GLASNIK MATEMATIČKI
Vol. 41(61)(2006), 275 – 282

SECOND-METACYCLIC FINITE p-GROUPS FOR ODD PRIMES

Vladimir Ćepulić, Olga Pyliavska and Elizabeta Kovač Striko
University of Zagreb, Croatia and National University Kyiv-Mohyla Academy, Ukraine

Abstract. A second-metacyclic finite p-group is a finite p-group which possesses a nonmetacyclic maximal subgroup, but all its subgroups of index p^2 are metacyclic. In this article we determine up to isomorphism all second-metacyclic p-groups for odd primes p. There are ten such groups of order p^4, for each prime $p \geq 3$, and two such groups of order 3^5.

1. Introduction

A second-metacyclic group is a group possessing a maximal nonmetacyclic subgroup, but it's second-maximal subgroups are all metacyclic. All second-metacyclic finite 2-groups were determined in [1]. The aim of this article is to determine all second-metacyclic finite p-groups for $p > 2$. We prove the following

Theorem 1.1. *Let G be a second-metacyclic finite p-group for some odd prime p. Then either $|G| = p^4$, or $|G| = 3^5$.*

(1) *If $|G| = p^4$, then G is isomorphic to one of the following groups:*

$$\begin{aligned} G_1 &= \langle a,b,c,d \mid a^p = b^p = c^p = d^p = 1, [a,b] = [a,c] = [a,d] = [b,c] \\ &\qquad = [b,d] = [c,d] = 1\rangle \cong E_{p^4}, \\ G_2 &= \langle a,b,c \mid a^{p^2} = b^p = c^p = 1, [a,b] = [a,c] = [b,c] = 1\rangle \cong Z_{p^2} \times E_{p^2}, \\ G_3 &= \langle a,b,c,d \mid a^p = b^p = c^p = d^p = 1, [a,b] = [a,c] = [a,d] = [b,c] \\ &\qquad = [b,d] = 1, [c,d] = a\rangle = \langle b,c,d\rangle, \\ &\qquad Z(G_3) = \langle a,b\rangle,\ G_3' = \langle a\rangle,\ \mho_1(G_3) = 1, \end{aligned}$$

2000 *Mathematics Subject Classification.* 20D15.

Key words and phrases. Finite group, p-group, metacyclic, second-maximal subgroup, second-metacyclic subgroup.

$$\begin{aligned}
G_4 &= \langle a,b,c,d \mid a^p=b^p=c^p=1, d^p=a, [a,b]=[a,c]=[a,d]=[b,c] \\
&\qquad = [b,d]=1, [c,d]=a\rangle = \langle b,c,d\rangle, \\
&\quad Z(G_4)=\langle a,b\rangle,\ G_4'=\langle a\rangle,\ \mho_1(G_4)=\langle a\rangle, \\
G_5 &= \langle a,b,c,d \mid a^p=b^p=c^p=1, d^p=b, [a,b]=[a,c]=[a,d]=[b,c] \\
&\qquad = [b,d]=1, [c,d]=a\rangle = \langle b,c,d\rangle, \\
&\quad Z(G_5)=\langle a,b\rangle,\ G_5'=\langle a\rangle,\ \mho_1(G_5)=\langle b\rangle, \\
G_6 &= \langle a,b,c,d \mid a^p=b^p=c^p=d^p=1, [a,b]=[a,c]=[a,d]=[b,c]=1, \\
&\qquad [b,d]=a, [c,d]=b\rangle = \langle c,d\rangle, \\
&\quad Z(G_6)=\langle a\rangle,\ G_6'=\langle a,b\rangle,\ \mho_1(G_6)=1, \\
G_7 &= \langle a,b,c,d \mid a^p=b^p=c^p=1, d^p=a, [a,b]=[a,c]=[a,d]=[b,c]=1, \\
&\qquad [b,d]=a, [c,d]=b\rangle = \langle c,d\rangle, \\
&\quad Z(G_7)=\langle a\rangle,\ G_7'=\langle a,b\rangle,\ \mho_1(G_7)=\langle a\rangle, \\
G_8 &= \langle a,b,c,d \mid a^p=b^p=c^p=1, d^p=a, [a,b]=[a,c]=[a,d]=[b,d] \\
&\qquad = [c,d]=1, [b,c]=a\rangle = \langle b,c,d\rangle, \\
&\quad Z(G_8)=\langle d\rangle,\ G_8'=\langle a\rangle,\ \mho_1(G_8)=\langle a\rangle, \\
G_9 &= \langle a,b,c,d \mid a^p=b^p=c^p=1, d^p=a, [a,b]=[a,c]=[a,d]=[b,d]=1, \\
&\qquad [b,c]=a, [c,d]=b\rangle = \langle c,d\rangle, \\
&\quad Z(G_9)=\langle a\rangle,\ G_9'=\langle a,b\rangle,\ \mho_1(G_9)=\langle a\rangle, \\
G_{10} &= \langle a,b,c,d \mid a^p=b^p=c^p=1, d^p=a^\sigma, [a,b]=[a,c]=[a,d]=[b,d]=1, \\
&\qquad [b,c]=a, [c,d]=b\rangle = \langle c,d\rangle,
\end{aligned}$$

$\sigma -$ *being the minimal quadratic nonresidue modulo* p,

$Z(G_{10})=\langle a\rangle,\ G_{10}'=\langle a,b\rangle,\ \mho_1(G_{10})=\langle a\rangle.$

Here, the groups $G_1 - G_7$ *contain an elementary abelian subgroup* $\langle a,b,c\rangle \cong E_{p^3}$, *and the groups* $G_8 - G_{10}$ *contain none such subgroup. For these groups* $\langle a,b,c\rangle \cong Sp(p^3)$, *the special group of order* p^3.

(2) *If* $|G| = 3^5$, *then* G *is isomorphic to one of the following groups of maximal nilpotent class:*

$$\begin{aligned}
G_{11} &= \langle a,b,c,d,f \mid a^3=b^3=1, c^3=d^3=a, f^3=ab, [a,b]=[a,c]=[a,d] \\
&\qquad = [a,f]=[b,c]=[b,f]=[c,f]=1, [b,d]=a^2, [c,d]=b, [d,f]=c\rangle \\
&= \langle d,f\rangle,\ Z(G_{11})=\langle a\rangle,\ Z_2(G_{11})=\langle a,b\rangle,\ G_{11}'=\langle a,b,c\rangle.
\end{aligned}$$

Here, $M=\langle a,b,c,f\rangle$ *is the unique abelian maximal subgroup of* G_{11}.

$$\begin{aligned}
G_{12} &= \langle a,b,c,d,f \mid a^3=b^3=1, c^3=d^3=a, f^3=b, [a,b]=[a,c]=[a,d] \\
&\qquad = [a,f]=[b,c]=[b,f]=1, [b,d]=a^2, [c,d]=b, [c,f]=a^2, [d,f]=c\rangle \\
&= \langle d,f\rangle,\ Z(G_{12})=\langle a\rangle,\ Z_2(G_{12})=\langle a,b\rangle,\ G_{12}'=\langle a,b,c\rangle.
\end{aligned}$$

G_{12} *does not contain any abelian maximal subgroup.*

2. Notation and preliminaries

At the beginning we recall some definitions and known results.

Definition 2.1. *A group G is metacyclic, $G \in \mathcal{MC}$, if it possesses a cyclic normal subgroup N such that the factorgroup G/N is also cyclic. We say that G is proper metacyclic if G is metacyclic but not cyclic.*

Definition 2.2. *G is minimal nonmetacyclic, $G \in \mathcal{MC}_1$, if G is not metacyclic, but all its maximal subgroups are metacyclic.*

Definition 2.3. *G is second-metacyclic group, $G \in \mathcal{MC}_2$, if it possesses a nonmetacyclic maximal subgroup, but all its second-maximal subgroups are metacyclic.*

Proposition 2.4. *If G is a proper metacyclic group of order $|G| = 3^3$, then either*

$$G = \langle x \mid x^9 = 1\rangle \times \langle y \mid y^3 = 1\rangle \cong Z_9 \times Z_3, \text{ or}$$

$$G = \langle x, y \mid x^9 = y^3 = 1, x^y = x^4\rangle,$$

for some $x, y \in G$, and it is $\mho_1(G) = \langle x^3\rangle, \Omega_1(G) = \langle x^3, y^3\rangle$ in both cases.

Proposition 2.5. *If G is a p-group of order $|G| = p^2$, then G is abelian and $|AutG|$, the order of its automorphism group, is divisible by p, but not by p^2.*

Proposition 2.6. *For $S \subseteq G$, denote by $\langle\langle S\rangle\rangle$ the normal closure of S in G, that is the minimal normal subgroup of G containing S. For $G = \langle a_1, a_2, ..., a_t\rangle$ is $G' = \langle\langle [a_i, a_j] \mid 1 \leq i < j \leq t\rangle\rangle$.*

Theorem 2.7 (Blackburn [1], Huppert [3, III.14.2]). *Let G be a p-group of maximal class of order p^n. Then for each $k, 0 \leq k \leq n-2$, there exist exactly one normal subgroup N of G of order p^k and $N = Z_k(G) = K_{n-k}(G)$.*

Theorem 2.8 (Blackburn [1], Huppert [3, III.11.11]). *Let G be a p-group, which is minimal nonmetacyclic. Then one of the following assertions holds:*

(a) *G is elementary abelian of order p^3.*
(b) *It is $p > 2$ and G is nonabelian of exponent p and order p^3.*
(c) *G is a 3-group of class 3 and of order 3^4.*
(d) *G is a 2-group with $|G| \leq 2^5$.*

Theorem 2.9 (Blackburn [1], Huppert [3, III.11.12]). *Let $p > 2$ and $|G| = p^5$, and let all subgroups of order p^3 in G be generated by two elements. Then one of the following assertions holds:*

(a) *G is metacyclic.*
(b) *G is a 3-group of maximal class.*
(c) *The group $\Omega_1(G)$ is of order p^3 and exponent p and $G/\Omega_1(G)$ is cyclic.*

THEOREM 2.10. *If G is a nonabelian p-group, possessing an abelian maximal subgroup, then* $|G| = p \cdot |G'| \cdot |Z(G)|$.

PROOF. Let A be a maximal subgroup of G which is abelian, and $g \in G \setminus A$. The mapping $\varphi : A \to A, \varphi(a) = [a, g]$, is homomorphism with $\operatorname{Im} \varphi = G', \operatorname{Ker} \varphi = Z(G)$, and thus $A/Z(G) \cong G'$.

Therefore $|A| = |G| : p = |Z(G)| \cdot |G'|$ which yields to the above formula. □

3. PROOF OF THEOREM 1.1

Let $G \in \mathcal{MC}_2$, G being a p-group for some odd prime p. By definition of $\mathcal{MC}_2$, the group G contains some maximal subgroup $H \in \mathcal{MC}_1$, and by Theorem 2.8, H is of order p^3 or 3^4. Therefore $|G| = p^4$ or $|G| = 3^5$.

In representing groups by generator orders and commutators, we shall omit, for brevity, those commutators of generators which equal 1 (that is for pairs of commuting generators).

A. CASE $|G| = p^4$.

Obviously, for $G' = 1$, G must contain a subgroup H isomorphic to E_{p^3} and $\exp(G) \leq p^2$. Thus, in this case G is isomorphic either to

$$G_1 = \langle a, b, c, d \mid a^p = b^p = c^p = d^p = 1 \rangle \cong E_{p^4}, \text{ or}$$

$$G_2 = \langle a, b, c \mid a^{p^2} = b^p = c^p = 1 \rangle \cong Z_{p^2} \times E_{p^2}.$$

In the following we assume that $G' > 1$. By Theorem 2.8 either $H' = 1$ and $H = \langle a, b, c \rangle \cong E_{p^3}$, or $H' \neq 1$ and $H = \langle a, b, c \mid a^p = b^p = c^p = 1, [b, c] = a \rangle \cong Sp(p^3)$.

*A*1. $H' = 1$:

Now, $G = \langle H, d \mid d^p \in H \rangle$. By Theorem 2.10, $|G| = p \cdot |G'| \cdot |Z(G)| = p^4$ and so either $|Z(G)| = p^2, |G'| = p$, or $|Z(G)| = p, |G'| = p^2$.

*A*1.1 $|Z(G)| = p^2, |G'| = p$:

Here, we may assume $G' = \langle a \rangle < Z(G) = \langle a, b \rangle$, with $[c, d] = a$. Since $[d, d^p] = 1$, it is $d^p \in Z(G)$ and we have, without loss, three different possibilities: $d^p = 1, d^p = a^\alpha \neq 1$ and $d^p = b$. But if $d^p = a^\alpha$, then $[c^\alpha, d] = [c, d]^\alpha = a^\alpha$, as $[c, d] \in Z(G)$, and substituting a^α for a and c^α for c, we get $[c, d] = d^p = a$ in this case. Thus we obtain three different $\mathcal{MC}_2$-groups:

$$G_3 = \langle H, d \mid d^p = 1, [c, d] = a \rangle,$$
$$G_4 = \langle H, d \mid d^p = a, [c, d] = a \rangle,$$
$$G_5 = \langle H, d \mid d^p = b, [c, d] = a \rangle.$$

*A*1.2 $|G'| = p^2, |Z(G)| = p$:

We may assume that $Z(G) = \langle a \rangle < G' = \langle a, b \rangle$. Denoting $\overline{x} = x\langle a \rangle$ for $x \in G$, we see that $(G/\langle a \rangle)' = G'/\langle a \rangle = \langle \overline{b} \rangle = Z(G/\langle a \rangle)$, as $|G/\langle a \rangle| = p^3$ and

$G/\langle a\rangle$ is not abelian. Therefore, $b^d = a^\alpha b, \alpha \neq 0$ and $c^d = a^\gamma b^\beta c, \beta \neq 0$, as $b \in G' \setminus Z(G)$. Substituting a^α for a and $a^\gamma b^\beta$ for b, we get

$$G = \langle a, b, c, d \mid d^p \in Z(G), [b, d] = a, [c, d] = b\rangle.$$

We have two possibilities: $d^p = 1$ or $d^p = a^\delta, \delta \neq 0$. If $d^p = 1$, then we obtain

$$G_6 = \langle a, b, c, d \mid d^p = 1, [b, d] = a, [c, d] = b\rangle.$$

If $d^p = a^\delta$, then for $c' = c^\delta, b' = [c', d] = [c^\delta, d] = [c, d]^\delta = b^\delta, a' = [b', d] = [b^\delta, d] = [b, d]^\delta = a^\delta$, and we get, after substituting a', b', c' for a, b, c, the group

$$G_7 = \langle a, b, c, d \mid d^p = a, [b, d] = a, [c, d] = b\rangle.$$

*A*2. $H \cong Sp(p^3)$ and if $M < G$, then $M \ncong E_{p^3}$:

Now, $H = \langle a, b, c \mid a^p = b^p = c^p = 1, [b, c] = a\rangle, Z(H) = \langle a\rangle$.

Let $K \leq H, K \lhd G, |K| = p^2$. Then $K \cong E_{p^2}$ and $N_G(K)/C_G(K) = G/C_G(K) \leq AutK$. We may assume without loss that $K = \langle a, b\rangle$.

As $p^2 \nmid |AutK|$, by Proposition 2.5 it is $|G/C_G(K)| \leq p$ and there exists $A, A < G$, such that $K < A \leq C_G(K) \leq G$ and $|A| = p^3$. Since $K \leq Z(A)$, A is abelian. By assumption $A \ncong E_{p^3}$ and also $A \ncong Z_{p^3}$, as G is not metacyclic. Thus $A \cong Z_{p^2} \times Z_p$. From $\langle a\rangle$ *char* $H \lhd G$ it follows $\langle a\rangle \lhd G$ and so $\langle a\rangle \leq Z(G)$. Therefore $a \in K$, since otherwise $\langle a\rangle \times K \cong E_{p^3}$, contradicting our assumption. Now, we may assume without loss that $K = \langle a, b\rangle = \Omega_1(A)$. Because of $\mho_1(A)$ *char* $A \lhd G$, it is $\mho_1(A) \leq Z(G)$ too. But $Z(H) = \langle a\rangle$ only, and so $\mho_1(A) = \langle a\rangle$. Thus $A = \langle a, b, d \mid d^p = a^\alpha, \alpha \neq 0\rangle$, for some $d \in A \setminus K$. The group A is an abelian maximal subgroup in G, and by Theorem 2.10 it must be either $|Z(G)| = p^2, |G'| = p$, or $|Z(G)| = p, |G'| = p^2$ again.

*A*2.1 $|Z(G)| = p^2, |G'| = p$:

Now, $Z(G) \leq A$ because G is not abelian, and we may assume without loss that $d \in Z(G) = \langle a, d\rangle = \langle d\rangle$. Let γ be such that $\alpha\gamma \equiv 1 \pmod p$. Then $(d^\gamma)^p = (d^p)^\gamma = a^{\alpha\gamma} = a$, and substituting d by d^γ, we get the group

$$G_8 = \langle a, b, c, d \mid d^p = a, [b, c] = a\rangle.$$

*A*2.2 $|Z(G)| = p, |G'| = p^2$:

Now, $Z(H) = \langle a\rangle = Z(G) < G'$. Since G/K is abelian, it is $G' \leq K$ and so $G' = K = \langle a, b\rangle$. Clearly, $[c, d] \in K \setminus \langle a\rangle$, as $G' > \langle a\rangle$. Substituting $b' = [c, d]$ for b, and $a' = [b', c]$ for a, we get

$$G = \langle a, b, c, d \mid a^p = b^p = c^p = 1, [b, c] = a, [c, d] = b, d^p = a^\alpha\rangle,$$

for some $\alpha \neq 1$. Substituting c by $c' = c^\gamma, \gamma \neq 0$, we get $[c', d] = [c^\gamma, d] = b^\gamma a^\varepsilon = b'$, for some ε, since $(G/\langle a\rangle)' = \langle b, a\rangle/\langle a\rangle = Z(G/\langle a\rangle)$, and so $[c^\gamma, d] \equiv [c, d]^\gamma \equiv b^\gamma (\mathrm{mod}\ \langle a\rangle)$. Now $[b', c'] = [b^\gamma a^\varepsilon, c^\gamma] = [b^\gamma, c^\gamma] = a^{\gamma^2} = a'$. Substituting a', b', c' for a, b, c we get

$$G = \langle a, b, c, d \mid a^p = b^p = c^p = 1, [b, c] = a, [c, d] = b, d^p = a^{\frac{\alpha}{\gamma^2}}\rangle,$$

with $\gamma \neq 0$, arbitrarily choosable. If $\alpha \equiv \tau^2 \pmod p$, for some τ, replacing γ by τ we get

$$G_9 = \langle a, b, c, d \mid a^p = b^p = c^p = 1, [b, c] = a, [c, d] = b, d^p = a \rangle.$$

Otherwise, $\alpha \equiv \sigma\tau^2$, for some τ and, without loss, the minimal quadratic nonresidue σ modulo p. Replacing again γ by τ we get the group

$$G_{10} = \langle a, b, c, d \mid a^p = b^p = c^p = 1, [b, c] = a, [c, d] = b, d^p = a^\sigma \rangle,$$

σ being the minimal quadratic nonresidue modulo p.

B. CASE $|G| = 3^5$.

Now, G contains a maximal subgroup $H \in \mathcal{MC}_1, |H| = 3^4$. According to Theorem 2.8 the group H is of maximal class. As H is not metacyclic, its maximal subgroups cannot be cyclic. Thus, by Theorem 2.7 and Theorem 2.8, for each $L < H$ with $|L| = 3^3$, it is $exp(L) = 3^2, \mho_1(L) = Z(H) \cong Z_3, \Omega_1(L) = H' = \Phi(H) = \Omega_1(H) \cong E_9$, since $\mho_1(L), \Omega_1(L) \lhd H$, as $L \lhd H$. We set $H' = \langle a, b \mid a^3 = b^3 = 1 \rangle, Z(H) = \langle a \rangle$ for some $a, b \in H$. Because of $|AutE_9| = 2^4 \cdot 3$, we have $|H/C_H(H')| = 3$, and the group $A = C_H(H') = \langle a, b, c \rangle$, where $c \in A \setminus H'$, is abelian. Let K be an other maximal subgroup of H, $K = \langle a, b, d \rangle$. Now, $H = \langle A, K \rangle = \langle a, b, c, d \rangle$. If K is abelian, then $\langle a, b \rangle \leq Z(H)$, a contradiction. Thus $K' = Z(H) = \langle a \rangle$. We may assume without loss that $c^3 = d^3 = a$, as by Theorem 2.7 and Proposition 2.4, $\mho_1(K) = \langle a \rangle$ too. From $H = \langle \Phi(H), c, d \rangle$ and $[c, d] \in H' \setminus Z(H)$, as $H' > Z(H)$, it follows that we can set $[c, d] = b$ and $[b, d] = a^\alpha, \alpha \in \{1, 2\}$. Now, $(cd)^3 = cd^3c^{d^2}c^d = cacbba^\alpha cb = a^{\alpha+1}c^3 = a^{\alpha+2}$. Since $cd \in H \setminus \Omega_1(H)$, it is $(cd)^3 \neq 1$ implying $\alpha = 2$. Therefore,

$$H = \langle a, b, c, d \mid a^3 = b^3 = 1, c^3 = d^3 = a, [b, d] = a^2, [c, d] = b \rangle.$$

We proceed to determine G. Since $|G| = 3^5$ and all its subgroups of order 3^3 are metacyclic, we can apply Theorem 2.9. As there is none subgroup of order 3^3 and exponent 3 in G, it follows that G is also of maximal class. Now, Theorem 2.7 implies that $Z(G) = Z(H) = \langle a \rangle, Z_2(G) = H', G' = \Phi(G) = C_H(H') = A$, as all these groups are characteristic in H, which is normal in G. Moreover, $\Omega_1(G) = H'$, as $G \in \mathcal{MC}_2$ and $H' \cong E_9$. Consider now, $M = C_G(H') = \langle a, b, c, f \rangle$, for some $f \in G \setminus H$. As $Z(M) \geq H' \cong E_9$, and all maximal subgroups of M, being of order 3^3, are metacyclic, M must also be metacyclic, since otherwise it would be isomorphic to H, a contradiction because of $Z(H) \cong Z_3$. If $exp(M) = 3^3$, then we may choose f such that $\langle f \rangle \cong Z_{27}$ and there is some $x \in \langle a, b \rangle \leq Z(M)$ such that $x \notin \langle f \rangle$ and therefore $M = \langle f \rangle \times \langle x \rangle \cong Z_{27} \times Z_3$. Now, $\mho_1(M) = \langle f^3 \rangle \lhd G$, and $\langle f^3 \rangle \cong Z_9$, which contradicts Theorem 2.7, as $Z_2(G) \cong E_9$. Therefore $exp(M) = 3^2$, and $M \cong \langle x, y \mid x^9 = y^9 = 1, [x, y] \in \langle x^3 \rangle \rangle$, for some $x, y \in M$. Now, $M' \leq \langle x^3 \rangle \cong Z_3$ and $M' \lhd G$, which implies $M' \leq \langle a \rangle$. Since $c^3 = a$, it is $\langle c \rangle > M'$ and so $\langle c \rangle \lhd M$. Therefore, we may assume $c = x, f = y$ and $\mho_1(M) = \langle a, b \rangle$ implies $f^3 = a^\alpha b^\beta$, with $\beta \in \{1, 2\}$. Because of $G' = \langle a, b, c \rangle = \Phi(G)$, it is $G = \langle a, b, c, d, f\rangle = \langle d, f\rangle$, and $G' = \langle\langle [d, f]\rangle\rangle$ the normal closure of $\langle [d, f]\rangle$. Therefore $[d, f] \in A \setminus H'$, since otherwise $G' = H'$, a contradiction. Substituting f by f^2, if necessary, and substituting c by $[d, f]$ we get again $[b, d] = a^2$ for these new generators. Moreover, $[c, f] = a^\gamma, \gamma \in \{0, 1, 2\}$ because $[c, f] \in M' \leq \langle a\rangle$. Thus G can be written as:

$$\begin{aligned} G \quad = \quad & \langle a, b, c, d, f \mid a^3 = b^3 = 1, c^3 = d^3 = a, f^3 = a^\alpha b^\beta, \\ & [d, f] = c, [c, d] = b, [c, f] = a^\gamma, [b, d] = a^2\rangle. \end{aligned}$$

Now,

$$\begin{aligned} [d, f^3] \quad &= \quad [d, f][d, f^2]^f = [d, f][d, f]^f[d, f]^{f^2} \\ &= \quad c \cdot c^f \cdot c^{f^2} = c \cdot ca^\gamma \cdot ca^\gamma a^\gamma = c^3 = a, \end{aligned}$$

and

$$\begin{aligned} [d, f^3] \quad &= \quad [d, a^\alpha b^\beta] = [d, b^\beta] = [b^\beta, d]^{-1} \\ &= \quad (b^{-\beta}(b^\beta)^d)^{-1} = (b^{-\beta}b^\beta a^{2\beta})^{-1} = a^\beta, \end{aligned}$$

implying $\beta = 1$ and therefore $f^3 = a^\alpha b$.

Next, we calculate $(df)^3$ and $(d^2f)^3$, which should both be different from 1 :

$$\begin{aligned} (df)^3 \quad &= \quad df^3d^{f^2}d^f = da^\alpha b \cdot dcca^\gamma \cdot dc = d^2(a^\alpha b)^d c^2 a^\gamma dc \\ &= \quad a^{\alpha+\gamma}d^3(a^\alpha b)^{d^2}(c^2)^d c = a^{\alpha+\gamma} \cdot a \cdot ba \cdot c^2b^2c = a^{\alpha+\gamma}a^2 \cdot a = a^{\alpha+\gamma}, \\ (d^2f)^3 \quad &= \quad (d \cdot df)^3 = d(df)^3 d^{(df)^2} d^{df} = da^{\alpha+\gamma} \cdot dc \cdot ca^\gamma b \cdot dc \\ &= \quad a^{\alpha+2\gamma}d^3(c^2b)^d c = a^{\alpha+2\gamma+1}c^2b^2ba^2c = a^{\alpha+2\gamma+1}. \end{aligned}$$

Therefore, $\alpha + \gamma \not\equiv 0 \pmod 3), \alpha + 2\gamma + 1 \not\equiv 0 \pmod 3)$, which gives possibilities $(\alpha, \gamma) \in \{(1, 0), (1, 1), (0, 2), (2, 2)\}$. In the case $(\alpha, \gamma) = (1, 0)$, $M' = C_G(H') = \langle a, b, c, f\rangle' = 1$ and in other cases $M' > 1$. Replacing f by f^2, d by df, c by c^2, b by a^2b^2, and a by a^2, the case $(\alpha, \gamma) = (1, 1)$ goes over in $(\alpha, \gamma) = (0, 2)$, and replacing d by df, c by a^2c and b by a^2b the case $(\alpha, \gamma) = (2, 2)$ goes over in $(\alpha, \gamma) = (0, 2)$ as well. Thus there remain only the cases $(\alpha, \gamma) = (1, 0)$ and $(\alpha, \gamma) = (0, 2)$, giving the groups

$$\begin{aligned} G_{11} \quad = \quad & \langle a, b, c, d, f \mid a^3 = b^3 = 1, c^3 = d^3 = a, f^3 = ab, [d, f] = c, [c, d] = b, \\ & [b, d] = a^2\rangle \end{aligned}$$

and

$$\begin{aligned} G_{12} \quad = \quad & \langle a, b, c, d, f \mid a^3 = b^3 = 1, c^3 = d^3 = a, f^3 = b, [d, f] = c, [c, d] = b, \\ & [c, f] = a^2, [b, d] = a^2\rangle. \end{aligned}$$

Our Theorem 1.1 is proved.

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V. Ćepulić
Department of Mathematics
Faculty of Electrical Engineering and Computing
University of Zagreb
Unska 3, HR-10000 Zagreb
Croatia
E-mail: vladimir.cepulic@fer.hr

O. Pyliavska
National University Kyiv-Mohyla Academy
Skorovody 2, Kyiv 04070
Ukraine

E. Kovač Striko
Faculty of Transport and Traffic Engineering
University of Zagreb
Vukelićeva 4, HR-10000 Zagreb
Croatia
E-mail: elizabeta.kovac@fpz.hr

Received: 28.9.2005.