UDC 532.5:629.51/.58

Dario BAN
Branko BLAGOJEVIĆ
Jani BARLE

Ship Geometry Description Using Global 2D RBF Interpolation

Authors' Address (Adresa autora):
University of Split, Faculty of Electrical Engineering, Mechanical Engineering and Naval Architecture (FESB),
Ruđera Boškovića b.b., 21000 Split, Croatia
E-mail: dario.ban@fesb.hr; bblag@fesb.hr; jani.barle@fesb.hr,

Received (Primljeno): 2010-01-20
Accepted (Prihvaćeno): 2010-02-085
Open for discussion (Otvoreno za raspravu): 2011-09-30

Original scientific paper

Radial basis function (RBF) networks are the new, recently developed, meshless explicit, piecewise geometry description methods. Among many useful properties the RBFs have, they belong to Reproducing Kernel Hilbert Spaces and have the best approximating property, and they therefore might be suitable for describing complex ship geometry. Moreover, they are the solution of scattered data interpolation problem and can achieve high accuracy. Various types of radial basis functions and their parameters will be investigated and their applicability in 2D ship geometry description studied in this paper.

Key words: *global interpolation, two-dimensional, high precision, piecewise, RBF, ship frames description*

Opisivanje brodske geometrije globalnom 2D RBF interpolacijom

Izvorni znanstveni rad

Mreže radijalnih osnovnih funkcija (RBF) su nove, nedavno razvijene bezmrežne eksplicitne metode opisivanja geometrije po dijelovima. Između brojnih korisnih svojstava koje imaju, RBF pripadaju Reprodukcijskim jezgrama Hibertovog prostora i imaju svojstvo najbolje aproksimacije, pa bi stoga mogle biti prikladne za opisivanje složene brodske geometrije. Štoviše, one su rješenje problema interpolacije raštrkanih podataka te mogu postići visoku točnost. U ovom radu će se istražiti razni tipovi radijalnih osnovnih funkcija te njihova svojstva, i ispitati njihova primjenjivost na dvodimenzionalno opisivanje brodske geometrije.

Ključne riječi: *globalna interpolacija, dvodimenzionalno, visoka preciznost, po dijelovima, RBF, opisivanje brodskih rebara*

1 Introduction

In the last three decades several authors like Micchelli [1], Schaback [2], Wendland [3], and Wu [4] continued the work of Bochner [5] and Schoenberg [6], [7] on positive definite functions and work from the middle of the 20th century, and developed new calculating meshless techniques applicable in geometry description. Additionally, in 1950 [8], Nachman and Aronszajn defined the concept of Reproducing Kernel Hilbert Spaces, setting thus foundations for linear radial basis function (RBF) networks.

Among meshless methods, the radial basis functions (RBF) are recognized as the solution of scattered data interpolation problem and are therefore applicable for high precision mathematical representations of 2D and 3D objects. They are a direct, explicit, interpolating representation method, in nature opposite to approximating parametric methods based on Bezier, Basis (B-spline) or Non-Uniform Rational Basis splines (NURB spline), mostly used in shipbuilding industry computer programs today.

In general, explicit representation methods have problems with accurate description of non-bijective parts and form breaks, together with singularity of inversion matrix. Usually, complex geometries like ship's hull form cannot be described properly using explicit representations without decomposition to bijective parts called manifolds, or transformation methods.

The applicability of different types of RBFs to ship geometry representation using global RBF interpolation procedures is the subject of this paper. The characteristics of different radial basis functions and their parameters will be observed, checking their accuracy and properties for 2D representation of ship's test sections without and with camber. The quality of RBF representation will be thus tested for the description of form discontinuity as one of the major advantages of NURB representation.

2 Reproducing Kernel Hilbert Spaces

The theoretical background for RBF network to be defined as linear combination of certain basis functions came from the theory on Reproducing Kernel Hilbert Spaces (RKHS) introduced by Nachman and Aronszajn [8]. RKHSs are positive definite kernels that ensure pointwise convergence and ortonormal bases defined with the statement:

$$\hat{f}(x)=\sum_{i=1}^{O} w_i K\left(x,x_i\right)=\sum_{i=1}^{O} w_i B_i=\sum_{i=1}^{O} w_i \Phi_i\left(x\right) \qquad (1)$$

where: $x_j, j = 1,...,N$; $x \in IR^s$ is input data set, K are reproducing kernels, B_1 are basis functions, Φ_i are radial basis functions, t_i are the development centres of RBF with $i = 1,...,O$, where O is the number of centres, w_i are RBF network weight coefficients, φ is radial basis function based on Euclidian norm between input data and centres, and $\hat{f}(x)$ is the generalized interpolation/approximation function.

3 RBF networks definition

3.1 Neural networks analogy

The RBFs are multivariate functions that can be used for calculation of required number of output variables, at once. They can be, by analogy with neural networks, defined as direct, feed-forward single-layered neural networks with possibly infinite input and output data sets, and their dimension, as shown in Figure 1. Their input and output variables are connected with weighted sum of radial basis functions translated around the points called centres, whose number depends on mathematical procedure chosen for object representation.

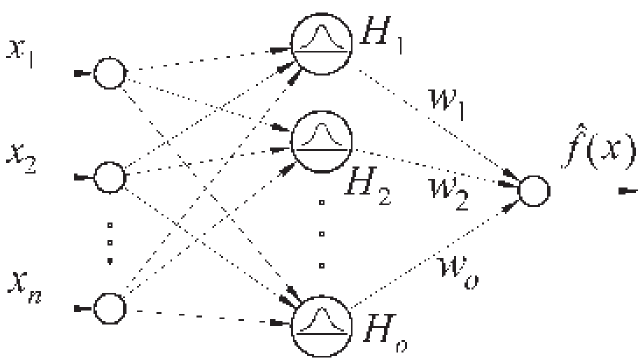


Figure 1 **Single-layer feed-forward RBF neural network**
Slika 1 **Jednoslojna, unaprijedna RBF neuronska mreža**

3.2 Interpolation matrix invertibility

For basis functions B_j to be invertible their interpolation matrix must be in Haar space, i.e. satisfy the condition:

$$\det\left(B_j\left(x_i\right)\right) \neq 0 \tag{2}$$

When the number of elements of input data set equals the number of elements in centres set, the interpolation network with an interpolating matrix is obtained. Otherwise, the approximation network is obtained with a corresponding approximation matrix.

The interpolation procedures with high generalization accuracy are the target of this paper, and therefore the number of centres O will always equal the number of elements in input data set N and centres set equals input data set.

3.3 Calculation procedure

The solution of scattered data interpolation problem based on RKHS is unique RBF network weight coefficient parameters w_i, equalling the number of development centres.

The neural network's weight coefficient values can be obtained by direct inversion of the neural network interpolation (activation) matrix **H** multiplied by target vector **y**, i.e. with:

$$\mathbf{w} = \mathbf{H}^{-1} \cdot \mathbf{y} \tag{3}$$

where: y - target vector (output data set), H – neural network activation (interpolation) matrix, $N \times N$, with elements r_{ji}:

$$\mathbf{H} = \begin{bmatrix} \varphi(r_{11}) & \dots & \varphi(r_{1j}) & \cdots & \varphi(r_{1N}) \\ \vdots & \ddots & \vdots & \iddots & \vdots \\ \varphi(r_{i1}) & \dots & \varphi(r_{ij}) & \dots & \varphi(r_{iN}) \\ \vdots & \iddots & \vdots & \ddots & \vdots \\ \varphi(r_{N1}) & \dots & \varphi(r_{Nj}) & \dots & \varphi(r_{NN}) \end{bmatrix} \tag{4}$$

where: r_{ji} is the norm, $\left\|x_j - x_i\right\|$, $j, i = 1, \dots, N$.

The main disadvantage of RBF networks is the problem with possible singularity of above interpolation matrix, (4). That matrix must be well-posed and the main criterion for it is that interpolation matrix is positive definite.

3.4 Generalization and accuracy

The RBF network weights w_i calculation procedure consists of two parts: *evaluation* and *generalization*. In the evaluation phase shown above, the network weight coefficients have to be calculated and corresponding accuracy on the input data set checked. After the weights are calculated, the RBF network goodness test is performed checking generalization accuracy over testing data set.

3.4.1 Evaluation accuracy

The usual accuracy measure used is RMSE (Root Mean Squared Error):

$$RMSE = \sqrt{\frac{\sum_{i=1}^{N}(f(x) - y_i)^2}{N}} \tag{5}$$

with N - input data set number, y_i, 1,...,N - output data set, $f(x)$ - radial basis function, and it will be used here, too.

3.4.2 Generalization accuracy

Usually, the evaluation accuracy does not ensure achieving overall required accuracy for geometric object to be described, so generalization accuracy has to be performed using additional data set.

$$Err_{\max} = \max_{x \in T}\left(f\left(x_T\right) - y_T\right) \tag{6}$$

where T is test data set $\left\{x_T, y_T\right\}$.

In the case of ship hull forms, the generalization accuracy checking will be performed calculating local error on the places of interest, like the bilge or transition from the bilge to the flat of the side, with acceptable value set to 0.1 mm.

4 Radial basis functions and their characteristics

The radial basis functions are defined as the functions based on norms (usually Euclid's L_2 norm) between input set data points and centres of development points.

$$\Phi(x) = \varphi(\|x\|) \quad (7)$$

They are usually defined with only one parameter called shape parameter, c, and their definition for the interpolation case is:

$$\varphi = \varphi(x,t;c) = \varphi(x,x_i;c) = \varphi\left(\|x - x_i\|_2, c\right),\ x \in IR^s \quad (8)$$

RBFs have some good intrinsic properties, being invariant on:

– translation,
– rotation, and
– reflexion.

These properties will be used in the selection of coordinate system representation.

The main criterion for the basis function to be acceptable as RBF is that it ensures interpolation matrix invertibility. That criterion can be fulfilled if the function chosen is positive definite and radial.

4.1 Positive definite functions

Positive definite functions were first studied by Mathias, 1923, [9] and Stewart was in 1976 [10] the first who waded corresponding theory to strictly positive definite functions. Finally, Micchelli [1] first connected scattered data interpolation with positive definite function.

4.1.1 Positive definite functions

The complex defined function $\Phi: IR^s \to IC$ is called *positive definite* on IR^s if for any N different points $x_1,\ldots,x_N \in IR^s$ and $\mathbf{c} = [w_1,\ldots,w_N]^T \in IC^N$ holds:

$$\sum_{j=1}^{N}\sum_{k=1}^{N} w_j \overline{w_k}\Phi(x_j - x_k) \geq 0 \quad (9)$$

The basis properties of positive definite functions are:

1) Non-negative linear combination of positive definite functions is positive definite. If $\Phi_1,\ldots,\Phi_N$ are positive definite on IR^s and $w_j \geq 0,\ j = 1,\ldots,N$ then:

$$\Phi(x) = \sum_{j=1}^{n} w_j \Phi_j(x) \geq 0,\ x \in IR^s$$

is positive definite, also. Moreover, if any Φ_j is strictly positive definite and corresponding weight coefficient $c_j > 0$ then Φ is positive definite.

2) $\Phi(\mathbf{0}) \geq 0$,
3) $\Phi(-x) = \overline{\Phi(x)}$,
4) Any positive definite function is bounded, i.e. $|\Phi(x)| \leq \Phi(\mathbf{0})$,
5) If Φ is positive definite with $\Phi(\mathbf{0}) = 0$ then $\Phi \equiv 0$,
6) The product of (strictly) positive definite functions is (strictly) positive definite function.

The properties 1) and 2) follow directly from the definition of positive definite functions (9). The property 5) follows from 4), and 6) is the result of the Schur theorem from linear algebra theory, described by Wendland [11].

The functions are positive definite if they satisfy one of the following three criteria:

– strictly positive definite,
– completely monotone, and
– multiply monotone functions.

When Fourier transform is not available, there are two alternative criteria for decision whether a function is strictly positive and radial on IR^s:

– complete monotone for:

$(-1)^l \varphi^{(l)}(r) \geq 0,\ r > 0,\ l = 0,\ 1,\ 2,\ \ldots$ (but not constant), see Schoenberg [7], (converse also holds, see Wendland [11]) for the case of all s, and

– multiply monotone:

$\varphi'' \geq 0$ (non-negative, non-increasing, and convex, Williamson [12]), for some fixed s.

4.2 Strictly positive definite radial basis functions

When choosing basis functions B_i that generate strictly positive definite RBF interpolation matrix, Micchelli [1], a well-posed interpolation problems are always produced. According to Wendland's theorem [11], the radial basis function $\varphi(\|x\|)$ is strictly positive definite and radial on IR^S if and only if $\varphi(\|x\|)$ its s-dimensional Fourier transform is non-negative and not identically equal to zero.

The examples of strictly positive definite radial functions are stated in Fasshauer [13]:

– Gaussian functions – radial functions,
– Laguerre-Gaussians – infinitely differentiable, oscillatory functions (not strictly positive definite and radial on IR^S for all s), Andrews at al. [14],
– Matérn functions – depending on the modified Bessel function of the second kind (sometimes called modified Bessel

Table 1 **Strictly positive definite radial functions based on Gaussian function**
Tablica 1 **Striktno pozitivno definirane funkcije temeljene na Gaussovoj funkciji**

Basis Functions, Φ(x)	Equations
Gaussian	$e^{-c\|x\|^2}, c > 0$
Laguerre-Gaussian	$e^{-\|x\|^2} L_n^{s/2}\left(\|x\|^2\right)$, $L_n^{s/2}(t) = \sum_{i=0}^{N} \frac{(-1)^k}{k!}\binom{n+s/2}{n-k} t^k$
Matern	$\frac{K_{\beta-s/2}(\|x\|)\|x\|^{\beta-s/2}}{2^{\beta-1}\Gamma(\beta)}, s \geq 2$
Poisson	$\frac{J_{s/2-1}(\|x\|)}{\|x\|^{s/2-1}}, s \geq 2$

function of the third kind), or MacDonald's function, or Sobolev splines of order v, Schaback [2],

– Poisson functions - oscillatory function that is radial and strictly positive definite on IR^S (and all $IR^\sigma \le s$), not defined in origin, but can be extended to be infinitely differentiable in all of IR^S, Fornberg at al. [15].

Above functions are Gaussian generalizations and can be set in one category, i.e. only basis Gaussian will be investigated in this paper. Table 1 shows the functions that are generalizations of Gaussian function.

Another group of strictly positive definite and radial basis functions that are not Gaussian based functions are as stated in Fasshauer [13]:

– Inverse Multiquadrics – infinitely differentiable, see Hardy [16],

– Generalized Multiquadrics, see Fornberg and Wright [17],

– Potentials and Whittaker's radial functions, Abramowitz and Stegun [18].

– Truncated power functions – the functions with compact support.

Table 2 shows strictly positive definite radial functions that are not Gaussian generalizations.

Table 2 **Strictly positive definite radial functions not based on Gaussian function**
Tablica 2 **Striktno pozitivno definirane funkcije koje nisu temeljene na Gausovoj funkciji**

Basis Functions, $\varphi(r)$	Equations
Inverse Multiquadrics	$\left(r^2+c^2\right)^\beta, \beta \le 0, \beta \notin 2IN, x \in IR^s$
Generalized Multiquadrics	$\left(1+r^2/c^2\right)^\beta, \beta > 0, \beta \notin 2IN, x \in IR^s$
Potentials and Whitteker	$\int_0^\infty (1-rt)_+^{k-1} f(t)dt, k \ge \lfloor s/2 \rfloor + 2$
Truncated Powers	$(1-r)_+^l$

where $r = \|x - x_i\|_2$.

4.3 Conditionally positive definite radial functions

Another class of radial basis functions are conditionally positive definite functions of order m, see Micchelli [1], and Guo at al. [19]. These are the functions that provide the natural generalization of RBF interpolation with polynomial precision, important for high accuracy required for hull geometry description.

According to the theory of conditionally positive definite radial functions the RBF network definition can be changed to:

$$\hat{f}(x) = \sum_{i=1}^{N} w_i \varphi\left(\|x, t_i\|\right) + \sum_{l=1}^{M} \omega_l p_l(x), \quad x \in IR^s \qquad (10)$$

where: $p_1 \cdots p_M$ form the basis for the $M = \binom{s+m-1}{m-1}$ – dimensional linear space Π_{m-1}^s of polynomials of total degree less than or equal to $m - 1$ in s variables.

There are conditionally and strictly conditionally positive definite functions. The condition for function Φ to be conditionally positive definite is that it possesses a generalized Fourier transform of order m, continuous on $IR^s / \{0\}$, i.e. if $\hat{\Phi}$ is non-negative and non-vanishing.

To ensure unique solution M, additional conditions need to be added:

$$\sum_{i=1}^{N} w_i p_l(\mathbf{x}_i) = 0, \quad l = 1, \ldots, M \qquad (11)$$

with polynomial degree at most $m - 1$.

The function $\hat{\Phi}$ is strictly conditionally positive definite function of order m on IR^S if its quadratic form is zero only for $\mathbf{w} \equiv 0$.

If their conditional positive definiteness can be connected to complete monotone and multiple monotone functions and not generalized Fourier transform, we are obtaining the criteria for function Φ to be strictly conditionally positive definite radial function.

The examples of conditionally positive definite functions are shown in Fasshauer [13]:

– Generalized multiquadrics, Hardy [16],

– Thin plate splines (without shape parameter c, 2D polyharmonic splines), see Duchon [20],

– Radial powers (without shape parameter c, 3D polyharmonic splines), with no even powers.

Table 3 shows the list of some conditionally positive definite radial functions.

Table 3 **Conditionally positive definite radial functions**
Tablica 3 **Uvjetno pozitivno definirane radijalne funkcije**

Functions $\varphi(r)$	Equations
Generalized Multiquadrics	$\left(1+r^2/c^2\right)^\beta, \beta > 0, \beta \notin 2IN, x \in IR^s$
Thin-plate spline	$(-1)^{\beta+1} r^{2\beta} \log r, \beta \in IN, x \in IR^s$
Radial Powers	$(-1)^\beta r^\beta, \beta > 0, \beta \notin 2IN, x \in IR^s$

5 Polynomial precision

In general, solving interpolation problem with extended expansion with polynomial term leads to solving a system of linear equations of the form:

$$\begin{bmatrix} \mathbf{H} & \mathbf{P} \\ \mathbf{P}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \omega \end{bmatrix} = \begin{bmatrix} \mathbf{y} \\ \mathbf{0} \end{bmatrix} \qquad (12)$$

where $H_{ji} = B_j(x_i)$, $j, i = 1, \ldots, N$, $P_{jl} = p_l(x_i)$, $l = 1, \ldots, M$, $\mathbf{w} = [w_1, \ldots, w_N]^T$, $\boldsymbol{\omega} = [\omega_1, \ldots, \omega_M]^T$, $\mathbf{y} = [y_1, \ldots, y_N]^T$ and $\mathbf{0}$ is a zero vector of length M.

The above system of linear equations can be solved using criterion (11) only, and therefore unique solution can be obtained by imposing this criterion.

There is the minimal polynomial degree to be used depending on particular RBF selected [21] as shown in Table 4.

Table 4 **The minimal degree of the polynomial depending on the RBF chosen**
Tablica 4 **Minimalni stupanj polinoma ovisno o odabranoj RBF**

Basis function $\varphi(r)$	Default degree	Minimal required degree	
		2D	3D
Biharmonic	1	1	0
Triharmonic	2	2	1
Multiquadric	2β	$2\beta<0$: No polynomial $2\beta \geq 0 : d=\frac{2\beta-1}{2}$	

The goal of this paper is to select RBFs with high accuracy so polynomial precision is required. The RBFs having this property are acceptable only, and the basis functions will be selected among them.

6 Compactly supported radial basis functions

Compactly supported RBFs are strictly conditionally positive functions and radial of order $m > 0$, but not for all s on IR^S, see Micchelli [1]. The acceptable range of compactly supported RBFs is defined for some restricted s range, with condition $\lfloor s/2 \rfloor \leq k+m-2$, i.e. maximal s value. This restriction ensures that Φ is integrable and therefore possesses classical Fourier transform Φ that is continuous. For integrable functions, the generalized Fourier transform coincides with the classical Fourier transform.

The compactly supported functions $\varphi_{s,k}$ are all supported on [0,1] and have polynomial representation there, with minimal degree for given space dimension s and smoothness $2k$.

These functions $\varphi_{s,k}$ are strictly positive definite and radial on IR^s and are of the form:

$$\varphi_{s,k}=\begin{cases} p_{s,k}(r), & r\in[0,1], \\ 0, & r>1 \end{cases} \tag{13}$$

with a univariate polynomial $\varphi_{s,k}$ of degree $\lfloor s/2 \rfloor + 3k+1$.

Table 5 **Compactly supported functions**
Tablica 5 **Funkcije s kompaktnom podrškom**

Functions $\varphi(r)$	Equations
Wendland	$\varphi_{s,0}=(1-r)_+^l$, with $l=\lfloor s/2 \rfloor+k+1$ $\varphi_{s,1}=(1-r)_+^{l+1}\left[(l+1)r+1\right]$ $\varphi_{s,2}=(1-r)_+^{l+2}\left[(l^2+4l+3)r^2+(3l+6)r+3\right]$
Wu	$\varphi_{s,2}=(1+r)_+^5\left(8+40r+48r^2+25r^3+5r^4\right)$ $\varphi_{s,3}=(1+r)_+^4\left(16+29r+20r^2+5r^3\right)$
Buhmann	$\Phi=12r^4\log r-21r^4+32r^3-12r^2+1$

In order to obtain compact support the distance between points needs to be divided by some value, d, that can be larger than points distance.

For example, the Wendland's CSRBFs [3], are obtained from truncated power function $\varphi_l=(1-r)_+^l$ by dimension walk and repeatedly applied operator I, and we obtain: $\varphi_{s,k}=I^k\varphi_{\lfloor s/2 \rfloor+k+1}$.

Table 5 shows the list of some compactly supported functions, see Wendland [3], Wu [4], Buhmann [22].

The greatest advantage of CSRBFs is sparsing interpolation matrix H to some quasi-diagonal form, i.e. compact support ensures that many elements of matrix H become zero. In that way, the inversion of interpolation matrix becomes easier, reducing quadratic matrix to some more computable one.

7 Global interpolation of 2D ship geometry with radial basis functions

The conditionally positive definite and radial functions have polynomial precision when applied to scattered interpolation problem that is required for high accurate hull form geometry description.

When used in global interpolation those functions have global support. Globally supported functions chosen are:

– MQs,
– Inverse MQs,
– Generalized MQs,
– Thin-plate splines.

Additionally, compactly supported functions have polynomial representation on [0,1] and are suitable also for achieving high accuracy, and the functions to be used are:

– Wendland CSRBFs.

Therefore, the functions from Tables 3 and 4 are chosen for ship geometry modelling in this paper, with compact supported functions limited to Wendland's functions for $s = 3$. Table 6 shows selected functions to be tested for accuracy in ship geometry description using RBF interpolation procedures.

Table 6 **Testing radial basis functions**
Tablica 6 **Radijalne osnovne funkcije koje će se testirati**

Functions $\varphi(r)$	Equations
MQ & Inverse MQ	$\left(r^2+c^2\right)^\beta, \beta\notin 2IN, x\in IR^s$
Generalized MQ	$\left(1+r^2/c^2\right)^\beta, \beta>0, \beta\notin 2IN, x\in IR^s$
Thin-plate splines	$(-1)^{k+1}r^{2k}\log r, k\in IN, x\in IR^s$
Wendland's Compactly Supported RBFs	$\varphi_{3,0}=(1-r)_+^2$ $\varphi_{3,1}=(1-r)_+^4\left[4r+1\right]$ $\varphi_{3,2}=(1-r)_+^6\left[35r^2+18r+3\right]$

8 RBF parameters selection

The radial basis functions chosen have a few parameters depending on their type. Wendland's CSRBFs have compact support diameter d as parameter together with space dimension s and smoothness $2k$. The rest of the functions have shape parameter c as parameter, together with the function exponent β.

8.1 MQRBFs

The MQRBFs are the functions described with one shape parameter c and exponent β value. Their high accuracy is requested in this paper, and therefore they are used together with polynomial term.

8.1.1 Exponent β values

The properties of multiquadric radial basis function can be clearly observed by graphical plot of Gamma and inverse Gamma function.

MQRBFs generalized Fourier transform has form for $x \in IR^s$:

$$\hat{\Phi}(\omega) = \frac{2^{1+\beta}}{\Gamma(-\beta)}\left(\frac{\|\omega\|}{c}\right)^{-\beta-\frac{s}{2}} K_{\beta+\frac{s}{2}}\left(c\|\omega\|\right), \ \omega \neq 0 \qquad (14)$$

where: Γ is Gamma function, K_V is modified Bessel function of the second kind of order v (MacDonald's function), shape parameter $c > 0$and x is input variable.

Corresponding plot of Gamma and inverse Gamma function is:

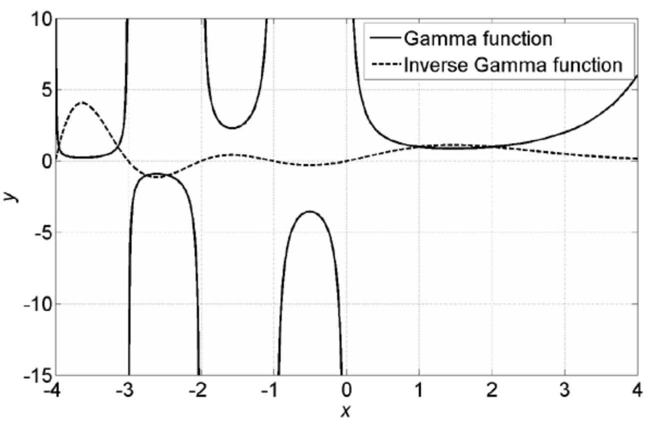


Figure 2 **Gamma and inverse Gamma functions plot**
Slika 2 **Graf Gama i inverzne Gama funkcije**

If we observe the Gamma function in the denominator, its plot self-explanatory shows efficient exponent β values to be used for MQRBFs, Figure 2. Obviously, integer β values should be avoided when using L_2 norm because of zero values.

8.1.2 Shape parameter *c*

The shape parameter c determines the shape of MQRBF chosen. The sensitivity diagram, Figure 3, shows the typical relations between shape parameter c and corresponding *RMSE* values.

It can be seen from the c – *RMSE* sensitivity diagram in Figure that the acceptable values of shape parameter c for high accuracy to be obtained are near zero value. Therefore, those values will be applied for MQRBFs in ship geometry description of 2D sections.

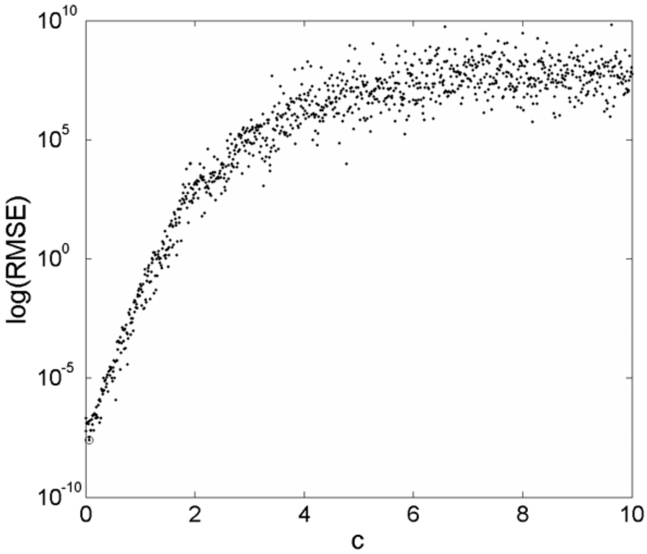


Figure 3 **Sensitivity diagram *c – RMSE***
Slika 3 **Dijagram senzitivnosti *c – RMSE***

8.2 Wendland's CSRBFs

The selection of different smoothness $2k$, and the diameter of compact support are crucial for the representation properties of CSRBFs.

Corresponding compact support diameter is the parameter that quality of RBFs description mostly depends on and it will be varied in the results study in the next chapter

9 Results

The results of the RBF interpolation of the test section for the above chosen representative functions from Table 6 are extended with the polynomial of minimum degree. The accuracy of the description is checked on ship's midship section with and without camber for chosen RBFs.

The RBF network quality of 2D ship section description will be observed for three corresponding description capabilities:

1. the description of form breaks,
2. the description of rounded form parts like the bilge,
3. the transition from rounded to flat parts, like the transition from the bilge to the flat of the side.

The break of the form between the deck and the side is checked against large oscillations characteristic for explicit interpolation procedures ("Gibbs phenomenon") for general cargo ship test-section.

The quality of the representation of the rounded bilge part and its transition to the flat of the side is checked on the main frame of one tanker with a flat bottom, flat sides and a rounded bilge with constant radius. This frame is chosen in order to check the quality and the accuracy of the RBF description of rounded geometry parts connected by the flat of the side by 90 degrees.

9.1 Calculation accuracy

The results of RBF interpolation of the test sections for the ship geometry with and without camber are shown in Table 7 in order to observe RBF interpolation accuracy and quality of description of the ship's test section.

Table 7 **RBF interpolation results of the ship's test section**
Tablica 7 **Rezultati RBF interpolacije brodskog rebra za provjeru**

Function	Without Camber		With Camber and Additional points	
	RMSE	Err_{max}	RMSE	Err_{max}
MQ, $\beta = 1/2$, $c = 0.01$	$1.57\cdot10^{-10}$	$1.34\cdot10^{-4}$	$4.69\cdot10^{-9}$	$8.51\cdot10^{-3}$
MQ, $\beta = 3/2$, $c = 0.01$	$6.16\cdot10^{-8}$	$9.69\cdot10^{-5}$	$1.45\cdot10^{-1}$	$2.65\cdot10^{-1}$
MQ, $\beta = 5/2$, $c = 0.01$	$2.06\cdot10^{-5}$	$1.78\cdot10^{-2}$	$1.15\cdot10^{4}$	$2.06\cdot10^{4}$
Generalized MQ, $\beta = 1/2$, $c = 0.01$	$1.09\cdot10^{-10}$	$3.65\cdot10^{-7}$	$6.04\cdot10^{-9}$	$8.51\cdot10^{-3}$
Generalized MQ, $\beta = 3/2$, $c = 0.01$	$5.39\cdot10^{-8}$	$9.69\cdot10^{-5}$	$2.62\cdot10^{-1}$	-
Generalized MQ, $\beta = 5/2$, $c = 0.01$	$2.43\cdot10^{-5}$	$1.78\cdot10^{-3}$	$4.33\cdot10^{3}$	$7.75\cdot10^{3}$
Inverse MQ, $\beta = 1/2$, $c = 0.11$	$3.10\cdot10^{-12}$	$1.50\cdot10^{0}$	$3.58\cdot10^{-9}$	$1.48\cdot10^{0}$
Inverse MQ, $\beta = 3/2$, $c = 0.11$	$1.33\cdot10^{-12}$	$5.36\cdot10^{0}$	$4.34\cdot10^{-10}$	$5.36\cdot10^{0}$
Thin-plate spline, $k = 2$,	$1.03\cdot10^{-9}$	$1.55\cdot10^{-3}$ Osc.	$3.13\cdot10^{-5}$	$1.47\cdot10^{-1}$ Osc.
Thin-plate spline, $k = 4$	$9.07\cdot10^{-6}$	$3.75\cdot10^{-3}$ Osc.	8.27	$1.48\cdot10^{2}$ Osc.
Wendland $k = 0$, $d = 2.5$	$6.55\cdot10^{-14}$	$3.64\cdot10^{-1}$	$4.00\cdot10^{-13}$	$3.64\cdot10^{-1}$
Wendland $k = 1$, $d = 2.5$	$2.02\cdot10^{-9}$	$4.33\cdot10^{-2}$	$1.06\cdot10^{-7}$	$1.66\cdot10^{-2}$
Wendland $k = 2$, $d = 2.5$	$1.22\cdot10^{-8}$	$5.74\cdot10^{-2}$	$4.35\cdot10^{-4}$	$2.40\cdot10^{0}$
Wendland $k = 0$, $d = 5$	$9.20\cdot10^{-14}$	$9.22\cdot10^{-2}$	$1.84\cdot10^{-12}$	$9.17\cdot10^{-2}$
Wendland $k = 1$, $d = 5$	$3.80\cdot10^{-9}$	$9.54\cdot10^{-3}$	$1.07\cdot10^{-4}$	$4.44\cdot10^{-3}$
Wendland $k = 2$, $d = 5$	$1.98\cdot10^{-7}$	$9.94\cdot10^{-3}$	$2.94\cdot10^{0}$	$4.94\cdot10^{0}$

(Note: "Osc." marks large bottom end oscillations)

The acceptable values of global and local representation are obtained for the test-section without camber only, with MQRBF and generalized MQRBF with $\beta = 3/2$ giving acceptable values.

The results for the section with camber do not have required Err_{max} accuracy, with Wendland's CSRBF with $s = 2$, $k = 1$ and $d = 5$ having results the closest to the required ones. After the diameter d is set to 6.8, better values are obtained: $RMSE = 5.0\cdot10^{-4}$ and $Err_{max} = 9.66\cdot10^{-4}$, showing CSRBFs dependency on that parameter.

It has to be noted, that MQRBFs with $\beta = 1/2$ and Wendland's function with $k = 0$ give straight lines of the description with C^0 continuity, and therefore they can be neglected.

9.2 Section without camber

The results for the section without camber are acceptable for almost all RBF choices except thin-plate spline that oscillates near the bottom.

Figure 4 shows the acceptable MQRBF representation of the test-section with $\beta = 1.5$, and with *Ncrt* number of drawing points.

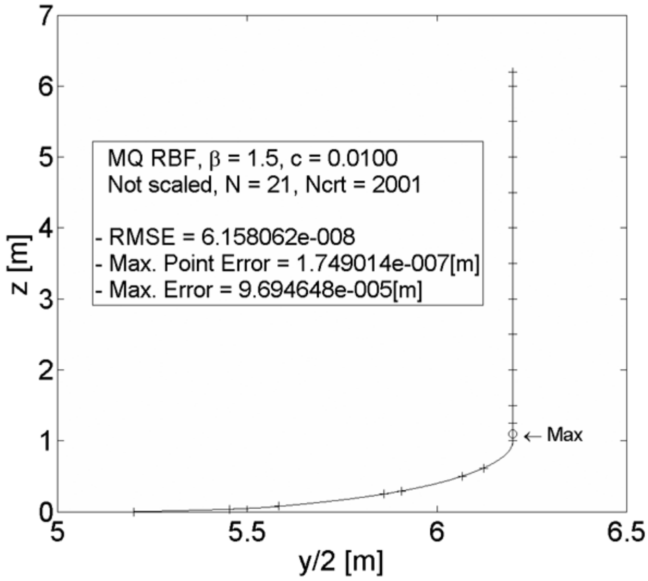


Figure 4 **The result of the MQRBF generalization of the ship test-section with $\beta = 3/2$, $c = 0.01$**
Slika 4 **Rezultat poopćavanja brodskog test-rebra pomoću MQRBF sa $\beta = 3/2$, $c = 0{,}01$**

The oscillations of the description at the transition from the rounded bilge to the flat side are shown in Figure 5.

Figure 5 **The zoom of MQRBF generalization of the ship test-section with $\beta = 3/2$, $c = 0.01$**
Slika 5 **Povećanje MQRBF poopćenja brodskog test-rebra sa $\beta = 3/2$, $c = 0{,}01$**

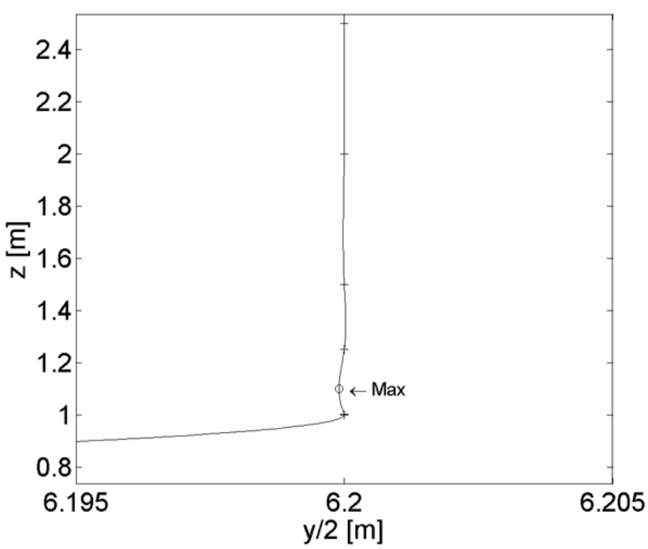


Figure 5 is Figure 4 zoom, and it shows slight oscillations near the curves transition from the rounded bilge to the straight side, as shown in Figure 5, of order 10^{-4} that can be taken as acceptable value. The additional points around transition can straighten the RBF curve in the flat part, this enabling required transition.

9.3 Section with camber

For the RBF description of the section with camber, the break of the form is the problem that cannot be solved acceptably for all chosen functions.

Because of large oscillations of RBF descriptions at the upper section end, at shear strike, a single point is added very near it to stabilize the generalization curve at the distance 10^{-4} from the top at the side. After stabilization, the only accurate and smooth functions are Wendland's functions with $k = 1$, $s = 2$ and $d = 5$, with a slightly lower local accuracy Err_{max} than required, as shown in Figure 6.

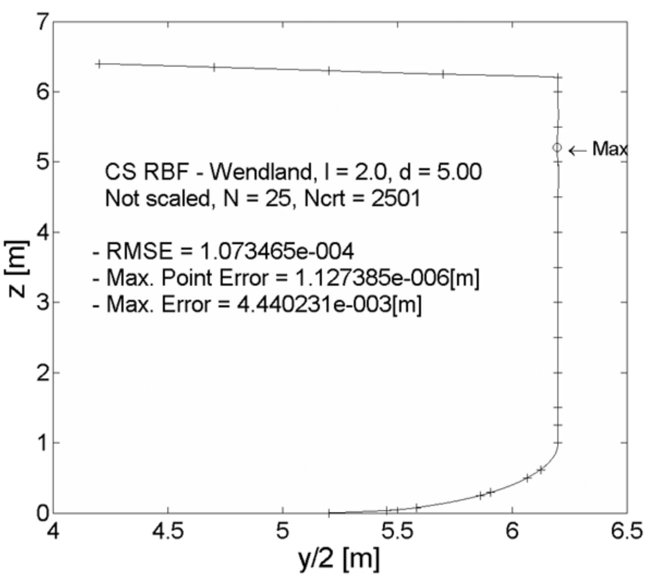


Figure 6 **Wendland's CSRBF generalization of the ship's test-section with camber**
Slika 6 **Poopćavanje brodskog test-rebra s prelukom Wendlandovom CSRBF**

In order to adequately describe curved and flat parts of the section, additional points are added, thus showing the need for adjustments in RBF description. The advantage of these adjustments is in the fact that the required accuracy is easy to achieve, and a small number of added points is needed for high quality of description, thus showing flexibility of RBF representation.

9.4 Description of the rounded bilge with transition to the flat of the side

The RBFs capability of rounded parts description, with some radius, is observed in the description of the midship section of the test tanker with transition to the flat of the side.

The bilge is described in two ways:

- with standard point distribution, and
- with Chebyshev points.

When described with standard point distribution, the section bilge is described on "standard" height point positions: 0, 0.01, 0.025, 0.100, 0.250, 0.500, 0.75, 1.000, 1.250, 1.500, 1.750, 2.000, 2.250, 2.500 metres. The rest of the section is described with equally distributed heights on the side and with equally distributed points on the bottom with some appropriate distance.

After calculations are made for chosen RBFs, the results are acceptable for standard shipbuilding point distribution only, and not for Chebyshev points. In mathematical sense, that means when the points are regular they will produce poor representation, with unexpected oscillations on the curved and flat parts for the description of the bilge transition to the flat of the side.

Moreover, the results are tolerant for MQRBFs only, showing the problems with compactly supported RBFs when applied in global context.

Figure 7 shows the MQRBF descriptions of the tanker test-section, without camber, and standard point distribution of the bilge.

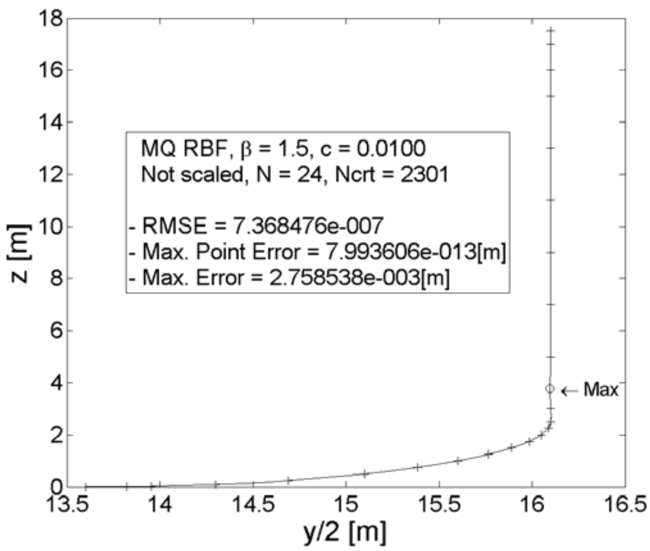


Figure 7 **Bilge and bilge transition description with MQRBF described by standard point distribution**
Slika 7 **Uzvoj i prijelaz s uzvoja na ravni bok opisan pomoću MQRBF sa standardnim rasporedom točaka**

It can be seen from Figure 7 that the quality of description is not good enough, with low Err_{max} value. Two additional points are added to improve the section description, on heights: 0.041 and 2.493, as shown in Figures 8 and 9 below.

After that, the representation has improved to required Err_{max} and *RMSE* values with better quality of the circle arc description of the bilge.

It can be seen that the RBF description of conic sections depends on the number and position of the input points and is not natural property like in NURB splines. The required accuracy cannot be obtained in the curved part of the section with the transition to straight parts, without local adjustments of shape parameter *c*. So, in order to obtain a higher accuracy, parameter *c* should be changed from global to local parameter, but that is out of the scope of this paper.

Once the required accuracy is obtained, scaling can be used to transform the RBF description to the actual bilge radius.

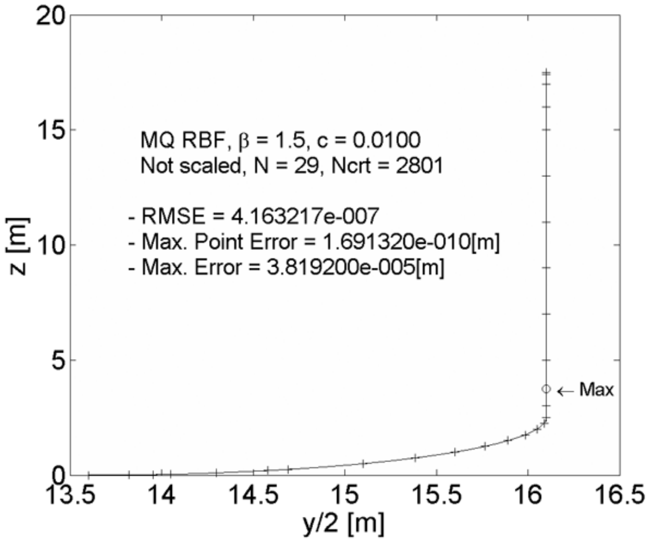


Figure 8 **Bilge and bilge transition description with MQRBF described by standard point distribution and additional two points**

Slika 8 **Opis uzvoja i prijelaza na ravni bok opisan MQRBF sa standardnim točkama i dvije dodatne točke**

10 Example of RBF representations of other ship frame

The above presented examples of frame description are focused to the accuracy of the description of some rounded bilge and to the transition to the flat of the side. The following example of frame description will show RBF description capabilities of

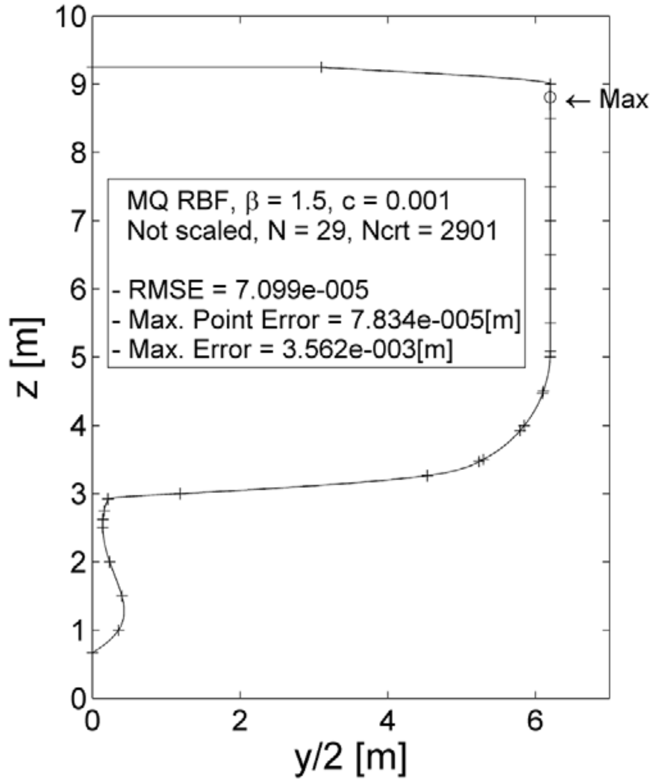
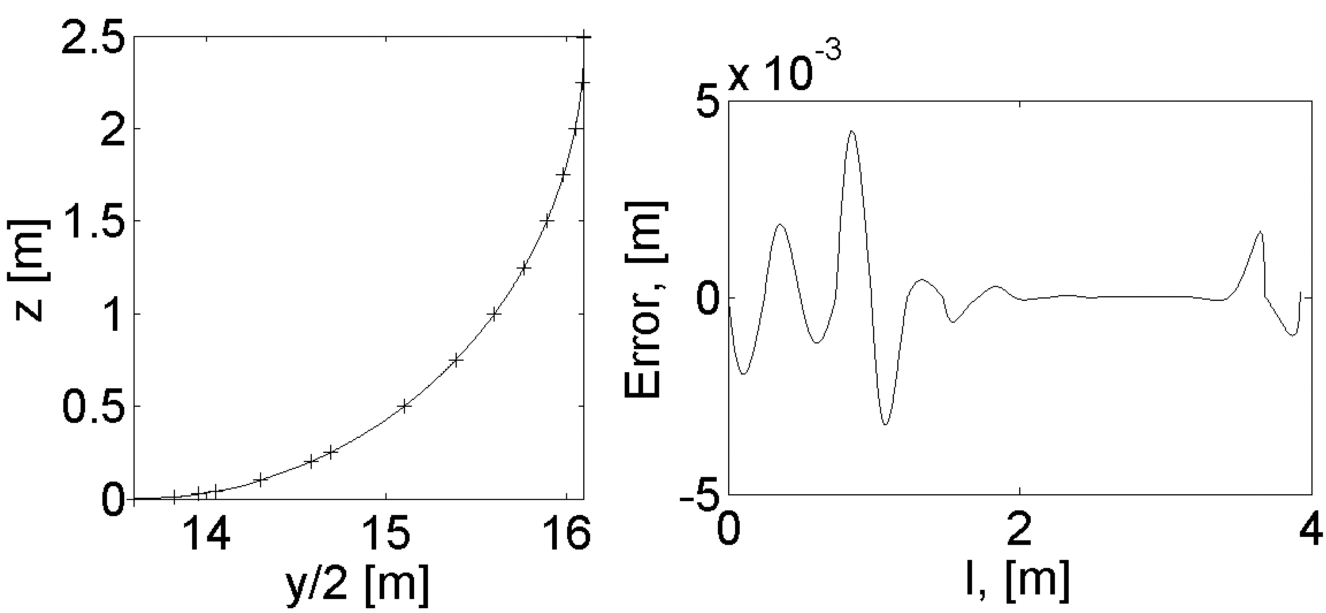


Figure 10 **MQRBF generalization with $\beta = 3/2$, $c = 0.001$ of multiple curved aft frame with camber and bulb**

Slika 10 **Opis višestruko zakrivljenog krmenog rebra sa prelukom i bulbom MQRBF poopćenjem sa $\beta = 3/2$ $c = 0,001$**

Figure 9 **Bilge zoom description, with MQRBF and 2 additional points added, and the discrepancy from ideal bilge circle arc by the bilge curve length**

Slika 9 **Uvećani uzvoj, opisan sa MQRBF sa 2 dodane točke, s odstupanjem od idealnog luka kružnice po duljini krivulje uzvoja**



multiply curved frame with camber. Figure 10 shows RBF description of the aft frame of the general cargo test ship with high curvature, bulb and camber.

This example shows the capability of RBF networks in very accurate global description of multiple curved frames with camber, and therefore it can be assumed that RBF networks are suitable for 2D ship frames description.

11 Conclusion

The RBF interpolation is applicable in 2D ship geometry description, extended with polynomial terms of minimal degree, when conditionally positive definite radial functions are used.

The high accuracy required can be obtained by input point data set adjustment, with less effort needed than in spline based methods. The transition from the rounded bilge to the flat of the side is comparable to NURB and B-splines, with possible adjustments using added points in order to achieve required smoothness of the rounded part, and flatness of the straight part of the section. The same can be concluded for the description of knuckles where the adjustment of input data set must be performed, too.

The bijection problems overall can be solved by geometric transformations, satisfying C continuity requirements.

Overall conclusion is that RBF interpolation procedures are comparable to those based on B-splines and NURB splines, and the effort should be done to further improve RBF representation of straight and rounded parts and to extend this work to corresponding 3D representations of the hull form.

References

[1] MICCHELLI, C. A.: “Interpolation of scattered data: Distance matrices and conditionally positive definite functions”, Constr. Approx. 2, p. 11-22, 1986.

[2] SCHABACK, R.: “Creating surfaces from scattered data using radial basis functions”, in “Mathematical Methods for Curves and Surfaces”, (M. Dæhlen et al. eds.), Vanderbilt University Press, p. 477-496, 1995.

[3] WENDLAND, H.: “Piecewise polynomial, positive definite and compactly supported radial functions of minimal degree”, Adv. in Comput. Math. 4, p. 389-396, 1995.

[4] WU, Z.: “Compactly supported positive definite radial functions”, Adv. in Comput. Math., p. 283-292, 1995.

[5] BOCHNER, S.: “Monotone Funktionen, Stieltjes Integrale un Harmonische Analyise”, Math. Ann. 108, p. 378-410, 1933.

[6] SCHOENBERG, I.J.: “Metric spaces and completely monotone functions”, Ann. of Math. 39, p. 811-841, 1938.

[7] SCHOENBERG, I.J.: “Metric spaces and positive definite functions”, Trans. Amer. Math. Soc. 44, p. 522-536, 1938.

[8] ARONSZAJN, N.: “Theory of reproducing kernels”, Trans. Amer. Math. Soc. 68, p. 337-404, 1950.

[9] MATHIAS, M.: “Über positive Fourier-Integrale”, Math. Zeit. 16, p. 103-125, 1923.

[10] STEWART, J.: “Positive definite functions and generalizations, an historical survey”, R. M. J. Math 6, p. 409-434, 1976.

[11] WENDLAND, H.: “Scattered Data Approximation”, Cambridge University Press (Cambridge), 2005.

[12] WILLIAMSON, R. E.: “Multiple monotone functions and their Laplace transform”, Duke Math. J. 23, p. 189-207, 1956.

[13] FASSHAUER, G. E.: “Meshfree Approximation Methods with Matlab”, Interdisciplinary Mathematical Sciences – Vol. 6, World Scientific Publishing Co. Pte. Ltd., 2007.

[14] ANDREWS, G. E., ASKEY, R., ROY, R.: “Special Functions”, Cambridge University Press, 1999.

[15] FORNBERG, B., LARSON, E., WRIGHT, G.: “A new class of oscillatory radial basis functions”, Com. Math. App. 51, p. 1209-1222, 2004.

[16] HARDY, R. L.: “Multiquadric equations to topography and other irregular surfaces”, J. Geophys. Res. 76, p. 1905-1915, 1971.

[17] FORNBERG, B., WRIGHT, G.: “Stable computation of multiquadric interpolants for all values of shape parameter”, Comp. Math. Appl. 47, p. 497-523, 2004.

[18] ABRAMOWITZ, M., STEGUN, I. A.: “Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables”, Dover (New York), 1972.

[19] GUO K. at al.: “Conditionally positive definite functions and Laplace-Stieltjes integrals”, J. Appr. Th. 74, p. 249-265, 1993.

[20] DUCHON. J.: “Interpolation des fonctions de deux variables suivant le principle de la flexion des plaques minces”, Rev. Franc. Autom. Inf. Rech. Oper. Anal. Numer. 10, p. 5-12, 1976.

[21] …: FastRBF Toolbox, Matlab Interface Version 1.4., Far-Field Technology, 2004.

[22] BUHMANN, M. D.: “Radial functions on compact support”, Proc. Edin. Math. Soc. II 41, p. 33-46, 1998.