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REAL RAMIFICATION POINTS AND REAL WEIERSTRASS POINTS OF REAL PROJECTIVE CURVES

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Abstract. Here we construct real smooth projective curves with prescribed genus, gonality and topological type or with a real Weierstrass point with prescribed first positive non-gap.

1. Introduction

For any smooth and connected projective curve X of genus $g \geq 0$ defined over $\mathbb{R}$ let $X(\mathbb{R})$ denote its set of real points and $n(X)$ the number of the connected components of $X(\mathbb{R})$. Hence $X(\mathbb{R})$ is the disjoint union of $n(X)$ circles. Set $a(X) = 1$ if $X(\mathbb{C})\backslash X(\mathbb{R})$ is connected and $a(X) = 0$ if $X(\mathbb{C})\backslash X(\mathbb{R})$ is not connected, i.e. if $X(\mathbb{C})\backslash X(\mathbb{R})$ has two connected components. The topological pair $(X(\mathbb{C}), X(\mathbb{R}))$ is uniquely determined by the triple of integers $(g, n(X), a(X))$ and such a triple of integers (g, n, a) is associated to some smooth real genus g curve if and only if either $a = 0$, $n \equiv g + 1 \pmod 2$ and $1 \leq n \leq g + 1$, or $a = 1$ and $0 \leq n \leq g$ ([5, Proposition 3.1]). Let X be a smooth and connected projective curve of genus $g \geq 2$. A point $P \in X$ is a Weierstrass point if and only if $h^0(X, \mathcal{O}_X(gP)) \geq 2$, i.e. (Riemann-Roch) if and only if $h^1(X, \mathcal{O}_X(gP)) > 0$. The first integer $t \geq 2$ such that $h^0(X, \mathcal{O}_X(tP)) = 2$ is called the first non-gap of P. The gonality $\mathrm{gon}(X)$ of X is the minimal integer $b > 0$ such that there is a degree b morphism $X \to \mathbf{P}^1$. It is known that $2 \leq \mathrm{gon}(X) \leq \lfloor (g + 3)/2 \rfloor$, that the general genus g curve has gonality $\lfloor (g + 3)/2 \rfloor$, and that for every integer k such that $2 \leq k < \lfloor (g + 3)/2 \rfloor$ the set of all k-gonal curves of genus g is parametrized one-to-one by an irreducible algebraic variety of dimension $2g + 2k - 5$ ([1,

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p. 346]). Here we prove the following results, the case $a = 1$ of Theorem 1.1 being previously known ([2, Theorem 0.1]).

THEOREM 1.1. *Fix integers g, k, a, n such that $g \geq 2k+3 \geq 9$, $a \in \{0,1\}$ and either $a = 1$ and $0 \leq n \leq g$, or $a = 0$, $1 \leq n \leq g+1$ and $n \equiv g+1 \pmod 2$. If $n = 0$ assume k is even. Then there exists a smooth projective k-gonal curve X of genus g defined over $\mathbb{R}$ with $n(X) = n$, $a(X) = a$ and such that there is a degree k pencil on X is defined over $\mathbb{R}$.*

REMARK 1.2. The case $g \geq 5$ and $k = \lfloor (g+3)/2 \rfloor$, i.e. the case of curves with general Brill-Noether theory for pencils, was proved by S. Chaudary ([3]).

THEOREM 1.3. *Fix integers g, k, a, n such that $g \geq 2k+3 \geq 9$, $a \in \{0,1\}$ and either $a = 1$ and $1 \leq n \leq g$ or $a = 0$ and $n = g+1$. Then there exist a smooth projective k-gonal curve X of genus g defined over $\mathbb{R}$ with $n(X) = n$, $a(X) = a$, and $P \in X(\mathbb{R})$ such that k is the first positive non-gap of P and $|kP|$ is a degree k pencil on X.*

Fix a smooth genus g curve such that there is a degree k morphism $f : X \to \mathbf{P}^1$ with a total ramification point, P. Hence $\mathcal{O}_X(kP) \geq 2$. Hence if $k < g$, then P is a Weierstrass point of X. This observation explains why Theorems 1.1 and 1.3 are also existence theorems for real Weierstrass points.

2. THE PROOFS

PROPOSITION 2.1. *Fix a topological type (g, n, a) for smooth real curves such that $n \neq 0$. Then there exist smooth curves A, B defined over $\mathbb{R}$ and of topological type (g, n, a), degree 3 morphisms $f : A \to \mathbf{P}^1$ and $h : B \to \mathbf{P}^1$ defined over $\mathbb{R}$ and $P \in A(\mathbb{R})$, $Q \in B(\mathbb{C}) \backslash B(\mathbb{R})$ such that P is a total ramification point of f and Q is a total ramification point of h. When $g = 1$ we may also find a real degree 3 morphism $\tau : A \to \mathbf{P}^1$ with one real total ramification point and two non-real complex conjugate total ramification points. In the case $g \geq 2$ and $n = g+1$ (and hence $a = 0$) we may even find a real curve of that topological type with a degree 3 real morphism with three total real ramification points and another one with one real total ramification point and two complex conjugate total ramification points.*

PROOF. We divide the proof into five steps.

(i) First assume $g = 0$. Hence $n = 1$ and $a = 0$. Take $P, P' \in \mathbf{P}^1(\mathbb{R})$, $P \neq P'$, $Q \in \mathbf{P}^1(\mathbb{C}) \backslash \mathbf{P}^1(\mathbb{R})$. Take as f the morphism induced by the linear system spanned by the effective divisors $3P$ and $3P'$ and as h the morphism induced by the linear system spanned by the effective divisors $3Q$ and $3\bar{Q}$. In this way we even get two total ramification points.

(ii) Now assume $g = 1$. If $n = 1$, then $a = 1$. If $n = 2$, then $a = 0$. Fix any real smooth genus 1 curve Z such that $Z(\mathbb{R}) \neq \emptyset$ and any degree 3 real embedding of Z into a plane. Any smooth real degree 3 plane curve has three real flexes and six non-real flexes (case $n = 1$) or five real flexes and four non-real flexes (case $n = 2$). Use the morphism induced by the linear system spanned by the triple of two of these flexes to obtain f and h. To obtain the morphism τ of the "Furthermore" part take a degree 3 cyclic covering of $\mathbf{P}^1$ either ramified (and hence totally ramified) over three real points, or ramified (and hence totally ramified) over a real point and two non-real complex conjugate points.

(iii) Now assume $g \geq 2$ and $n = g+1$. Hence $a = 0$. We first construct A, P, P' and f. Let (A_i, P_i, P_i', f_i), $i = 1, \dots, g$, be solutions for the case $g = 1$ and $n = 2$. Let E be the genus g stable curve of compact type constructed in the following way. E has g irreducible components and we will call them $Z_1, \dots, Z_g$ because they are isomorphic to the previously chosen smooth genus one curves. We will identify all the points P_i, P_i' with the corresponding points of E. E has $g-1$ singular points $O_1, \dots, O_{g-1}$. Notice that P_1 and P_g' are smooth points of E, while all other points P_i, P_i' are singular points of E. We have $Z_i \cap Z_j \neq \emptyset$ if and only if $|i-j| \leq 1$. E is obtained from these data just gluing the point P_i' of Z_i to the point P_{i+1} of Z_{i+1}, $1 \leq i \leq g-1$. Let $\mathbb{T}$ denote the genus 0 nodal connected curve with g irreducible components $\mathbb{E}_i \cong \mathbf{P}^1$, $1 \leq i \leq g$, and $g-1$ singular points $\mathbb{O}_i$, $1 \leq i \leq g-1$, with $\mathbb{O}_i = \mathbb{E}_i \cap \mathbb{E}_{i+1}$, $1 \leq i \leq g-1$. We identify (over $\mathbb{R}$) $\mathbb{E}_i \cong \mathbf{P}^1$ with the target of f_i. In this way the set of all f_i's induces a degree 3 admissible cover $\phi : E \to \mathbb{E}$ ([6, §4]) with P_1 and P_g' as total ramification points contained in E_{reg}. We prescribe that each $\mathbb{O}_i$ is obtained gluing together a real point of $\mathbb{E}_i$ with a real point of $\mathbb{E}_{i+1}$. In this way $\mathbb{E}$ gets a real structure compatible with ϕ. We get f smoothing (E, ϕ), but keeping P_1 and P_g' as total ramification points (see [3, Chapter 2] for real smoothings of admissible coverings). To construct h we take the same construction, except that for Z_1 and Z_g we use $\tau_1 : A_1 \to \mathbf{P}^1$ and $\tau_g : A_g \to \mathbf{P}^1$ with P_1' and P_g a real total ramification point, while P_1 and P_g' are non-real total ramification points. For the last assertion of Proposition 2.1 we take two of the curves Z_i with three real total ramification points.

(iv) Now assume $g \geq 2$, $a = 0$ and $n < g+1$. Hence $q := (g+1-n)/2$ is a positive integer. To get h we make the following construction. Take a solution (B, Q', h') for the topological type $(n-1, n, 0)$ and the non-real total ramification point Q'. Let U be any smooth complex genus q curve equipped with a degree 3 morphism $u : U \to \mathbf{P}^1$ with two distinct total ramification points W, W'. Let $\bar{U}$ denote the complex conjugate curve of U; if U is seen as a Riemann surface with local charts, then $\bar{U}$ is the same topological manifold with, as gluing data for the same atlas, the complex conjugation of the gluing data of U; if U is seen as the zero-locus of some homogeneous complex polynomials in some projective space, then $\bar{U}$ is the zero-locus in the same projective space of the complex conjugate polynomials, i.e. of the polynomials obtained taking the complex conjugate of all coefficients; if U is seen as an abstract algebraic scheme in the sense of Groethendieck or as a variety in the very old set-up of Weil, then $\bar{U}$ is obtained from U applying the action of the Galois group of the field extension $\mathbb{C}/\mathbb{R}$. Any of these equivalent descriptions gives the existence of a unique $\bar{W} \in \bar{U}(\mathbb{C})$ corresponding to W and a unique $\bar{W}' \in \bar{U}(\mathbb{C})$ corresponding to W'. Furthermore, u induces a degree 3 morphism $\bar{u} : \bar{U} \to \mathbf{P}^1$ such that $\bar{W}$ and $\bar{W}'$ are total ramification points of $\bar{u}$. Let Γ be the genus g nodal curve of compact type constructed in the following way. Γ has three irreducible components, respectively isomorphic to B (over $\mathbb{R}$), to U and to $\bar{U}$, and two singular points $O, \bar{O}$, with O obtained gluing together the point W' of U with the point Q' of B and $\bar{O}$ obtained gluing together the point $\bar{Q}'$ of B with the point $\bar{W}$ of $\bar{U}$. With these gluings the curve Γ has a real structure such that U and $\bar{U}$ are exchanged by the complex conjugation and hence $\Gamma(\mathbb{R}) = B(\mathbb{R})$. Thus $\Gamma(\mathbb{R})$ is the disjoint union of n circles. It is easy to check that $\Gamma(\mathbb{C})\backslash\Gamma(\mathbb{R})$ has two connected components. Taking as target a genus 0 nodal curves with three irreducible components and using $h', u, \bar{u}$ we get an admissible cover whose general smoothing gives a solution for h for the topological type $(g, n, 0)$: the non-real total ramification point comes from $W \in \Gamma$. Now we show the existence of f, A, P. We start with a real smooth curve A' of type $(n-1, n, 0)$ equipped with a degree 3 real morphism $f' : A' \to \mathbf{P}^1$ with one real total ramification point and two complex conjugate complex ramification points. We use the two complex conjugate ramification points to add two genus q tails as in the proof of the existence of h.

(v) Now we assume $g \geq 2$ and $a = 1$. By assumption we have $1 \leq n \leq g$. We fix g smooth real genus one curves A_i, $1 \leq i \leq g$, such that for all i there is a degree 3 real morphism $f_i : A_i \to \mathbf{P}^1$ with two real total ramification points P_i, P_i'. If $n(A_i) = 2$, i.e. if $a(A_i) = 1$, we also assume that the points P_i and P_i' are in different connected components of $A_i(\mathbb{R})$. We assume $n(A_i) = 1$ for $1 \leq i \leq g+1-n$ and $n(A_i) = 2$ for all $i > g+1-n$ (if any). Let E be the unique curve of compact type defined over $\mathbb{R}$ with arithmetic genus g built in the following way. E has g irreducible components B_i, $1 \leq i \leq g$, each of them defined over $\mathbb{R}$. We assume the existence of a real isomorphism $u_i : A_i \to B_i$ and we set $Q_i := u_i(P_i)$, $Q_i' := u_i(P_i')$. We assume $B_i \cap B_j \neq \emptyset$ if and only if $|i-j| \leq 1$. For all $1 \leq i \leq g-1$ the only point of $B_i \cap B_{i+1}$ is the point $Q_i' \in B_i$, which is identified (glued) to the point $Q_{i+1} \in B_{i+1}$. Since for all $1 \leq i \leq g-1$ the point P_i' is a total ramification point of f_i and P_{i+1} is a total ramification point of f_{i+1}, the g degree 3 coverings $f_1, \dots, f_g$ define a degree 3 admissible covering ϕ of E. Both E and ϕ are defined over $\mathbb{R}$. Notice that $E(\mathbb{R})$ has n connected components. Since $a(B_1) = 1$, the open set $E(\mathbb{C})\backslash E(\mathbb{R})$ is connected. Notice that Q_1 and Q_g' are total ramification points. To conclude the proof we take a real smoothing of the admissible covering ϕ. ∎

Proof of Theorem 1.1. If $a(X) = 1$ the result is [2, Theorem 0.1], except for the uniqueness part. The uniqueness part is true because it is true outside a proper closed subset (for the Zariski topology) of the variety parametrizing all smooth complex k-gonal curves of genus g. Hence we will only check the case $a = 0$.

(a) Here we will check the case $n = g + 1$. Let Y be a smooth hyperelliptic curve of genus g defined over $\mathbb{R}$ with $n(Y) = g+1$ and $a(Y) = 0$; we also assume that the hyperelliptic involution is the only non-trivial holomorphic automorphism of Y. The curve Y obviously exists and it corresponds to the case in which all $2g+2$ ramification points of the hyperelliptic pencil are real ([5, §6]) and not too special. The assumption on $\mathrm{Aut}(Y)$ implies that Y has a unique real structure. Let $\Delta(Y)$ be the local moduli space of Y. The germ $\Delta(Y)$ is a smooth $3g - 3$ dimensional germ of a complex space around the point $[Y]$ representing Y. The complex conjugation σ acts on $\Delta(Y)$. Since $Y(\mathbb{R}) \neq \emptyset$ and near $[Y]$ all complex curves have at most one complex structure, the set of real curves near Y is parametrized by the fix point set $\Delta(Y)^\sigma$ of σ. Since $\Delta(Y)$ is smooth, every connected component of $\Delta(Y)^\sigma$ is a real $3g - 3$ dimensional differential manifold. Since $[Y] \in \Delta(Y)^\sigma$, we see that the image of $\Delta(Y)^\sigma$ in the moduli space $\mathcal{M}_g$ is a Zariski dense subset of the moduli space $\mathcal{M}_g$ containing the element $[[Y]]$ representing Y. Let $\mathcal{M}_g^\circ$ denote the open subset of $\mathcal{M}_g$ parametrizing all smooth curves without non-trivial automorphism. The variety $\mathcal{M}_g^\circ$ is smooth. Since each $C \in \mathcal{M}_g^\circ$ has no holomorphic involution, each $C \in \mathcal{M}_g^\circ$ has at most one real structure. There is an anti-holomorphic involution σ on $\mathcal{M}_g^\circ$ whose fixed locus $\mathcal{M}_g^\circ(\mathbb{R})$ is the set of all $C \in \mathcal{M}_g^\circ$ with a real structure. Hence every connected component of $\mathcal{M}_g^\circ(\mathbb{R})$ is a $(3g-3)$-dimensional differentiable manifold. Set $\mathcal{M}_{g;k}^\circ := \{C \in \mathcal{M}_g^\circ : C$ is k-gonal$\}$. $\mathcal{M}_{g;k}^\circ \neq \emptyset$ because $3 \le k \le (g-3)/2$. Furthermore, $\mathcal{M}_{g;k}^\circ$ is smooth, irreducible and of dimension $2g + 2k - 5$. The closure $\Gamma(k)$ of $\mathcal{M}_{g;k}^\circ$ in $\mathcal{M}_g$ contains the hyperelliptic locus. Hence in $\Delta(Y)$ there is an irreducible σ-invariant $2g+2k-5$ complex space with $[Y]$ in its closure and parametrizing k-gonal curves. Hence $[Y]$ is in the closure in $\Delta(Y)^\sigma$ of a family of real k-gonal curves, each of them with the real topological type of Y, concluding the proof of the case $n = g + 1$.

(b) Here we will check the case $a = 0$, $1 \le n \le g - 1$ and $n \equiv g + 1 \pmod 2$. Just use the previous proof starting from the case $k = 3$ (proved in Proposition 2.1) instead of the hyperelliptic case lifted from [5, §6]. □

Proof of Theorem 1.3. We modify the proof of Proposition 2.1 taking degree k morphisms (and their total ramification points) instead of degree 3 morphisms. In the case $g = 0$ we just write k instead of 3. First assume $a = 0$. If $k \ge 4$, there are real genus 1 smooth curves with two total real ramification points, but by Riemann-Hurwitz formula there is no genus 1 smooth curve with a degree k morphism with three or more total ramification points. The proof of the case $n = g + 1$ of Proposition 2.1 works verbatim when $k \geq 4$, and this is the reason of our restriction in the case $a = 0$ of Theorem 1.3. If $a = 1$ part (v) of the proof of Proposition 2.1 works verbatim. ∎

References

[1] E. Arbarello and M. Cornalba, *Footnotes to a paper of Beniamino Segre: The number of g^1_d's on a general d-gonal curve, and the unirationality of the Hurwitz spaces of 4-gonal and 5-gonal curves*, Math. Ann. **256** (1981), 341–362.

[2] E. Ballico, *Real algebraic curves and real spanned bundles*, Ricerche Mat. **50** (2001), 223-241.

[3] S. Chaudary, The Brill-Noether theorem for real algebraic curves, Ph. D. Thesis, Duke University, 1995.

[4] D. Eisenbud and J. Harris, *Limit linear series: basic theory*, Invent. Math. **85** (1986), 337-371.

[5] B. H. Gross and J. Harris, *Real algebraic curves*, Ann. Scient. École Norm. Sup. (4) **14** (1981), 157-182.

[6] J. Harris and D. Mumford, *On the Kodaira dimension of the moduli space of curves*, Invent. Math. **67** (1982), 23-88.

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