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SOME RATIONAL DIOPHANTINE SEXTUPLES

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Abstract. A famous problem posed by Diophantus was to find sets of distinct positive rational numbers such that the product of any two is one less than a rational square. Some sets of six such numbers are presented and the computational algorithm used to find them is described. A classification of quadruples and quintuples with examples and statistics is also given.

1. Historical introduction

In the third century AD, Diophantus of Alexandria studied various problems of indeterminate equations with rational or integer solutions. One of these was to find sets of distinct positive rational numbers such that the product of any two is one less than a rational square [14].

He found examples of four such numbers, e.g.

$$\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}.$$

When Fermat revisited the Diophantine problems in the seventeenth century, he became more interested in integer solutions and found:

$$1, 3, 8, 120$$

(see [11]). It was proved in 1969 by Baker and Davenport [3] that a fifth positive integer cannot be added to this set (see [6, 10, 12] for generalizations), and no Diophantine quintuple of integers has been found. In 2004, Dujella [8] proved that there does not exist a Diophantine sextuple, and there are only finitely many Diophantine quintuples. However, Euler discovered that a

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fifth *rational* number can be added to give the following rational Diophantine quintuple:

$$1,\ 3,\ 8,\ 120,\ \frac{777480}{8288641}$$

(see [4]).

Rational sextuples with two equal elements have been found previously [1], but the following example is the first full rational Diophantine sextuple with distinct elements:

$$\frac{11}{192}, \frac{35}{192}, \frac{155}{27}, \frac{512}{27}, \frac{1235}{48}, \frac{180873}{16}.$$

Some more are given below.

2. Definition and terminology

A *Diophantine m-tuple* is a set of m different positive integers such that the product of any two is one less than a square. In particular, we talk about Diophantine triples, quadruples, quintuples and sextuples for $m = 3, 4, 5$ and 6. We also say that a set of m positive integers has the *property of Diophantus of order n*, written $D(n)$, if the product of any two elements plus n is a square.

A *rational Diophantine m-tuple* is a set of m different positive rational numbers such that the product of any two is one less than a rational square. Finding a rational Diophantine m-tuple with a common denominator of l is equivalent to finding an integer set with the property $D(l^2)$, since the elements of the rational m-tuple can be multiplied by l^2 to give such a set, and inversely.

For brevity we will sometimes omit the words "Diophantine" and "rational", when the context removes any ambiguity.

3. Rational Diophantine quadruples

In 1979 it was demonstrated by Arkin, Hoggatt and Straus [2] that every Diophantine triple $\{a, b, c\}$ can be extended to a Diophantine quadruple $\{a, b, c, d\}$ by taking

$$d = a + b + c + 2abc + 2rst,$$

where

$$ab + 1 = r^2, \quad ac + 1 = s^2, \quad bc + 1 = t^2.$$

This gives

$$ad + 1 = (at + rs)^2, \quad bd + 1 = (bs + rt)^2, \quad cd + 1 = (cr + st)^2.$$

An alternative extension formula is

$$d = a + b + c + 2abc - 2rst,$$

but this value may fail to extend the triple to a Diophantine quadruple because it can be zero or, in the rational case, a repeat of one of the first three numbers.

We will refer to this as *AHS-extension*. It can be used to extend Diophantine triples in integers or rational numbers. Diophantine quadruples constructed in this way satisfy an equation

$$(a+b-c-d)^2 = 4(ab+1)(cd+1)$$

(see [13]). This equation is invariant under permutations of a, b, c and d. We will say that a Diophantine quadruple or rational Diophantine quadruple is *regular* if it satisfies this equation and *irregular* otherwise. It is also useful to define a regular Diophantine triple as one which satisfies this equation with $d = 0$.

All known Diophantine quadruples are regular and it has been conjectured that there are no irregular Diophantine quadruples [2, 13] (this is known to be true for polynomials with integer coefficients [9]). If this is correct, then there are no Diophantine quintuples. However, there are infinitely many irregular *rational* Diophantine quadruples. The smallest is

$$\frac{1}{4},\ 5,\ \frac{33}{4},\ \frac{105}{4}.$$

Many of these irregular quadruples are examples of another common type for which two of the sub-triples are regular, i.e. $\{a, b, c, d\}$ is an irregular rational Diophantine quadruple, while $\{a, b, c\}$ and $\{a, b, d\}$ are regular Diophantine triples. These are known as *semi-regular* rational Diophantine quadruples. There are only finitely many of these for any given common denominator l and they can be readily found.

If $a = A/l$, $b = B/l$, $c = C/l$ and $d = D/l$, then we can assume that

$$\begin{aligned} C &= A + B - 2\sqrt{AB + l^2}, \\ D &= A + B + 2\sqrt{AB + l^2}, \\ CD &= X^2 - l^2, \end{aligned}$$

which implies

$$C + D = 2(A + B), \qquad CD = (A - B)^2 - 4l^2,$$

and

$$(X - A + B)(X + A - B) = 3l^2.$$

Since l is given, we can readily list all factorizations of $3l^2$ into two factors rs of the same parity and then take

$$\begin{aligned} X &= \frac{r+s}{2}, \\ A - B &= \frac{r-s}{2}, \\ CD &= \frac{(r-s)^2}{4} - 4l^2. \end{aligned}$$

Now form all possible factorizations of the right hand side of this last equation into two factors of the same parity to give C and D. The rest of the equations can now be solved to give the corresponding values of A and B. The result is an irregular rational Diophantine quadruple unless any of the numbers are not positive or not different.

Not all irregular rational Diophantine quadruples are semi-regular. An example of a quadruple containing only one regular sub-triple is

$$\frac{2}{7}, \frac{36}{7}, \frac{60}{7}, 588.$$

The simplest example of an irregular quadruple containing no regular sub-triples is

$$\frac{1}{12}, \frac{13}{3}, \frac{385}{4}, \frac{2464}{3}.$$

4. Rational Diophantine quintuples

In 1997 Dujella showed that AHS-extension can be generalized to extend quadruples to rational quintuples [5]. If $\{a, b, c, d\}$ is a Diophantine quadruple in positive integers or rational numbers, quintuples $\{a, b, c, d, e\}$ are given by

$$\begin{aligned} e =& \big((abcd+1)(a+b+c+d) + 2abc + 2abd + 2bcd + 2acd \\ &+ 2\sqrt{(ab+1)(ac+1)(ad+1)(bc+1)(bd+1)(cd+1)}\big)/(abcd-1)^2, \end{aligned}$$

or

$$\begin{aligned} e =& \big((abcd+1)(a+b+c+d) + 2abc + 2abd + 2bcd + 2acd \\ &- 2\sqrt{(ab+1)(ac+1)(ad+1)(bc+1)(bd+1)(cd+1)}\big)/(abcd-1)^2. \end{aligned}$$

If $\{a, b, c, d\}$ is a regular quadruple, the second formula gives zero. In other circumstances it might be negative or a repeat of a number from the quadruple. It is also possible for the first formula to fail by giving a repeat of a number already present in the quadruple. For example, this regular quadruple can not be extended to a quintuple using the extension formula:

$$\frac{7}{20}, \frac{3}{5}, \frac{32}{5}, \frac{75}{4}.$$

However, the extension does work in a wide class of cases (see [5]).

We will refer to this as *Dujella extension*. Quintuples constructed in this way are *regular* and others are *irregular* (see [7] for characterizations of regular quadruples and quintuples in terms of elliptic curves). Regular rational Diophantine quintuples satisfy the equation

$$(abcde + 2abc + a + b + c - d - e)^2 = 4(ab+1)(ac+1)(bc+1)(de+1).$$

Further knowledge of quintuples comes from a computational search. The lowest known common denominator is 16 in these two examples:

$$\frac{5}{16}, \frac{21}{16}, 4, \frac{285}{16}, 420,$$

$$\frac{1}{16}, \frac{33}{16}, \frac{105}{16}, 20, 1140.$$

Both of these are irregular quintuples but contain two regular quadruples. These are semi-regular quintuples. In general, we will say that quintuples are *semi-regular* when they are irregular and every element is contained in a regular sub-triple or regular sub-quadruple.

The smallest denominator regular quintuples are

$$\frac{1}{5}, \frac{21}{20}, \frac{69}{20}, \frac{25}{4}, \frac{96}{5},$$

$$\frac{7}{20}, \frac{3}{5}, \frac{63}{20}, \frac{32}{5}, \frac{75}{4}.$$

Both of these also contain a regular quadruple. The smallest regular quintuple containing no regular quadruples is

$$\frac{3}{28}, \frac{19}{28}, \frac{20}{7}, \frac{195}{28}, 588,$$

and we know of only one example of a regular quintuple with two regular quadruples, which is

$$\frac{3}{104}, \frac{209}{312}, \frac{448}{39}, \frac{39}{2}, \frac{2093}{24}.$$

Finding quintuples with less regularity is also difficult. This example is an irregular quintuple with only one regular quadruple:

$$\frac{21}{40}, \frac{21}{10}, \frac{25}{8}, \frac{429}{40}, \frac{128}{5},$$

while the first example of an irregular quintuple with no regular quadruples is

$$\frac{11}{192}, \frac{35}{192}, \frac{155}{27}, \frac{512}{27}, \frac{180873}{16}.$$

The classification of quintuples by their regularity and their number of regular quadruples is complete with these six cases. It is not possible to have quintuples containing three or more regular quadruples. As an indication of the frequency of each case, we have classified and counted cases for the first 688 examples which my numerical search produced. The results are displayed in Table 1. Notice that about half are semi-regular quintuples with two regular quadruples. This result was also apparent in other runs of the search program. The quintuples can be further classified according to the number of regular triples they contain and how they overlap. This reveals that even most of the irregular quintuples with only one regular quadruple are semi-regular. Only

Table 1. Classification of quintuples by their regularity.

# of regular quadruples	regular cases	irregular cases
0	168	2
1	49	127
2	1	341

about five to six percent of quintuples found in the brute force search were neither regular nor semi-regular.

It is also worth remarking that none of the quintuples had a common denominator which is prime and we only found one case where the common denominator is twice a prime, i.e.

$$\frac{6}{43}, \frac{285}{86}, \frac{505}{86}, \frac{3696}{43}, 6880.$$

5. Computational algorithms

We have combined two basic computational algorithms. The first is a brute force search. Pairs of numbers A and B are generated with $A < B$. The product AB is factorized into two factors r and s of the same parity to give

$$X = \frac{r+s}{2},$$
$$l = \frac{|r-s|}{2},$$
$$AB + l^2 = X^2.$$

Next, a search for a third number C is conducted such that

$$AC + l^2 = Y^2,$$
$$BC + l^2 = Z^2.$$

This search can be done simply by brute force or more rapidly using complex continued fraction and factorization algorithms for solving the Pellian equation

$$BY^2 - AZ^2 = l^2(B-A).$$

Once a list of solutions for C have been found it is straight forward to compare them in pairs to find Diophantine quadruples, quintuples and sextuples with common denominator l. This brute force algorithm was sufficient to find the first sextuple using an overnight search on a 90MHz Pentium PC.

Six more sextuples were found by extending quintuples. We used two methods of extension which both worked in each case. The first method was a brute force attempt to extend quintuples by searching through rational numbers of the form Fl/k^2 and simply checking against all the elements of the quintuple to see if the extension made a sextuple. The second method was to apply Dujella extension to all sub-quadruples contained in the quintuple and check to see whether the new number also worked against the one which was left out.

Application of AHS-extension and Dujella extension is the second basic computational algorithm which we have tried. It is reasonable to try to extend progressively quadruples, quintuples and sextuples using these methods. So far this has not provided any sextuples which the brute force algorithm did not also find.

6. Rational Diophantine sextuples

There is no equation for regular sextuples which generalizes the definition of regular triples, quadruples and quintuples. However, it is interesting to examine the rational Diophantine sextuples which we have found to see which regular quintuples, quadruples and triples they contain. The Table 2 shows the results.

Table 2. Rational sextuples and their regularity structure.

a	b	c	d	e	f
11/192	35/192	155/27	512/27	1235/48	180873/16
$(a\,b\,c\,d\,f)$ $(a\,b\,d\,e)$ $(c\,d\,e\,f)$					
17/448	265/448	2145/448	252	23460/7	2352/7921
$(a\,b\,d\,e\,f)$ $(b\,c\,d\,e)$					
9/44	91/132	60/11	44/3	1265/12	4420/3993
$(a\,b\,d\,e\,f)$ $(a\,b\,c\,d)$ $(a\,c\,d\,e)$ $(a\,b\,c\,f)$					
3/80	55/16	28/5	1683/80	1680	2220/6889
$(a\,c\,d\,e\,f)$ $(a\,b\,c\,d)$ $(b\,c\,d\,e)$ $(a\,b\,c)$					
47/60	287/240	225/64	1463/60	512/15	225/1156
$(a\,b\,d\,e\,f)$ $(a\,b\,c\,d)$ $(a\,d\,e)$					
27/1856	2065/5568	116/3	23693/192	12880/87	21420/229709
$(a\,c\,d\,e\,f)$ $(a\,b\,d\,e)$					
21/352	237/352	280/33	1573/96	4680/11	398090/236883
$(a\,b\,c\,e\,f)$ $(a\,b\,c\,d)$ $(b\,c\,d\,e)$					

In all cases, the sextuple contains a regular quintuple and at least one regular quadruple. In several cases there is also a semi-regular quintuple with two regular quadruples. Indeed, all the sextuples are rich in regular sub-tuples. They are also all semi-regular when we define this to mean that all elements are contained in a regular sub-triple, sub-quadruple or sub-quintuple. On the other hand, each sextuple also contains an irregular sub-quintuple with no regular sub-triples or sub-quadruples. This might be more surprising since such quintuples were very rarely found in the brute force search.

7. Further work

It will be interesting to continue the search for more sextuples. Perhaps a rational Diophantine septuple can be found using the algorithms described here.

Further progress may be made by analyzing some of the semi-regular quintuple cases. In particular, the common case of quintuples with two regular sub-quadruples might be soluble, in the sense that all examples for a given common denominator can be found efficiently.

Some of the new questions raised by this work which remain to be answered include:

- Are there any rational Diophantine septuples?
- Are there finitely many quintuples with a given common denominator?
- Are there any quintuples with a prime common denominator?
- Are there as many quintuples containing two regular quadruples as there are others (when placed in sequence given some natural ordering)?

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