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Dirac operators and unitarizability of Harish-Chandra modules*

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Abstract. Let G be a simple noncompact Lie group. Let K be a maximal compact subgroup of G, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition of the complexified Lie algebra $\mathfrak{g}$ of G. We give a criterion for a $(\mathfrak{g}, K)$-module M to be unitary in terms of the action of the Dirac operator D on $M \otimes S$, where S is a spin module for the Clifford algebra $C(\mathfrak{p})$. More precisely, we show that an arbitrary Hermitian inner product on M will be invariant if and only if D is symmetric with respect to the corresponding inner product on $M \otimes S$.

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1. Introduction

The problem of classifying the unitary dual of a real reductive Lie group G is an important unsolved problem of representation theory. Its algebraic version asks to classify all irreducible $(\mathfrak{g}, K)$-modules M which admit an invariant Hermitian inner product.

There are two, in some sense opposite, possible approaches to the algebraic version of the problem. They both rely on decomposing the module M into its K-isotypic components:

$$M = \oplus_{\delta \in \hat{K}} M(\delta).$$

Here $M(\delta) = M_\delta \otimes V_\delta$, where V_δ is an irreducible K-module of type δ, and $M_\delta = \operatorname{Hom}_K(V_\delta, M)$ is the multiplicity space, whose dimension measures the number of occurrences of V_δ in M.

The first approach to searching for an invariant inner product on M is to take any inner product and integrate it over K to make it K-invariant. On each V_δ the inner product is determined up to a constant, and we can modify it freely on each M_δ, trying to make it invariant. This approach works for some examples like $G = SL(2, \mathbb{R})$, but in general the prospects are not good.

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The other approach, which has been more successful, is to first look for invariant, but possibly indefinite nondegenerate Hermitian forms on M. There is at most one such form up to scalar, and it is known exactly for what M such a form exists. It remains to check when the form is positive definite. One can assume it is positive definite on each V_δ, and try to calculate the signature on each M_δ. There are many general results and many classes of examples of M for which the problem has been settled, but the solution to the classification problem is still out of reach.

In this note we will try to revisit the first approach, but in a slightly modified setting in which M is first tensored by the spin module S for the Clifford algebra $C(\mathfrak{p})$. Here $\mathfrak{p}$ is the (-1)-eigenspace of the Cartan involution on the complexified Lie algebra $\mathfrak{g}$ of G (see the next section). There is a standard inner product on S and we can combine it with any inner product on M to get an inner product on $M \otimes S$. The advantage of $M \otimes S$ over M is the fact that it admits an action of the Dirac operator $D \in U(\mathfrak{g}) \otimes C(\mathfrak{p})$. We show that under appropriate assumptions, the $\mathfrak{g}$-action can be reconstructed from the action of $C(\mathfrak{p})$, which is well understood and independent of M, and the action of D. It follows that a given inner product on M is $\mathfrak{g}$-invariant (so M is unitary) if and only if D is symmetric with respect to the corresponding inner product on $M \otimes S$.

2. Some structural results

Let us first quickly review the setting. More details can be found for example in [3]. Other good sources for Clifford algebras and spinors are for example [1, 4] and [9].

Let G be a connected real reductive Lie group with a Cartan involution Θ such that the group $K = G^\Theta$ of fixed points of Θ is a maximal compact subgroup of G. We denote the Lie algebras of G and K by $\mathfrak{g}_0$ respectively $\mathfrak{k}_0$, and their complexifications by $\mathfrak{g}$ respectively $\mathfrak{k}$. The Cartan decompositions of $\mathfrak{g}_0$ and $\mathfrak{g}$ with respect to $\theta = d\Theta$ are

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0, \quad \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

These decompositions are orthogonal with respect to any non-degenerate invariant symmetric bilinear form B on $\mathfrak{g}_0$ and $\mathfrak{g}$. We fix such a form in the following.

Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and let $C(\mathfrak{p})$ be the Clifford algebra of $\mathfrak{p}$ with respect to B. Recall that $U(\mathfrak{g})$ is an associative algebra with a unit generated by $\mathfrak{g}$, with relations

$$XY - YX = [X, Y], \quad X, Y \in \mathfrak{g},$$

while $C(\mathfrak{p})$ is an associative algebra with a unit generated by $\mathfrak{p}$, with relations

$$XY + YX = -2B(X, Y), \quad X, Y \in \mathfrak{p}.$$

Let $b_1, \dots, b_p$ be any basis of $\mathfrak{p}$ and let $d_1, \dots, d_p$ be the dual basis with respect to B. In other words,

$$B(b_i, d_j) = \delta_{ij}, \quad i, j = 1, \dots, p.$$

The Dirac operator is

$$D = \sum_{i=1}^{p} b_i \otimes d_i \in U(\mathfrak{g}) \otimes C(\mathfrak{p}).$$

It is an easy exercise to see that D is independent of the basis b_i, and K-invariant for the adjoint K-action on $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ in both factors. This Dirac operator was first introduced in [6]; see also [8, 2, 5].

It is clear that $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ is generated by $\mathfrak{g} \otimes 1$ and $1 \otimes \mathfrak{p}$. The following lemma was implicit in [2]; it shows that $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ is in fact generated by D, $\mathfrak{k} \otimes 1$ and $1 \otimes \mathfrak{p}$.

Lemma 1. *For any $Z \in \mathfrak{p}$, $D(1 \otimes Z) + (1 \otimes Z)D = -2Z \otimes 1$.*

Proof. We can assume $Z = b_j$ is one of the basic elements of $\mathfrak{p}$. Then

$$\begin{aligned} D(1 \otimes Z) + (1 \otimes Z)D &= \sum_{i=1}^{p} b_i \otimes d_i b_j + \sum_{i=1}^{p} b_i \otimes b_j d_i \\ &= \sum_{i=1}^{p} b_i \otimes (d_i b_j + b_j d_i) = \sum_{i=1}^{p} b_i \otimes (-2\delta_{ij}) \\ &= -2b_j \otimes 1 = -2Z \otimes 1. \end{aligned}$$

□

To eliminate also $\mathfrak{k} \otimes 1$ from the list of generators of $U(\mathfrak{g}) \otimes C(\mathfrak{p})$, we use the following well known lemma.

Lemma 2. *Assume that G is simple noncompact. Then $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$.*

Proof. It is easy to check that $[\mathfrak{p}, \mathfrak{p}] \oplus \mathfrak{p}$ is an ideal of $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Since G is noncompact, this ideal is nonzero, and since $\mathfrak{g}$ is simple, it must be all of $\mathfrak{g}$. □

Note that the lemma is also true if G is semisimple with no compact factors. As noted earlier, we can now conclude the following.

Corollary 1. *If G is simple noncompact, then $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ is generated by D and $1 \otimes \mathfrak{p}$.*

3. Unitarizability of Harish-Chandra modules

Let M be a $(\mathfrak{g}, K)$-module, and let S be a spin module for $C(\mathfrak{p})$. Then $M \otimes S$ is a $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ module in the obvious way. $M \otimes S$ is also a module for the spin double cover $\tilde{K}$ of K, obtained by pulling back the double cover $\mathrm{Spin}(\mathfrak{p}_0) \to SO(\mathfrak{p}_0)$ by the action map $K \to SO(\mathfrak{p}_0)$. The Lie algebra of $\tilde{K}$ embeds into $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ diagonally, by

$$\Delta : \mathfrak{k}_0 \to U(\mathfrak{g}) \otimes C(\mathfrak{p}), \quad \Delta(X) = X \otimes 1 + 1 \otimes \alpha(X),$$

where

$$\alpha : \mathfrak{k}_0 \to \mathfrak{so}(\mathfrak{p}_0) \cong \textstyle\bigwedge^2 \mathfrak{p}_0 \to C(\mathfrak{p})$$

is the action map followed by the usual skew-symmetrization map given by $X \wedge Y \mapsto \frac{1}{2}(XY - YX)$.

Recall that M is called unitarizable (or unitary) if it admits a positive definite Hermitian form (inner product) $\langle\,,\rangle = \langle\,,\rangle_M$ such that all elements of $\mathfrak{g}_0$ are skew symmetric with respect to this form, i.e.,

$$\langle Xm_1, m_2\rangle = -\langle m_1, Xm_2\rangle, \quad X \in \mathfrak{g}_0, \quad m_1, m_2 \in M.$$

For $X \in \mathfrak{g}$, this means that the adjoint of X with respect to $\langle\,,\rangle$ is $-\bar{X}$, where the bar denotes the complex conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$.[§]

There is a well known Hermitian inner product $\langle\,,\rangle_S$ on S so that the elements of $\mathfrak{p}_0 \subset C(\mathfrak{p})$ are skew symmetric with respect to $\langle\,,\rangle_S$. See [9], or [3], 2.3.9. We equip $M \otimes S$ with the tensor product of these two forms, denoted by $\langle\,,\rangle_{M\otimes S}$. It is now clear that D is symmetric with respect to $\langle\,,\rangle_{M\otimes S}$. In particular, $D^2 \geq 0$. This is the famous Parthasarathy's Dirac inequality, which is a very useful necessary condition for unitarity of M. It can be written out more explicitly using the following important formula for D^2, also due to Parthasarathy [6]:

$$D^2 = -(\mathrm{Cas}_{\mathfrak{g}} \otimes 1 + \|\rho_{\mathfrak{g}}\|^2) + (\Delta(\mathrm{Cas}_{\mathfrak{k}}) + \|\rho_{\mathfrak{k}}\|^2). \tag{1}$$

Here $\mathrm{Cas}_{\mathfrak{g}}$ and $\mathrm{Cas}_{\mathfrak{k}}$ denote the Casimir elements of $U(\mathfrak{g})$ and $U(\mathfrak{k})$, respectively. See [9] or [3].

Theorem 1. *Assume that G is simple and noncompact. Let $\langle\,,\rangle_M$ be any positive definite Hermitian form on a $(\mathfrak{g}, K)$-module M. Tensor this form with $\langle\,,\rangle_S$ described above to get a positive definite form $\langle\,,\rangle_{M\otimes S}$ on $M \otimes S$. Then M is unitary with respect to $\langle\,,\rangle_M$ if and only if D is symmetric with respect to $\langle\,,\rangle_{M\otimes S}$.*

Proof. We already saw that if M is unitary, then D is symmetric. Conversely, assume that D is symmetric. Since any element of $1 \otimes \mathfrak{p}_0 \subset U(\mathfrak{g}) \otimes C(\mathfrak{p})$ is skew symmetric on $M \otimes S$, Lemma 1 implies that all elements of $\mathfrak{p}_0 \otimes 1$ are also skew symmetric on $M \otimes S$. Namely, if we denote by a star the adjoint of an operator with respect to $\langle\,,\rangle_{M\otimes S}$, then

$$(D(1\otimes Z) + (1\otimes Z)D)^* = (1\otimes Z)^* D^* + D^*(1\otimes Z)^* = -(D(1\otimes Z) + (1\otimes Z)D).$$

Now Lemma 2 and a similar calculation as above imply that elements of $\mathfrak{k}_0$ are skew symmetric on $M \otimes S$ as well. So all elements of $\mathfrak{g}_0$ are skew symmetric on M, i.e., M is unitary. □

As for Lemma 2, we can also get the same conclusion if G is semisimple with no compact factors.

In practice, we often know the K-decomposition of M:

$$M = \bigoplus_{\delta} M_\delta \otimes V_\delta,$$

where V_δ, $\delta \in \hat{K}$ are the K-types, and M_δ are the multiplicity spaces. Suppose that we are given some K-invariant inner product on M. We can assume that this inner

[§]We remark here that connectedness of G and K plays no role in this definition. The same is true for most of the statements in this paper. It is however customary to assume connectedness when studying the Dirac operator actions because some of the structural results depend on it.

product is fixed and K-invariant on each V_δ, and we can modify it freely on each M_δ. M will be unitary if and only if we can find a modification such that elements of $\mathfrak{p}_0$ are skew symmetric.

In view of Theorem 1, we are trying to modify the inner product on M so that D becomes symmetric.

By (1), D^2 is a scalar on each $\tilde{K}$-component of $M \otimes S$. By Parthasarathy's Dirac inequality, if there is any chance to modify the form and make M unitary, these scalars must be ≥ 0, so they are of the form λ^2, $\lambda \in \mathbb{R}$.

On those $\tilde{K}$-components where $D^2 = \lambda^2 \neq 0$, D has two real eigenvalues, λ and $-\lambda$. So D is symmetric on $\operatorname{Ker}(D^2 - \lambda^2)$ if and only if the eigenspaces corresponding to $\pm\lambda$ are orthogonal. On the kernel of D^2, D is symmetric if and only if it is 0.

Let us now assume that $\mathfrak{p}$ is even-dimensional. For example, this is true if G and K have equal rank. In that case, S is $\mathbb{Z}_2$-graded, i.e., $S = S^+ \oplus S^-$, where $S^\pm$ are invariant under $\operatorname{Spin}(\mathfrak{p}_0)$ and hence also under $\tilde{K}$. If S is constructed as $\bigwedge U$ for a maximal isotropic subspace U of $\mathfrak{p}$, then S^+ consists of elements of $\bigwedge U$ of even degree, while S^- consists of elements of $\bigwedge U$ of odd degree.

It is clear that D switches $M \otimes S^+$ and $M \otimes S^-$. We call elements of $M \otimes S^+$ even and elements of $M \otimes S^-$ odd. Let us take some even $v \in M \otimes S$, such that $D^2 v = \lambda^2 v (\neq 0)$. Then the space spanned by v and Dv is two-dimensional (since Dv is odd), invariant for D, and it contains a D-eigenvector for λ and a D-eigenvector for $-\lambda$. These are just $\lambda v \pm Dv$.

This implies that D-eigenspaces for $\pm\lambda$ are of equal dimension, and they are described as span of $\lambda v + Dv$ (respectively $\lambda v - Dv$) for even $v \in \operatorname{Ker}(D^2 - \lambda^2)$. Hence, the eigenspaces will be orthogonal if and only if

$$\langle \lambda v + Dv, \lambda w - Dw \rangle = 0,$$

for all even $v, w \in \operatorname{Ker}(D^2 - \lambda^2)$. Since even and odd elements are orthogonal, this is equivalent to

$$\langle Dv, Dw \rangle = \lambda^2 \langle v, w \rangle, \quad \text{for all even } v, w \in \operatorname{Ker}(D^2 - \lambda^2). \tag{2}$$

This is also the right condition for $\lambda = 0$, since it is equivalent to D being 0 on $\operatorname{Ker} D^2$. So we proved

Corollary 2. *Let G be a connected reductive group so that the maximal compact subgroup K has rank equal to the rank of G. Let M be an irreducible $(\mathfrak{g}, K)$-module with a K-invariant Hermitian inner product.*

Then M is unitary if and only if the following conditions hold:

1. *All eigenvalues of D^2 on $M \otimes S$ are nonnegative,*
2. *Condition (2) holds for each eigenvalue λ^2 of D^2.*

4. Example $G = SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$

As an example, let us consider the case of the spherical principal series of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ with real infinitesimal character. The reason for skipping the smallest example, $G = SL(2,\mathbb{R})$, is the fact that the situation there is a little too simple: the Dirac inequality is not only necessary but also sufficient for unitarity.

We need some notation from the $SL(2,\mathbb{R})$-case; see [7], or [3], 1.3.10. We will use the following basis of $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$:

$$H = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad E = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad F = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}.$$

Then H spans $\mathfrak{k} = \mathfrak{so}(2,\mathbb{C})$ while E and F span $\mathfrak{p}$, and the commutation relations are the usual ones

$$[H,E] = 2E, \quad [H,F] = -2F, \quad [E,F] = H.$$

We normalize the form B so that $B(E,F) = 1$.

Let $\mu \in \mathbb{R}$ be the parameter of a spherical principal series representation of $SL(2,\mathbb{R})$. Because of equivalences, we can assume $\mu \geq 0$. The corresponding $(\mathfrak{sl}(2,\mathbb{C}), SO(2))$-module $M = M_\mu$ is spanned by the $SO(2)$-isotypic vectors v_n, $n \in 2\mathbb{Z}$, where v_n is of weight n for H. The action of E and F is given by

$$Ev_n = \frac{1}{2}(\mu + (n+1))v_{n+2}, \quad Fv_n = \frac{1}{2}(\mu - (n-1))v_{n-2}.$$

Since D for $SL(2,\mathbb{R})$ can be written as $E \otimes F + F \otimes E$, and since the action of $C(\mathfrak{p})$ on the spin module $S(\mathfrak{p}) = \mathbb{C}\,1 \oplus \mathbb{C}\,E$ is given by

$$E \cdot 1 = E,\ E \cdot E = 0; \quad F \cdot 1 = 0,\ F \cdot E = -2,$$

it follows that

$$\begin{aligned} D(v_n \otimes 1) &= Fv_n \otimes E = \frac{1}{2}(\mu - (n-1))v_{n-2} \otimes E \\ D(v_n \otimes E) &= -2Ev_n \otimes 1 = -(\mu + (n+1))v_{n+2} \otimes 1. \end{aligned} \tag{3}$$

Let us get back to $G = SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, $K = SO(2) \times SO(2)$ and $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$. Consider the $(\mathfrak{g},K)$-module $M = M_{\mu_1} \otimes M_{\mu_2}$. If E_1, E_2 denote the "E" of each factor of $\mathfrak{g}$, then S is the span of 1, E_1, E_2 and $E_1 \wedge E_2$. Moreover, these four elements are an orthogonal basis for S with respect to $\langle\,,\rangle_S$, which we can take to be orthonormal after rescaling. The weight vectors for $\mathfrak{k}$ in M are $v_{n,m} = v_n \otimes v_m$, where n and m are even integers. Any K-invariant inner product on M is given by setting

$$\|v_{n,m}\|^2 = c_{n,m}$$

for some positive scalars $c_{n,m}$, and requiring different $v_{n,m}$ to be orthogonal to each other.

We write condition (2) for orthogonal even vectors $v_{0,0} \otimes 1$ and $v_{-2,-2} \otimes E_1 \wedge E_2$. Since $D = F_1 \otimes E_1 + E_1 \otimes F_1 + F_2 \otimes E_2 + E_2 \otimes F_2$, we see using (3) that

$$\begin{aligned} D(v_{0,0} \otimes 1) &= \frac{1}{2}(\mu_1 + 1)v_{-2,0} \otimes E_1 + \frac{1}{2}(\mu_2 + 1)v_{0,-2} \otimes E_2, \\ D(v_{-2,-2} \otimes E_1 \wedge E_2) &= -(\mu_1 - 1)v_{0,-2} \otimes E_2 + (\mu_2 - 1)v_{-2,0} \otimes E_1. \end{aligned}$$

These vectors must be orthogonal:

$$-\frac{1}{2}(\mu_2+1)(\mu_1-1)c_{0,-2}+\frac{1}{2}(\mu_1+1)(\mu_2-1)c_{-2,0}=0,$$

so

$$(\mu_1+1)(\mu_2-1)c_{-2,0}=(\mu_2+1)(\mu_1-1)c_{0,-2}.$$

Since $c_{m,n}$ are positive and since μ_1, μ_2 can also be taken positive, it follows that μ_1-1 and μ_2-1 are of the same sign. They cannot both be positive since by Parthasarathy's Dirac inequality $\mu_1^2+\mu_2^2\leq 2$. So $\mu_1-1<0$ and $\mu_2-1<0$, i.e., the point (μ_1,μ_2) lies in the unit square.

If this is the case, then we can get e.g. $c_{0,-2}$ from $c_{-2,0}$. Similar considerations show that all $c_{m,n}$ can be reconstructed from one of them, say $c_{0,0}$, provided that the point (μ_1,μ_2) lies in the unit square. So this condition is also sufficient for unitarity.

In this example, there is another way to get the same result. Namely, $M=M_{\mu_1}\otimes M_{\mu_2}$ is unitary if and only if M_{μ_1} and M_{μ_2} are both unitary $(\mathfrak{sl}(2,\mathbb{C});SO(2))$-modules, and that is the case if and only if μ_1 and μ_2 are both less than 1. These are just the complementary series for $SL(2,\mathbb{R})$.

References

[1] C. Chevalley, *The algebraic theory of spinors*, Columbia University Press, New York, 1954.

[2] J.-S. Huang, P. Pandžić, *Dirac cohomology, unitary representations and a proof of a conjecture of Vogan*, J. Amer. Math. Soc. **15**(2002), 185–202.

[3] J.-S. Huang, P. Pandžić, *Dirac Operators in Representation Theory*, Birkhäuser, Boston, 2006.

[4] B. Kostant, *Clifford algebra analogue of the Hopf-Koszul-Samelson theorem, the ρ-decomposition $C(\mathfrak{g})=\operatorname{End}V_\rho\otimes C(P)$, and the $\mathfrak{g}$-module structure of $\bigwedge\mathfrak{g}$*, Adv. Math. **125**(1997), 275–350.

[5] B. Kostant, *A cubic Dirac operator and the emergence of Euler number multiplets of representations for equal rank subgroups*, Duke Math. Jour. **100**(1999), 447–501.

[6] R. Parthasarathy, *Dirac operator and the discrete series*, Ann. of Math. **96**(1972), 1–30.

[7] D. A. Vogan, *Representations of real reductive Lie groups*, Birkhäuser, Boston, 1981.

[8] D. A. Vogan, *Dirac operators and unitary representations*, Three talks at MIT Lie groups seminar, Fall, 1997.

[9] N. R. Wallach, *Real Reductive Groups–Vol. I*, Academic Press, New York, 1988.