GLASNIK MATEMATIČKI
Vol. 41(61)(2006), 271 – 274

FINITE NONABELIAN 2-GROUPS IN WHICH ANY TWO NONCOMMUTING ELEMENTS GENERATE A SUBGROUP OF MAXIMAL CLASS

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Abstract. We determine here the structure of the title groups. It turns out that such a group G is either quasidihedral or G=HZ(G), where H is of maximal class or extraspecial and $\mho_1(Z(G)) \leq Z(H)$. This solves a problem stated by Berkovich. The corresponding problem for $p > 2$ is open but very difficult since the p-groups of maximal class are not classified for $p > 2$.

1. Introduction and known results

We determine here the structure of all finite nonabelian 2-groups in which any two noncommuting elements generate a subgroup of maximal class. More precisely, we prove the following result.

Theorem 1.1. *Let G be a finite nonabelian 2-group in which any two noncommuting elements generate a subgroup of maximal class. Then one of the following holds:*

(a) $|G : H_2(G)| = 2$ *and* $H_2(G)$ *is noncyclic (i.e., G is quasidihedral but not dihedral);*
(b) $G = HZ(G)$, *where H is of maximal class and* $\mho_1(Z(G)) \leq Z(H)$;
(c) $G = HZ(G)$, *where H is extraspecial of order* $\geq 2^5$ *and* $\mho_1(Z(G)) \leq Z(H)$.

Conversely, each group in (a), (b) *and* (c) *satisfies the assumption of the theorem.*

2000 *Mathematics Subject Classification.* 20D15.

Key words and phrases. Finite 2-groups, 2-groups of maximal class, minimal nonabelian 2-groups, quasidihedral 2-groups, Hughes H_p-subgroups.

We consider here only finite p-groups and our notation is standard. In particular, a 2-group S is quasidihedral if S has an abelian subgroup T of exponent > 2 so that $|S : T| = 2$ and there is an involution in $S - T$ which inverts each element in T. It turns out that T is a characteristic subgroup of S.

We state three known results which are used in the proof of Theorem 1.1.

Proposition 1.2 (Berkovich [1, Lemma 4.2]). *Let G be a p-group with $|G'| = p$. Then $G = (A_1 * A_2 * ... * A_s)Z(G)$ ($*$ denotes a central product), where $A_1, A_2, ..., A_s$ are minimal nonabelian subgroups.*

Proposition 1.3 (Berkovich [1, §58] and Kazarin [2]). *Let G be a nonabelian 2-group all of whose cyclic subgroups of composite order are normal in G. Then we have either $|G : H_2(G)| = 2$ (and then G is quasidihedral) or $|G'| = 2$ and the Frattini subgroup $\Phi(G)$ is cyclic.*

Proposition 1.4 (Janko [3, Proposition 2.3]). *A 2-group is of maximal class if and only if G is dihedral, semidihedral or generalized quaternion.*

From Proposition 1.4 follows at once that if $G = \langle a, b\rangle$ is a 2-group of maximal class, then at least one of a and b is of order ≤ 4 and G possesses exactly one involution z (where $\langle z\rangle = Z(G)$) which is a square in G. We shall use freely this remark in the proof of Theorem 1.1.

2. Proof of Theorem 1.1

Let G be a nonabelian 2-group in which any two noncommuting elements generate a subgroup of maximal class. We may assume that G is not of maximal class.

(i) First we assume that $\exp(G) > 4$. Suppose for a moment that each element of order ≥ 8 lies in $Z(G)$. Let $x, y \in G$ with $[x, y] \neq 1$. Since $\langle x, y\rangle$ is of maximal class, we have in our case $\langle x, y\rangle \cong D_8$ or $\langle x, y\rangle \cong Q_8$. Let k be an element of order 8 so that here $k \in Z(G)$. But then kx and ky are elements of order 8 with $[kx, ky] = [x, y] \neq 1$ and therefore $\langle kx, ky\rangle$ is of maximal class, a contradiction. We have proved that G possesses a cyclic subgroup A of order ≥ 8 such that $A \not\leq Z(G)$.

It is easy to see that any cyclic subgroup X of order ≥ 8 is normal in G. Indeed, let $g \in G$ so that g either centralizes X or $\langle X, g\rangle$ is of maximal class in which case g normalizes X.

Let $y \in G$ be such that $[A, y] \neq 1$ and so $\langle A, y\rangle$ is of maximal class. Then $\langle A, y\rangle$ contains a subgroup of maximal class $\langle B, y\rangle$ of order 2^4, where $B = \langle b\rangle \cong C_8$, $B \leq A$, and $y^2 \in \Omega_1(B) = \langle z\rangle$. We know that B is normal in G. Set $M = C_G(B)$ so that $G/M \neq \{1\}$ is elementary abelian of order ≤ 4. If $G/M \cong E_4$, then there is $l \in G - M$ such that $l^2 \in M$, l^2 centralizes B and $b^l = bz$. But then $\langle b, l\rangle' = \langle z\rangle$ and so $\langle b, l\rangle$ is not of maximal class, a contradiction. Thus $|G : M| = 2$.

For each $x \in G - M$, $x^2 \in \langle z \rangle$. Indeed, $[b, x] \neq 1$ and so $\langle b, x \rangle$ is of maximal class and therefore $x^2 \in \Omega_1(B) = \langle z \rangle$. Consider $\bar{G} = G/\langle z \rangle$. Then all elements in $\bar{G} - \bar{M}$ are involutions which implies that $M/\langle z \rangle$ is abelian and for each $m \in M$, $m^y = m^{-1} z^{\epsilon}$, $\epsilon = 0, 1$.

Suppose that M is nonabelian. Then $M' = \langle z \rangle$ and let $m, n \in M$ with $[m, n] = z$. In that case (since $\langle m, n \rangle$ is of maximal class), $\langle m, n \rangle \cong D_8$ or $\cong Q_8$. But then bm and bn are elements of order 8 with $[bm, bn] = [m, n] = z$ and so $\langle bm, bn \rangle$ is of maximal class, a contradiction. Hence M is abelian. If M is cyclic, then $\langle M, y \rangle = G$ is of maximal class, a contradiction. Thus, M is noncyclic abelian.

If all elements in $G - M$ are involutions, then $H_2(G) = M$ and we have obtained a group in part (a) of our theorem.

We may assume that not all elements in $G - M$ are involutions and so we may suppose $y^2 = z$. Let t be any involution in $M - \langle z \rangle$ and assume that t is a square in M, i.e., there is $k \in M$ such that $k^2 = t$. Since $k^y = k^{-1} z^{\epsilon}$ ($\epsilon = 0, 1$), $\langle y, k \rangle$ is nonabelian. In that case $\langle y, k \rangle$ is of maximal class containing two distinct involutions z and t which are squares in $\langle y, k \rangle$, a contradiction. We have proved that M is abelian of type $(2^s, 2, ..., 2)$, $s \geq 3$. Setting $E = \Omega_1(M)$, we get $M = \langle b' \rangle E$, where $o(b') = 2^s$, $|E| \geq 4$, and $\langle b' \rangle \cap E = \Omega_1(\langle b' \rangle) = \langle z \rangle$ since z is the unique involution in M which is a square in M. Since $(b')^y = (b')^{-1} z^{\eta}$ ($\eta = 0, 1$), $H = \langle b', y \rangle$ is of maximal class and $G = HE$. For each $t \in E$, we have either $t^y = t$ or $t^y = tz$. If y centralizes E, then $E = Z(G)$. If y does not centralize E, then $E_0 = C_E(y)$ is of index 2 in E. Let v be an element of order 4 in $\langle b' \rangle$ and let $u \in E - E_0$. In that case

$$(vu)^y = v^{-1}(uz) = (vz)(uz) = vu \textit{ and } (vu)^2 = z,$$

and so $Z(G) = E_0 \langle vu \rangle$ with $\mho_1(Z(G)) = \langle z \rangle$. In any case we get $G = HZ(G)$, $Z(G) > Z(H) = \langle z \rangle$, and $\mho_1(Z(G)) \leq \langle z \rangle$. We have obtained a group in part (b) of our theorem.

(ii) We examine now the case $\exp(G) = 4$. Let $\langle x \rangle$ be a cyclic subgroup of order 4 and $y \in G$. Then either $[x, y] = 1$ or $\langle x, y \rangle \cong D_8$ or Q_8. In any case y normalizes $\langle x \rangle$ and so each cyclic subgroup of order 4 is normal in G. We may use Proposition 1.3. It follows that either $|G : H_2(G)| = 2$ (and we get a group of part (a) of our theorem) or $|G'| = 2$ and $\Phi(G)$ is cyclic.

Suppose that we are in the second case. Since G does not possess elements of order 8, we have $|\Phi(G)| = 2$ and then $\Phi(G) = G'$. The fact $|G'| = 2$ implies that $G = H_1 * H_2 * ... * H_n Z(G)$, where H_i ($i = 1, ..., n$) is minimal nonabelian (Proposition 1.2). In our case $H_i \cong D_8$ or Q_8 and so $H = H_1 * H_2 * ... * H_n$ is extraspecial. Also, $\Phi(G) = G' = H' = Z(H)$ implies that $\mho_1(Z(G)) \leq Z(H)$. If $n = 1$, we have obtained a group of part (b) of our theorem and so we may assume that $n > 1$. In that case $|H| \geq 2^5$ and we have obtained a group of part (c) of our theorem.

It is necessary to prove only for groups (b) and (c) of our theorem that any two noncommuting elements generate a group of maximal class. Indeed, let h_1z_1 and h_2z_2 be any noncommuting elements in G, where $h_1, h_2 \in H$ and $z_1, z_2 \in Z(G)$. Then $[h_1z_1, h_2z_2] = [h_1, h_2] \neq 1$ and so $H_0 = \langle h_1, h_2 \rangle \leq H$ is a group of maximal class with $H_0' \geq Z(H)$. On the other hand, a 2-group $\langle h_1, h_2 \rangle$ is of maximal class if and only if $[h_1, h_2] \neq 1$, $\langle [h_1, h_2] \rangle$ is normal in H_0 and $h_1^2, h_2^2 \in \langle [h_1, h_2] \rangle$. Hence $H_1 = \langle h_1z_1, h_2z_2 \rangle$ is of maximal class since $[h_1z_1, h_2z_2] = [h_1, h_2] \neq 1$, h_1z_1 and h_2z_2 normalize $\langle [h_1z_1, h_2z_2] \rangle = \langle [h_1, h_2] \rangle$ and $(h_1z_1)^2$, $(h_2z_2)^2$ are contained in $\langle [h_1, h_2] \rangle$ (noting that $z_1^2, z_2^2 \in Z(H) \leq H_0' = H_1'$).

References

[1] Y. Berkovich, *Groups of prime power order, Parts I, II, and III (with Z. Janko)* in preparation.

[2] L. S. Kazarin, *Groups with certain conditions imposed on the normalizers of subgroups*, Perm. Gos. Univ. Učen. Zap. **218** (1969), 268–279 (Russian).

[3] Z. Janko, *Finite 2-groups with small centralizer of an involution*, J. Algebra **241** (2001), 818-826.

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Received: 5.4.2006.