpha_5)^2}{4\alpha_4^2(\alpha_1\alpha_2 - \alpha_3)^2}, \frac{\varepsilon^2\alpha_2}{4\delta^2}\right],$$

$$|f_z(z,w)| \le M \quad \text{and} \quad |f_w(z,w)| \le L.$$

Then, the trivial null solution of Eq. (1) is globally asymptotically stable provided that

$$r < \min\left\{\frac{\varepsilon\alpha_4}{\delta(L+M)}, \frac{\varepsilon\alpha_2(\alpha_1\alpha_4 - \alpha_5)}{\alpha_4(L+M)(\alpha_1\alpha_2 - \alpha_3)}, \frac{\varepsilon}{2\alpha_1(L+M) + 4\rho}, \frac{\varepsilon_0}{L+M+2\lambda}\right\}.$$

Proof. We define the Lyapunov functional $V = V(x_t, y_t, z_t, w_t, u_t)$ as:

$$
\begin{aligned}
2V =\, & u^2+2\alpha_1 uw+\frac{2\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}uz+2\delta uy+2\int_0^w f(z,\xi)d\xi \\
& +\left[\alpha_1^2-\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\right]w^2+2\left[\alpha_3+\frac{\alpha_1\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}-\delta\right]wz \\
& +2\alpha_1\delta wy+2\alpha_4 wy+2\alpha_5 wx+\alpha_1\alpha_3 z^2+\left[\frac{\alpha_2\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}-\alpha_4-\alpha_1\delta\right]z^2 \\
& +2\delta\alpha_2 yz+2\alpha_1\alpha_4 zy-2\alpha_5 zy+2\alpha_1\alpha_5 zx+\frac{\alpha_4^2(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}y^2 \\
& +(\delta\alpha_3-\alpha_1\alpha_5)y^2+\frac{2\alpha_4\alpha_5(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}yx+\delta\alpha_5 x^2 \\
& +2\rho\int_{-r}^{0}\int_{t+s}^{t} w^2(\theta)d\theta ds+2\lambda\int_{-r}^{0}\int_{t+s}^{t} u^2(\theta)d\theta ds,
\end{aligned}
\tag{5}
$$

where s is a real variable such that the integrals

$$\int_{-r}^{0}\int_{t+s}^{t} w^2(\theta)d\theta ds$$

and

$$\int_{-r}^{0}\int_{t+s}^{t} u^2(\theta)d\theta ds$$

are non-negative, ρ and λ are some positive constants which will be determined later in the proof and δ is a positive constant defined by

$$\delta := \frac{\alpha_5(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}+\varepsilon. \tag{6}$$

It is clear that $V(0,0,0,0,0)=0$. Since $f(z,0)=0$ and $\frac{f(z,w)}{w}\geq\alpha_2$, $(w\neq 0)$, we have $2\int_0^w f(z,\xi)d\xi = 2\int_0^w \frac{f(z,\xi)}{\xi}\xi d\xi \geq 2\int_0^w \alpha_2\xi d\xi = \alpha_2 w^2$. Hence, the Lyapunov functional $V = V(x_t, y_t, z_t, w_t, u_t)$ defined by (5) can be recast as:

$$
\begin{aligned}
2V \geq\, & \left[u+\alpha_1 w+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}z+\delta y\right]^2+\frac{\alpha_4\delta_0}{(\alpha_1\alpha_4-\alpha_5)^2}\left(z+\frac{\alpha_5}{\alpha_4}y\right)^2 \\
& +\frac{(\alpha_1\alpha_4-\alpha_5)}{(\alpha_1\alpha_2-\alpha_3)}\left[\frac{\alpha_5(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}x+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{(\alpha_1\alpha_4-\alpha_5)}y+\alpha_1 z+w\right]^2 \\
& +\Delta_2\left(w+\alpha_1 z\right)^2+\frac{\varepsilon}{\alpha_1}w^2+2\varepsilon\left(\frac{\alpha_3\alpha_4-\alpha_2\alpha_5}{\alpha_1\alpha_4-\alpha_5}\right)yz \\
& +2\rho\int_{-r}^{0}\int_{t+s}^{t} w^2(\theta)d\theta ds+2\lambda\int_{-r}^{0}\int_{t+s}^{t} u^2(\theta)d\theta ds+\sum_{i=1}^{2}V_i,
\end{aligned}
\tag{7}
$$

where

$$V_1 := \delta\alpha_5 x^2 - \frac{\alpha_5^2(\alpha_1\alpha_2 - \alpha_3)}{(\alpha_1\alpha_4 - \alpha_5)}x^2$$

and

$$V_2 := \left[\delta\alpha_3 - \alpha_1\alpha_5 - \frac{\alpha_5^2\delta_0}{\alpha_4(\alpha_1\alpha_4 - \alpha_5)^2} - \delta^2\right]y^2.$$

By noting (6), it is clear that

$$V_1 = \varepsilon\alpha_5 x^2$$

and

$$V_2 \geq \frac{\alpha_5\delta_0}{4\alpha_4(\alpha_1\alpha_4 - \alpha_5)}y^2$$

provided that

$$\frac{\alpha_5\delta_0}{4\alpha_4(\alpha_1\alpha_4 - \alpha_5)} \geq \varepsilon\left[\varepsilon + \frac{2\alpha_5(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \alpha_3\right],$$

which we now assume.

Summing up the equality and inequality obtained for V_1 and V_2 into (7), we have

$$\begin{aligned} 2V \geq{}& \left[u + \alpha_1 w + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}z + \delta y\right]^2 + \frac{\alpha_4\delta_0}{(\alpha_1\alpha_4 - \alpha_5)^2}\left(z + \frac{\alpha_5}{\alpha_4}y\right)^2 \\ &+ \Delta_2 (w + \alpha_1 z)^2 + \varepsilon\alpha_5 x^2 + \frac{\alpha_5\delta_0}{4\alpha_4(\alpha_1\alpha_4 - \alpha_5)}y^2 + \frac{\varepsilon}{\alpha_1}w^2 \\ &+ 2\varepsilon\left(\frac{\alpha_3\alpha_4 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5}\right)yz + 2\rho\int_{-r}^{0}\int_{t+s}^{t} w^2(\theta)d\theta ds + 2\lambda\int_{-r}^{0}\int_{t+s}^{t} u^2(\theta)d\theta ds. \end{aligned} \tag{8}$$

Clearly, it follows from the first six terms included in (8) that there exist sufficiently small positive constants D_i, $(i = 1, 2, 3, 4, 5)$, such that

$$\begin{aligned} 2V \geq{}& D_1 x^2 + D_2 y^2 + D_3 z^2 + D_4 w^2 + D_5 u^2 + 2\varepsilon\left(\frac{\alpha_3\alpha_4 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5}\right)yz \\ &+ 2\rho\int_{-r}^{0}\int_{t+s}^{t} w^2(\theta)d\theta ds + 2\lambda\int_{-r}^{0}\int_{t+s}^{t} u^2(\theta)d\theta ds. \end{aligned} \tag{9}$$

Now, we consider the terms

$$V_3 =: \frac{D_2}{2}y^2 + 2\varepsilon\left(\frac{\alpha_3\alpha_4 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5}\right)yz + \frac{D_3}{2}z^2,$$

which are contained in (9).

Clearly, V_3 represents a quadratic expression. So, it can be easily seen that V_3 is positive semi-definite if the symmetric matrix

$$\begin{bmatrix} \frac{D_2}{2} & \varepsilon\left(\frac{\alpha_3\alpha_4-\alpha_2\alpha_5}{\alpha_1\alpha_4-\alpha_5}\right) \\ \varepsilon\left(\frac{\alpha_3\alpha_4-\alpha_2\alpha_5}{\alpha_1\alpha_4-\alpha_5}\right) & \frac{D_3}{2} \end{bmatrix}$$

is positive semi-definite. That is, $V_3 \geq 0$, provided that

$$\varepsilon^2 \leq \left(\frac{\alpha_1\alpha_4-\alpha_5}{\alpha_3\alpha_4-\alpha_2\alpha_5}\right)^2 \frac{D_2D_3}{4} = D_6, \quad D_6 > 0.$$

By using this fact, we get from (9) that

$$2V \geq D_1x^2 + \tfrac{D_2}{2}y^2 + \tfrac{D_3}{2}z^2 + D_4w^2 + D_5u^2$$

$$+2\rho \int_{-r}^{0} \int_{t+s}^{t} w^2(\theta)d\theta ds + 2\lambda \int_{-r}^{0} \int_{t+s}^{t} u^2(\theta)d\theta ds.$$

As a result, since the integrals

$$2\rho \int_{-r}^{0} \int_{t+s}^{t} w^2(\theta)d\theta ds$$

and

$$\lambda \int_{-r}^{0} \int_{t+s}^{t} u^2(\theta)d\theta ds$$

are non-negative, it is obvious that there exists a positive constant D_7 which satisfies the inequality

$$D_7(x^2(t)+y^2(t)+z^2(t)+w^2(t)+u^2(t)) \leq V(x_t,y_t,z_t,w_t,u_t), \tag{10}$$

where $D_7 = \frac{1}{2}\min\left\{D_1, \frac{D_2}{2}, \frac{D_3}{2}, D_4, D_5\right\}$.

Now, by a direct calculation from (5) and (2) one finds

$$\begin{aligned}\frac{d}{dt}V(x_t,y_t,z_t,w_t,u_t) = &-\left[\psi(x(t-r),y(t-r),z(t-r),w(t-r),u(t-r))-\alpha_1\right]\ u^2 \\ &-\left[\alpha_1\frac{f(z,w)}{w} - \left\{\alpha_3 + \frac{\alpha_1\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5} - \delta\right\}\right] w^2 \\ &-\left[\frac{\alpha_3\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5} - \{\delta\alpha_2 + (\alpha_1\alpha_4-\alpha_5)\}\right] z^2 \\ &-\left[\delta\alpha_4 - \frac{\alpha_4\alpha_5(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\right] y^2 \\ &-\alpha_1[\psi(x(t-r),y(t-r),z(t-r),w(t-r),u(t-r)) \\ &-\alpha_1]\ wu\end{aligned}$$

$$-\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\left[\psi(x(t-r),y(t-r),z(t-r),w(t-r),u(t-r))-\alpha_1\right]\ zu$$

$$-\delta\left[\psi(x(t-r),y(t-r),z(t-r),w(t-r),u(t-r))-\alpha_1\right]\ yu$$

$$-\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\left[\frac{f(z,w)}{w}-\alpha_2\right]wz-\delta\left[\frac{f(z,w)}{w}-\alpha_2\right]wy$$

$$+u\int_{t-r}^{t} f_z(z(s),w(s))w(s)ds+\alpha_1 w\int_{t-r}^{t} f_z(z(s),w(s))w(s)ds \tag{11}$$

$$+\delta y\int_{t-r}^{t} f_z(z(s),w(s))w(s)ds+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}z\int_{t-r}^{t} f_z(z(s),w(s))w(s)ds$$

$$+u\int_{t-r}^{t} f_w(z(s),w(s))u(s)ds+\alpha_1 w\int_{t-r}^{t} f_w(z(s),w(s))u(s)ds$$

$$+\delta y\int_{t-r}^{t} f_w(z(s),w(s))u(s)ds+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}z\int_{t-r}^{t} f_w(z(s),w(s))u(s)ds$$

$$+\rho w^2 r-\rho\int_{t-r}^{t} w^2(s)ds+\lambda u^2 r-\lambda\int_{t-r}^{t} u^2(s)ds.$$

Making use of the assumptions (i)-(iii) and (6), we get

$$\left[\psi(x(t-r),y(t-r),z(t-r),w(t-r),u(t-r))-\alpha_1\right]\ \geq 2\varepsilon_0,$$

$$\left[\alpha_1\frac{f(z,w)}{w}-\left\{\alpha_3+\frac{\alpha_1\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}-\delta\right\}\right]\geq\varepsilon,$$

$$\left[\frac{\alpha_3\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}-\{\delta\alpha_2+(\alpha_1\alpha_4-\alpha_5)\}\right]>\varepsilon\alpha_2$$

and

$$\left[\delta\alpha_4-\frac{\alpha_4\alpha_5(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\right]=\varepsilon\alpha_4.$$

By using the assumptions $|f_z(z,w)|\leq M$ and $|f_w(z,w)|\leq L$ of the theorem and the inequality $2\,|ab|\leq a^2+b^2$, we obtain the following inequalities:

$$u\int_{t-r}^{t} f_z(z(s),w(s))w(s)ds\leq\frac{M}{2}ru^2(t)+\frac{M}{2}\int_{t-r}^{t} w^2(s)ds,$$

$$\alpha_1 w \int\limits_{t-r}^{t} f_z(z(s),w(s))w(s)ds \leq \frac{\alpha_1 M}{2} r w^2(t) + \frac{\alpha_1 M}{2} \int\limits_{t-r}^{t} w^2(s)ds,$$

$$\delta y \int\limits_{t-r}^{t} f_z(z(s),w(s))w(s)ds \leq \frac{\delta M}{2} r y^2(t) + \frac{\delta M}{2} \int\limits_{t-r}^{t} w^2(s)ds,$$

$$\frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} z \int\limits_{t-r}^{t} f_z(z(s),w(s))w(s)ds \leq \frac{\alpha_4 M(\alpha_1\alpha_2 - \alpha_3)}{2(\alpha_1\alpha_4 - \alpha_5)} r z^2(t) + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)M}{2(\alpha_1\alpha_4 - \alpha_5)} \int\limits_{t-r}^{t} w^2(s)ds,$$

$$u \int\limits_{t-r}^{t} f_w(z(s),w(s))u(s)ds \leq \frac{L}{2} r u^2(t) + \frac{L}{2} \int\limits_{t-r}^{t} u^2(s)ds,$$

$$\alpha_1 w \int\limits_{t-r}^{t} f_w(z(s),w(s))u(s)ds \leq \frac{\alpha_1 L}{2} r w^2(t) + \frac{\alpha_1 L}{2} \int\limits_{t-r}^{t} u^2(s)ds,$$

$$\delta y \int\limits_{t-r}^{t} f_w(z(s),w(s))u(s)ds \leq \frac{\delta L}{2} r y^2(t) + \frac{\delta L}{2} \int\limits_{t-r}^{t} u^2(s)ds$$

and

$$\frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} z \int\limits_{t-r}^{t} f_w(z(s),w(s))u(s)ds \leq \frac{\alpha_4 L(\alpha_1\alpha_2 - \alpha_3)}{2(\alpha_1\alpha_4 - \alpha_5)} r z^2(t) + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)L}{2(\alpha_1\alpha_4 - \alpha_5)} \int\limits_{t-r}^{t} u^2(s)ds.$$

Replacing the last equality and the preceding inequalities into (11), we obtain

$$\begin{aligned}
\tfrac{dV}{dt} \leq &- \left[\frac{\varepsilon\alpha_4}{2} - \left(\frac{\delta(L+M)}{2}\right) r\right] y^2 - \left\{\frac{\varepsilon\alpha_2}{2} - \left(\frac{\alpha_4(L+M)(\alpha_1\alpha_2 - \alpha_3)}{2(\alpha_1\alpha_4 - \alpha_5)}\right) r\right\} z^2 \\
&- \left[\frac{\varepsilon}{4} - \left(\frac{\alpha_1(L+M)}{2} + \rho\right) r\right] w^2 - \left[\frac{\varepsilon_0}{2} - \left(\frac{L+M}{2} + \lambda\right) r\right] u^2 \\
&- \left[\rho - \left(\tfrac{M}{2} + \tfrac{\alpha_1 M}{2} + \tfrac{\delta M}{2} + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)M}{2(\alpha_1\alpha_4 - \alpha_5)}\right)\right] \int\limits_{t-r}^{t} w^2(s)ds \\
&- \left[\lambda - \left(\frac{L}{2} + \frac{\alpha_1 L}{2} + \frac{\delta L}{2} + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)L}{2(\alpha_1\alpha_4 - \alpha_5)}\right)\right] \int\limits_{t-r}^{t} u^2(s)ds - \sum_{k=4}^{8} V_k,
\end{aligned} \tag{12}$$

where

$$V_4 = \frac{1}{4}\left[\psi - \alpha_1\right]\ u^2 + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}\left[\psi - \alpha_1\right] zu + \frac{\varepsilon\alpha_2}{4}z^2,$$
$$V_5 = \frac{1}{4}\left[\psi - \alpha_1\right]\ u^2 + \alpha_1\left[\psi - \alpha_1\right] wu + \frac{\varepsilon}{4}\,w^2,$$
$$V_6 = \frac{1}{4}\left[\psi - \alpha_1\right]\ u^2 + \delta\left[\psi - \alpha_1\right] yu + \frac{\varepsilon\alpha_4}{4}y^2,$$
$$V_7 = \frac{\varepsilon}{4}\,w^2 + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}\left[\frac{f}{w} - \alpha_2\right] wz + \frac{\varepsilon\alpha_2}{4}z^2,$$
$$V_8 = \frac{\varepsilon}{4}\,w^2 + \delta\left[\frac{f}{w} - \alpha_2\right] wy + \frac{\varepsilon\alpha_4}{4}y^2.$$

It is clear that the expressions given by V_4, V_5, V_6, V_7 and V_8 represent certain specific quadratic forms, respectively. Making use of the basic information on the positive semi-definite of a quadratic form, one can easily conclude that $V_4 \geq 0$, $V_5 \geq 0$, $V_6 \geq 0$, $V_7 \geq 0$ and $V_8 \geq 0$ provided that

$$(\psi - \alpha_1) \leq \frac{\varepsilon\alpha_2(\alpha_1\alpha_4 - \alpha_5)^2}{4\alpha_4^2(\alpha_1\alpha_2 - \alpha_3)^2}, (\psi - \alpha_1) \leq \frac{\varepsilon}{4\alpha_1^2}, (\psi - \alpha_1) \leq \frac{\varepsilon\alpha_4}{4\delta^2},$$
$$\left(\frac{f}{w} - \alpha_2\right)^2 \leq \frac{\varepsilon^2\alpha_2(\alpha_1\alpha_4 - \alpha_5)^2}{4\alpha_4^2(\alpha_1\alpha_2 - \alpha_3)^2}$$

and

$$\left(\frac{f}{w} - \alpha_2\right)^2 \leq \frac{\varepsilon^2\alpha_4}{4\delta^2},$$

respectively.

Thus, in view of the above discussion and inequality (12), it follows that

$$\begin{aligned}\frac{dV}{dt} \leq &- \left[\frac{\varepsilon\alpha_4}{2} - \left(\frac{\delta(L+M)}{2}\right) r\right] y^2 - \left\{\frac{\varepsilon\alpha_2}{2} - \left(\frac{\alpha_4(L+M)(\alpha_1\alpha_2 - \alpha_3)}{2(\alpha_1\alpha_4 - \alpha_5)}\right) r\right\} z^2 \\ &- \left[\frac{\varepsilon}{4} - \left(\frac{\alpha_1(L+M)}{2} + \rho\right) r\right] w^2 - \left[\frac{\varepsilon_0}{2} - \left(\frac{L+M}{2} + \lambda\right) r\right] u^2 \\ &- \left[\rho - \left(\frac{M}{2} + \frac{\alpha_1 M}{2} + \frac{\delta M}{2} + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)M}{2(\alpha_1\alpha_4 - \alpha_5)}\right)\right] \int_{t-r}^{t} w^2(s)ds \\ &- \left[\lambda - \left(\frac{L}{2} + \frac{\alpha_1 L}{2} + \frac{\delta L}{2} + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)L}{2(\alpha_1\alpha_4 - \alpha_5)}\right)\right] \int_{t-r}^{t} u^2(s)ds.\end{aligned} \tag{13}$$

If we now choose the constants ρ and λ as

$$\rho = \left(\frac{M}{2} + \frac{\alpha_1 M}{2} + \frac{\delta M}{2} + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)M}{2(\alpha_1\alpha_4 - \alpha_5)}\right)$$

and

$$\lambda = \left(\frac{L}{2} + \frac{\alpha_1 L}{2} + \frac{\delta L}{2} + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)L}{2(\alpha_1\alpha_4 - \alpha_5)} \right),$$

then the inequality in (13) implies that

$$\begin{aligned} \frac{dV}{dt} \leq &- \left[\frac{\varepsilon\alpha_4}{2} - \left(\frac{\delta(L+M)}{2} \right) r \right] y^2 - \left\{ \tfrac{\varepsilon\alpha_2}{2} - \left(\frac{\alpha_4(L+M)(\alpha_1\alpha_2 - \alpha_3)}{2(\alpha_1\alpha_4 - \alpha_5)} \right) r \right\} z^2 \\ &- \left[\frac{\varepsilon}{4} - \left(\frac{\alpha_1(L+M)}{2} + \rho \right) r \right] w^2 - \left[\frac{\varepsilon_0}{2} - \left(\frac{L+M}{2} + \lambda \right) r \right] u^2. \end{aligned} \tag{14}$$

Hence, one can easily get from (14) that

$$\frac{d}{dt} V(x_t, y_t, z_t, w_t, u_t) \leq -D_8 y^2 - D_9 z^2 - D_{10} w^2 - D_{11} u^2 \leq 0$$

for some positive constants D_i, $(i = 8,\ 9,\ 10,\ 11,)$ provided that

$$r < \min \left\{ \frac{\varepsilon\alpha_4}{\delta(L+M)}, \frac{\varepsilon\alpha_2(\alpha_1\alpha_4 - \alpha_5)}{\alpha_4(L+M)(\alpha_1\alpha_2 - \alpha_3)}, \frac{\varepsilon}{2\alpha_1(L+M) + 4\rho}, \frac{\varepsilon_0}{L+M+2\lambda} \right\}.$$

Finally, it follows that $\frac{d}{dt}V(x_t, y_t, z_t, w_t, u_t) \equiv 0$ if and only if $y_t = z_t = w_t = u_t$ $=0$, $\frac{d}{dt}V(\phi) < 0$ for $\phi \neq 0$ and $V(\phi) \geq u(|\phi(0)|) \geq 0$. Thus all the conditions of the above Proposition are satisfied. This shows that the null solution of Eq. (1) is globally asymptotically stable. $\square$

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