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# ISOPERIMETRIC INEQUALITIES FOR AN ELECTROSTATIC PROBLEM

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Abstract. We study the problem of the (p-)capacity $c_p$ of a multiconnected configuration $\Omega = (G\backslash E)\backslash(\cup H_i)$ when $\partial G$ and $\partial E$ have given potentials. Here $\Omega$ represents a nonhomogeneous medium and the $H_i$, which separate the different connected components of $\Omega$, represent perfect conductors. By comparison with a similar configuration with spherical symmetry, we give isoperimetric inequalities for $c_p$ and the unknown potentials on $H_i$.

## 1. Introduction

In a recent paper [6], V. Ferone gives an isoperimetric inequality for the (p-)capacity $c_p$ of a configuration $\Omega = (G\backslash E)\backslash(\cup H_i)$, where $\Omega$ represents a nonhomogeneous isotropic medium, $\partial G$ and $\partial E$ have given potentials respectively equal to 0 and 1, and the $H_i$ have constant unknown potentials $k_i$. He shows that $c_p$ is not smaller than the (p-)capacity $c_p^*$ of a symmetrical configuration which has no interior conductor such as $H_i$. In this paper, we complete Ferone's result when $\Omega$ is multiconnected and the $H_i$ separate the different connected component of $\Omega$, we show that $c_p \geqslant \widetilde{c}_p \geqslant c_p^*$, where $\widetilde{c}_p$ is the (p-)capacity of a natural symmetrized configuration (having inner conductors). We also give isoperimetric estimates for the unknown potentials $k_i$. Our proof, which is different from Ferone's one, uses the notion of relative rearrangement introduced by J. Mossino and R. Temam [8] and developed in [12, 13].

Let us now present the problem we want to study. Let $1 < p < +\infty$, let $\Omega = \Omega_1 \cup \cdots \cup \Omega_n$, where $\Omega_i$ $(i = 1, \ldots, n)$ has the form $\Omega_i = \omega_{i+1}\backslash$

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$\overline{\omega_i'}$, $\omega_i$ and $\omega_i'$ are regular bounded open sets in $\mathbb{R}^N$ $(N \geq 2)$ such that: for $i \in \{1, \ldots, n\}$, $\omega_i \subset \omega_i' \Subset \omega_{i+1}$. Let $\Omega_0 = \omega_1$, $C_i = \overline{\omega_i'} \setminus \omega_i$ $(i = 1, \ldots, n)$ (note that $C_i$ may have empty interior), $\gamma_i = \partial\omega_i$ and $\gamma_i' = \partial\omega_i'$ $(i = 1, \ldots, n+1)$. Let $A(x) = (a_{ij}(x))_{i,j=1,\ldots,N}$ be a symmetric matrix such that:

$$(A(x)\xi, \xi) \geqslant a(x)|\xi|^2 \qquad \text{for a.e } x \in \Omega \text{ and for every } \xi \in \mathbb{R}^N, \tag{1.1}$$

where $(\cdot,\cdot)$ denotes the scalar product in $\mathbb{R}^N$, $|\cdot|$ denotes the Euclidian norm and $a : \Omega \to \mathbb{R}$ is a (a.e.) positive function (a condition which can be found in [9]) such that:

$$a \in L^{\frac{p}{2}}(\Omega), a^{\frac{-p}{2}} \in L^r(\Omega), r \geqslant \frac{N}{p} \text{ and } 1 + \frac{1}{r} < p < N\left(1 + \frac{1}{r}\right). \tag{1.2}$$

We suppose that for every $i, j \in \{1, \ldots, N\}$:

$$a_{ij}a^{-1} \in L^\infty(\Omega). \tag{1.3}$$

Let $W_a^{1,p}(\Omega)$ be the completion of $C^1\left(\overline{\Omega}\right)$ for the norm:

$$\|v\|_a^{1,p} = \left(\int_\Omega |v|^p \, dx\right)^{\frac{1}{p}} + \left(\int_\Omega a^{\frac{p}{2}} |\nabla v|^p \, dx\right)^{\frac{1}{p}}.$$

We consider $\Omega$ as a nonhomogeneous and anisotropic medium and we define a generalized (p-)capacity of $\Omega$ as the infimum of the following problem:

$$\inf\left\{\int_\Omega (A\nabla v, \nabla v)^{\frac{p}{2}} \, dx, \ v \in H\right\} \tag{1.4}$$

where

$$H = \left\{ \begin{array}{l} v \in W_a^{1,p}(\Omega) : v = 1 \text{ on } \gamma_1', v = 0 \text{ on } \gamma_{n+1} \text{ and} \\ v = k_i \text{ (undetermined constant) on } \partial C_i = \gamma_i \cup \gamma_i' \text{ for } i = 2, \ldots, n \end{array} \right\}.$$

In the following, denote by $u_i = u|_{\Omega_i}$ and $a_i = a|_{\Omega_i}$ $(i = 1, \ldots, n)$.

## 2. STUDY OF PROBLEM (1.4)

In this section we study the existence, uniqueness and characterization of solution of problem (1.4).

THEOREM 2.1. *Problem (1.4) admits one solution and one only.*

PROOF. Let $(u_n)$ be a minimizing sequence:

$$(u_n) \in H \text{ and } J(u_n) \to I = \inf\{J(v), v \in H\}$$

where

$$J(v) = \int_\Omega (A\nabla v, \nabla v)^{\frac{p}{2}} \, dx.$$

We have due to the coerciveness condition in (1.1):

$$c \geq J(u_n) \geqslant \int_\Omega a^{\frac{p}{2}} |\nabla u_n|^p \, dx \qquad \text{(where } c \text{ is a constant)}$$

hence, $(u_n)$ is bounded in $H$ (by using Poincaré's inequality) and then, up to extraction of a subsequence, we can suppose that:

$$u_n \rightharpoonup u \qquad \text{in } \ W_a^{1,p}(\Omega).$$

But $J$ is weakly l.s.c. Then

$$J(u) \leqslant \liminf_{n\to\infty} J(u_n) \leqslant \lim_{n\to\infty} J(u_n) = I \leqslant J(u)$$

and then by the strict convexity of $J$ and because $H$ is closed we deduce that $u$ is the unique solution of the problem (1.4). ☐

Remark 2.2. Let $u$ be the solution of (1.4). It is classical that $u$ is characterized by the variational formulation:

$$\int_\Omega (A\nabla u, \nabla u)^{\frac{p}{2}-1} (A\nabla u, \nabla \phi)\, dx = 0, \ \forall \phi \in H_0, \tag{1.5}$$

where

$$H_0 = \left\{ \begin{array}{l} v \in W_a^{1,p}(\Omega) : v = 0 \text{ on } \gamma_1' \cup \gamma_{n+1} \text{ and} \\ v = k_i \text{ (undetermined constant) on } \gamma_i \cup \gamma_i' \text{ for } i = 2, \ldots, n \end{array} \right\}$$

and then

$$\mathcal{A}u = \operatorname{div}\left[(A\nabla u, \nabla u)^{\frac{p}{2}-1} A\nabla u\right] = 0 \qquad \text{in } D^{'}(\Omega). \tag{1.6}$$

We deduce then the following proposition:

Proposition 2.3. *Let $u$ be the solution of (1.4), $k_1 = 1$, $k_{n+1} = 0$ and $k_i$ the value of $u$ on $\gamma_i \cup \gamma_i'$ $(i = 2, \ldots, n)$. Then we have:*

$$\begin{cases} \mathcal{A}u_i = 0 & \text{in } \ D'(\Omega_i) \\ u_1 = 1 & \text{on } \ \gamma_1' \\ u_n = 0 & \text{on } \ \gamma_{n+1} \\ u_i = k_i \text{ (unprescribed constant)} & \text{on } \ \gamma_i', \quad i = 2, \ldots, n \\ u_i = k_{i+1} \text{ (unprescribed constant)} & \text{on } \ \gamma_{i+1}, \quad i = 1, \ldots, n-1. \end{cases} \tag{1.7}$$

Lemma 2.4. *Let $v_i$ be the unique solution of the following problem:*

$$\inf\left\{\int_{\Omega_i} (A\nabla v, \nabla v)^{\frac{p}{2}}\, dx, \ v \in K_i\right\}$$

*where $K_i = \left\{v \in W_{a_i}^{1,p}(\Omega_i) \ : \ v = 1 \text{ on } \gamma_i', \ v = 0 \text{ on } \gamma_{i+1}\right\}$. Then $v_i$ satisfy:*

$$\int_{\Omega_i} (A\nabla v_i, \nabla v_i)^{\frac{p}{2}-1} (A\nabla v_i, \nabla \varphi)\, dx = 0\,, \forall \varphi \in K_i^0 \tag{1.8}$$

*where $K_i^0 = \left\{v \in W_{a_i}^{1,p}(\Omega_i) : v = 0 \text{ on } \gamma_i' \cup \gamma_{i+1}\right\}$, and hence,*

$$\mathcal{A}v_i = 0 \quad \text{in} \quad D'(\Omega_i). \tag{1.9}$$

Theorem 2.5. *Let us set $c_p = \displaystyle\int_\Omega (A\nabla u, \nabla u)^{\frac{p}{2}}\, dx$, let $u$ being the solution of the problem (1.4) and $v_i$ the function defined in Lemma 2.4. Then we have:*

a) $c_p > 0\,,$

b) $k_i \neq k_{i+1}, \qquad \forall i = 1, \ldots, n\,,$

c) $u_i = (k_i - k_{i+1})v_i + k_{i+1},\ \forall i = 1, \ldots, n\,,$

d) $$0 < k_i - k_{i+1} = \left( \frac{\displaystyle\int_{\Omega_i} (A\nabla v_i, \nabla v_i)^{\frac{p}{2}}\, dx}{c_p} \right)^{\frac{-1}{p-1}} , \forall i = 1, \ldots, n\,,$$

e) $$c_p = \left[ \sum_{i=1}^{n} \left( \int_{\Omega_i} (A\nabla v_i, \nabla v_i)^{\frac{p}{2}}\, dx \right)^{\frac{-1}{p-1}} \right]^{-(p-1)} ,$$

f) $$k_i = 1 - (c_p)^{\frac{1}{p-1}} \sum_{j=1}^{i-1} \left( \int_{\Omega_j} (A\nabla v_j, \nabla v_j)^{\frac{p}{2}}\, dx \right)^{\frac{-1}{p-1}}$$
$$= 1 - \frac{\displaystyle\sum_{j=1}^{i-1} \left( \int_{\Omega_j} (A\nabla v_j, \nabla v_j)^{\frac{p}{2}}\, dx \right)^{\frac{-1}{p-1}}}{\displaystyle\sum_{j=1}^{n} \left( \int_{\Omega_j} (A\nabla v_j, \nabla v_j)^{\frac{p}{2}}\, dx \right)^{\frac{-1}{p-1}}} \ .$$

PROOF. a) If $c_p = 0$ then $u =$ constant on $\Omega_i$ $(i = 1, \ldots, n)$ and by using transmission conditions (because $u \in H$) we obtain contradiction.

b) Suppose that: there exists $j \in \{1, \ldots, n\}$ such that $k_j = k_{j+1}$. Let

$$w = \begin{cases} k_j & \text{in} \quad \Omega_j \\ u & \text{otherwise} \end{cases} .$$

Then

$$w \neq u, w \in H \text{ and } \int_{\Omega} (A\nabla w, \nabla w)^{\frac{p}{2}}\, dx < \int_{\Omega} (A\nabla u, \nabla u)^{\frac{p}{2}}\, dx,$$

contradiction with $u$ is the solution of the problem (1.4).

c) Let $w_i = \dfrac{u_i - k_{i+1}}{k_i - k_{i+1}}$ $(i = 1, \ldots, n)$. Then from (1.7) we have

$$\begin{cases} \mathcal{A} w_i = 0 & \text{in} \quad D'(\Omega_i) \\ w_i = 1 & \text{on} \quad \gamma_i' \\ w_i = 0 & \text{on} \quad \gamma_{i+1} \end{cases}$$

hence $v_i = w_i$ and then $u_i = (k_i - k_{i+1})v_i + k_{i+1}$.

d) Let

$$w_i = \begin{cases} v_i & \text{in} \quad \Omega_i \\ \text{constant} & \text{otherwise} \end{cases} \quad (i = 1, \ldots, n)$$

be such that $w_i \in H$. Then $w_i - u \in H_0$ and, by using (1.5), we have

$$c_p = \int_{\Omega_i} (A\nabla u_i, \nabla u_i)^{\frac{p}{2}-1} (A\nabla u_i, \nabla v_i)\, dx,$$

and from c) we obtain the result.

e) By using $\sum_{i=1}^{n}(k_i - k_{i+1}) = 1$ and d).

f) By using $k_i = 1 - \sum_{j=1}^{i-1}(k_j - k_{j+1})$, d) and e). □

Remark 2.6. $0 = k_{n+1} < k_n < \cdots < k_1 = 1$.

Remark 2.7. If Green's formula is valid, then we have for all $i \in \{2, \dots, n\}$, see [4]:

$$\begin{aligned} 0 < c_p &= -\int_{\gamma_1'} \frac{\partial u_1}{\partial \nu^{\mathcal{A}}} d\gamma = -\int_{\gamma_i'} \frac{\partial u_i}{\partial \nu^{\mathcal{A}}} d\gamma = \cdots \\ &= -\int_{\gamma_{n+1}} \frac{\partial u_n}{\partial \nu^{\mathcal{A}}} d\gamma = -\int_{\gamma_i} \frac{\partial u_{i-1}}{\partial \nu^{\mathcal{A}}} d\gamma, \end{aligned}$$

where $\dfrac{\partial u}{\partial \nu^{\mathcal{A}}} = (A\nabla u, \nabla u)^{\frac{p}{2}-1} (A\nabla u, \nu)$ and $\nu$ be the normal to $\gamma_i$ pointing outside $\Omega_{i-1}$ $(i = 2, \dots, n+1)$.

## 3. Inequalities

Let us recall some notions of rearrangement (see for example, [2, 5, 7, 8, 12, 13]). In this paper, we use only the Lebesgue measure on $\mathbb{R}^N$. Let $|E|$ be the measure of a measurable set $E$. Let $u$ be a measurable function from $\Omega$ into $\mathbb{R}$ where $\Omega$ is a measurable set in $\mathbb{R}^N$.

The (unidimensional) decreasing rearrangement $u_*$ of $u$ is defined on $\overline{\Omega^*} = [0, |\Omega|]$ by:

$$u_*(s) = \begin{cases} \inf\{\alpha \in \mathbb{R} \;/\; |u > \alpha| \leqslant s\} & \text{if } s < |\Omega| \\ \operatorname{essinf}_\Omega u & \text{if } s = |\Omega| \end{cases}$$

where $|u > \alpha| = |\{x \in \Omega : \, u(x) > \alpha\}|$.

The increasing rearrangement of $u$, denoted $u^*$, is defined by $u^*(s) = u_*(|\Omega| - s)$. The functions $u$, $u_*$ and $u^*$ satisfy $|u > \alpha| = |u_* > \alpha| = |u^* > \alpha|$.

For $v$ in $L^1(\Omega)$ and for a measurable function $u$ from $\Omega$ into $\mathbb{R}$, we define the function $\mathcal{W}$ on $\overline{\Omega^*}$ by:

$$\mathcal{W}(s) = \begin{cases} \displaystyle\int_{u > u_*(s)} v(x)dx & \text{if } |u = u_*(s)| = 0 \\[2ex] \displaystyle\int_{u > u_*(s)} v(x)dx + \int_0^{s - |u > u_*(s)|} (v|_{P(s)})_*(\sigma)\, d\sigma & \text{otherwise,} \end{cases}$$

where $(v|_{P(s)})_*$ is the decreasing rearrangement of $v$ restricted to $P(s) = \{x \in \Omega : u(x) = u_*(s)\}$. The integrable function $\dfrac{d\mathcal{W}}{ds}$ denoted $v_{*_u}$ is called the relative rearrangement of $v$ with respect to $u$.

All the isoperimetric inequalities of this section are consequences of the following theorem.

Theorem 3.1. *Let $p'$ be such that $\frac{1}{p}+\frac{1}{p'}=1$, $\alpha_N$ be the measure of the unit ball in $\mathbb{R}^N$ and $v_i$ $(i=1,\dots,n)$ be the function defined in Lemma 2.4. Then for all $t, t'$ such that $0 \leqslant t \leqslant t' \leqslant 1$ we have*

$$t'-t \leqslant N^{-p'}(\alpha_N)^{\frac{-p'}{N}}\left(\int_{\Omega_i}(A\nabla v_i,\nabla v_i)^{\frac{p}{2}}\,dx\right)^{\frac{p'}{p}} \times \int_{|v_i>t'|}^{|v_i>t|}(|\omega_i'|+\sigma)^{\frac{p'}{N}-p'}\,(a_i^*)^{\frac{-p'}{2}}\,(\sigma-|v_i>t'|)d\sigma.$$

From theorems 2.5 and 3.1 we deduce the following corollary.

Corollary 3.2. *Let $i \in \{1,\dots,n\}$. For all $t$ and $t'$ such that $k_{i+1} \leqslant t \leqslant t' \leqslant k_i$ we have*

$$t'-t \leqslant N^{-p'}(\alpha_N)^{\frac{-p'}{N}}(c_p)^{\frac{p'}{p}} \times \int_{|u_i>t'|}^{|u_i>t|}(|\omega_i'|+\sigma)^{\frac{p'}{N}-p'}\,(a_i^*)^{\frac{-p'}{2}}\,(\sigma-|u_i>t'|)d\sigma. \tag{3.1}$$

From (3.1), for $t=k_{i+1}$ and $t'=k_i$ we deduce:

Corollary 3.3.

$$k_i-k_{i+1} \leqslant N^{-p'}(\alpha_N)^{\frac{-p'}{N}}(c_p)^{\frac{p'}{p}}\int_0^{|\Omega_i|}(|\omega_i'|+\sigma)^{\frac{p'}{N}-p'}\,(a_i^*)^{\frac{-p'}{2}}\,(\sigma)d\sigma.$$

Proof of Theorem 3.1. For $\theta \in [0,1]$, let us set

$$Z_i = \theta-(v_i-\theta)_- = \begin{cases} v_i & \text{if } v_i<\theta \\ \theta & \text{if } v_i \geqslant \theta \end{cases}.$$

We have $Z_i-\theta v_i \in K_i^0$ and then by using (1.8) we obtain:

$$\int_{v_i<\theta}(A\nabla v_i,\nabla v_i)^{\frac{p}{2}}dx = \theta\int_{\Omega_i}(A\nabla v_i,\nabla v_i)^{\frac{p}{2}}dx$$

hence,

$$\frac{d}{d\theta}\int_{v_i>\theta}(A\nabla v_i,\nabla v_i)^{\frac{p}{2}}dx = -\int_{\Omega_i}(A\nabla v_i,\nabla v_i)^{\frac{p}{2}}dx.$$

Moreover, by using (1.1), (1.2) and Hölder's inequality, we have for $h>0$ :

$$\frac{1}{h}\int_{\theta<v_i\leqslant\theta+h}|\nabla v_i|\,dx \leqslant \left[\frac{1}{h}\int_{\theta<v_i\leqslant\theta+h}{a_i}^{-\frac{p'}{2}}dx\right]^{\frac{1}{p'}}\left[\frac{1}{h}\int_{\theta<v_i\leqslant\theta+h}a_i^{\frac{p}{2}}\,|\nabla v_i|^p\,dx\right]^{\frac{1}{p}}$$

$$\leq \left[\frac{1}{h}\int_{\theta<v_i\leqslant\theta+h} {a_i}^{-\frac{p'}{2}} dx\right]^{\frac{1}{p'}} \left[\frac{1}{h}\int_{\theta<v_i\leqslant\theta+h} (A\nabla v_i, \nabla v_i)^{\frac{p}{2}} dx\right]^{\frac{1}{p}}$$

and letting $h$ tend to 0,

$$\begin{aligned} -\frac{d}{d\theta}\int_{v_i>\theta} |\nabla v_i|\, dx &\leqslant \left(-\frac{d}{d\theta}\int_{v_i>\theta} {a_i}^{-\frac{p'}{2}} dx\right)^{\frac{1}{p'}} \left(-\frac{d}{d\theta}\int_{v_i>\theta} (A\nabla v_i, \nabla v_i)^{\frac{p}{2}} dx\right)^{\frac{1}{p}} \\ &= \left(-\frac{d}{d\theta}\int_{v_i>\theta} {a_i}^{-\frac{p'}{2}} dx\right)^{\frac{1}{p'}} \left(\int_{\Omega_i} (A\nabla v_i, \nabla v_i)^{\frac{p}{2}} dx\right)^{\frac{1}{p}}. \end{aligned}$$

On the other hand, by using the following formula of derivation (see [13]) we have

$$\frac{d}{d\theta}\int_{v_i>\theta} a_i^{-\frac{p'}{2}} dx = \mathcal{W}'(\nu_i(\theta)){\nu_i}'(\theta)$$

where $\nu_i(\theta) = |v_i > \theta|$ and $\dfrac{d\mathcal{W}}{ds} = \left(a_i^{-\frac{p'}{2}}\right)_{*_{v_i}}$ is the relative rearrangement of $a_i^{-\frac{p'}{2}}$ with respect to $v_i$.

Hence,

$$-\frac{d}{d\theta}\int_{v_i>\theta} |\nabla v_i|\, dx \leqslant (-\mathcal{W}'(\nu_i(\theta))\nu_i'(\theta))^{\frac{1}{p'}} \left(\int_{\Omega_i} (A\nabla v_i, \nabla v_i)^{\frac{p}{2}}\, dx\right)^{\frac{1}{p}}.$$

Moreover, classically, by using theorems of De Georgi and Fleming-Rishel (see for example [7]) we have

$$-\frac{d}{d\theta}\int_{v_i>\theta} |\nabla v_i|\, dx \geqslant N\,(\alpha_{_N})^{\frac{1}{N}}\,(|\omega_i'| + \nu_i(\theta))^{1-\frac{1}{N}}$$

and then

$$N\,(\alpha_{_N})^{\frac{1}{N}}\,(|\omega_i'| + \nu_i(\theta))^{1-\frac{1}{N}} \leqslant (-\mathcal{W}'(\nu_i(\theta))\nu_i'(\theta))^{\frac{1}{p'}} \left(\int_{\Omega_i} (A\nabla v_i, \nabla v_i)^{\frac{p}{2}}\, dx\right)^{\frac{1}{p}}.$$

Hence,

$$\begin{aligned} 1 \quad &\leqslant \quad -N^{-p'}\,(\alpha_{_N})^{\frac{-p'}{N}} \left(\int_{\Omega_i} (A\nabla v_i, \nabla v_i)^{\frac{p}{2}}\, dx\right)^{\frac{p'}{p}} \\ &\qquad \times (|\omega_i'| + \nu_i(\theta))^{\frac{p'}{N}-p'}\, \mathcal{W}'(\nu_i(\theta))\nu_i'(\theta). \end{aligned}$$

By integrating between $t$ and $t'$ we have

$$\begin{aligned} t' - t \quad &\leqslant \quad N^{-p'}\,(\alpha_{_N})^{\frac{-p'}{N}} \left(\int_{\Omega_i} (A\nabla v_i, \nabla v_i)^{\frac{p}{2}}\, dx\right)^{\frac{p'}{p}} \\ &\qquad \times \int_0^{|\Omega_i|} 1_{[\nu_i(t'),\nu_i(t)]}(\sigma)\,(|\omega_i'| + \sigma)^{\frac{p'}{N}-p'} \left(a_i^{-\frac{p'}{2}}\right)_{*_{v_i}} (\sigma) d\sigma \end{aligned}$$

and by using in [12, Theorem 3] we have

$$\begin{aligned} t'-t &\leqslant N^{-p'}\left(\alpha_{_N}\right)^{\frac{-p'}{N}}\left(\int_{\Omega_i}(A\nabla v_i,\nabla v_i)^{\frac{p}{2}}\,dx\right)^{\frac{p'}{p}} \\ &\quad\times\int_0^{|\Omega_i|}\left(1_{[\nu_i(t'),\nu_i(t)]}(\cdot)\left(|\omega_i'|+\cdot\right)^{\frac{p'}{N}-p'}\right)_*(\sigma)\left(a_i^{-\frac{p'}{2}}\right)_*(\sigma)d\sigma \\ &= N^{-p'}\left(\alpha_{_N}\right)^{\frac{-p'}{N}}\left(\int_{\Omega_i}(A\nabla v_i,\nabla v_i)^{\frac{p}{2}}\,dx\right)^{\frac{p'}{p}} \\ &\quad\times\int_0^{|\Omega_i|}1_{[0,\nu_i(t)-\nu_i(t')]}(\sigma)\left(|\omega_i^{'}|+\nu_i(t')+\sigma\right)^{\frac{p'}{N}-p'}(a_i^*)^{\frac{-p'}{2}}(\sigma)d\sigma \\ &= N^{-p'}\left(\alpha_{_N}\right)^{\frac{-p'}{N}}\left(\int_{\Omega_i}(A\nabla v_i,\nabla v_i)^{\frac{p}{2}}\,dx\right)^{\frac{p'}{p}} \\ &\quad\times\int_0^{\nu_i(t)-\nu_i(t')}\left(|\omega_i^{'}|+\nu_i(t')+\sigma\right)^{\frac{p'}{N}-p'}(a_i^*)^{\frac{-p'}{2}}(\sigma)d\sigma. \end{aligned}$$

Therefore,

$$\begin{aligned} t'-t &\leqslant N^{-p'}(\alpha_{_N})^{\frac{-p'}{N}}\left(\int_{\Omega_i}(A\nabla v_i,\nabla v_i)^{\frac{p}{2}}\,dx\right)^{\frac{p'}{p}} \\ &\quad\times\int_{|v_i>t'|}^{|v_i>t|}(|\omega_i'|+\sigma)^{\frac{p'}{N}-p'}(a_i^*)^{\frac{-p'}{2}}(\sigma-|v_i>t'|)d\sigma \end{aligned}$$

for all $t,t'$ such that $0\leqslant t\leqslant t'\leqslant 1$. ◻

## 4. Symmetrized problem

Let $i\in\{1,\dots,n+1\}$, let $\widetilde{\omega}_i$ (resp. $\widetilde{\omega}_i'$) the ball of $\mathbb{R}^N$ centered at the origin such that $|\widetilde{\omega}_i|=|\omega_i|$ (resp. $|\widetilde{\omega}_i'|=|\omega_i'|$). Let $\Lambda_0=\widetilde{\omega}_1$, $\Lambda_i{=}\widetilde{\omega}_{i+1}\backslash\,\overline{\widetilde{\omega}_i'}$ $(i=1,\dots,n)$ and $\Lambda=\Lambda_1\cup\cdots\cup\Lambda_n$. Let $\widetilde{\gamma}_i=\partial\,\widetilde{\omega}_i$ and $\widetilde{\gamma}_i'=\partial\,\widetilde{\omega}_i'$ for $i\in\{1,\dots,n+1\}$. Let $\mu$ be the normal to $\widetilde{\gamma}_i$ pointing outside $\Lambda_{i-1}$ $(i=1,\dots,n+1)$.

Let $\widetilde{a}$ be the function defined by $\widetilde{a}(x)=a_i^*(\alpha_N|x|^N-|\omega_i'|)$ in $\Lambda_i$, where $a_i^*$ is the increasing rearrangement of $a_i$, $i\in\{1,\dots,n\}$. We consider the symmetrized problem defined as follows:

$$\inf\left\{\int_\Lambda\widetilde{a}^{\frac{p}{2}}\,|\nabla V|^p\,dx,\ V\in\widetilde{H}\right\} \tag{4.1}$$

where

$$\widetilde{H}=\left\{\begin{array}{l} V\in W^{1,p}_{\widetilde{a}}(\Lambda)\ :\ V=1\text{ on }\widetilde{\gamma}_1',V=0\text{ on }\widetilde{\gamma}_{n+1}\text{ and} \\ V=K_i\text{ (undetermined constant) on }\widetilde{\gamma}_i\cup\widetilde{\gamma}_i'\text{ for }i=2,\dots,n \end{array}\right\}.$$

REMARK 4.1. As the function $a$, $\widetilde{a}$ also satisfies (1.2) (with $\Lambda$ instead of $\Omega$), then from Theorem 2.1 the symmetrized problem (4.1) has unique solution.

PROPOSITION 4.2. *For* $j \in \{1, \dots, n\}$, *let us set*

$$f_j(\sigma) = \left(|\omega'_j| + \sigma\right)^{\frac{p'}{N} - p'} \left(a_j^*\right)^{\frac{-p'}{2}}(\sigma) \text{ for } \sigma \in [0, |\Omega_j|], I_j = \int_0^{|\Omega_j|} f_j(\sigma) d\sigma,$$

*and* $V_j$ *be the solution of the following problem:*

$$\begin{cases} \mathcal{B}V_j = 0 & \text{in } \Lambda_j \\ V_j = 0 & \text{on } \widetilde{\gamma}_{j+1} \\ V_j = 1 & \text{on } \widetilde{\gamma}'_j \end{cases}$$

*where* $\mathcal{B}V = -\operatorname{div}(\widetilde{a}^{\frac{p}{2}} |\nabla V|^{p-2} \nabla V)$. *Then we have*

a) $\quad V_j(x) = (I_j)^{-1} \displaystyle\int_{\alpha_N |x|^N - |\omega'_j|}^{|\Omega_j|} f_j(s) ds \qquad$ *for* $x \in \Lambda_j$ ,

b) $\quad -\displaystyle\int_{\widetilde{\gamma}'_j} \frac{\partial V_j}{\partial \mu^{\mathcal{B}}}\, d\gamma = N^p (\alpha_N)^{\frac{p}{N}} (I_j)^{1-p}$

*where* $\dfrac{\partial V}{\partial \mu^{\mathcal{B}}} = \widetilde{a}^{\frac{p}{2}} |\nabla V|^{p-2} \dfrac{\partial V}{\partial \mu}$ , $\dfrac{\partial V}{\partial \mu}$ *is the normal derivative.*

REMARK 4.3. A similar computation shows that one has also

$$-\int_{\widetilde{\gamma}_j} \frac{\partial V_j}{\partial \mu^{\mathcal{B}}}\, d\gamma = N^p (\alpha_N)^{\frac{p}{N}} (I_j)^{1-p}$$

that is Green's formula is valid for $V_j$ and it follows from c) of Theorem 2.5 that it is also valid for $U_j$ $(j = 1, \dots, n)$, where $U_j = U|\Lambda_j$ and $U$ is the solution of the symmetrized problem (4.1).

PROOF OF PROPOSITION 4.2. a) We have

$$-\mathcal{B}V_i = \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \left( (a_i^*)^{\frac{p}{2}} \left(\alpha_N |x|^N - |\omega'_i|\right) |\nabla V_i|^{p-2} \frac{\partial V_i}{\partial x_j} \right).$$

With $r = |x|$ we obtain

$$\begin{aligned} -\mathcal{B}V_i \;\; = \;\; & (-1)^p \, (a_i^*)^{\frac{p}{2}-1} \left(\alpha_N r^N - |\omega'_i|\right) \left(\frac{dV_i}{dr}\right)^{p-2} \\ & \times \left( \frac{p}{2} \frac{d}{dr} a_i^*(\alpha_N r^N - |\omega'_i|) \frac{dV_i}{dr} \right. \\ & \qquad \left. + (p-1) a_i^*(\alpha_N r^N - |\omega'_i|) \frac{d^2 V_i}{dr^2} + \frac{N-1}{r} a_i^*(\alpha_N r^N - |\omega'_i|) \frac{dV_i}{dr} \right) \end{aligned}$$

and then

$$\mathcal{B}V_i = 0 \quad \Longleftrightarrow \quad \frac{dV_i}{dr} = k r^{\frac{N-1}{1-p}} \, (a_i^*)^{\frac{-p'}{2}} \left(\alpha_N r^N - |\omega'_i|\right)$$

where $k$ is constant. Hence,

$$V_i = k' \int_r^{(\alpha_{_N})^{\frac{-1}{N}} |\omega_{i+1}|^{\frac{1}{N}}} s^{\frac{N-1}{1-p}} \, (a_i^*)^{\frac{-p'}{2}} \, (\alpha_{_N} s^N - |\omega_i'|) ds + k''$$

where $k'$ and $k''$ are constants. With $\sigma = \alpha_{_N} s^N - |\omega_i'|$ we have for $x \in \Lambda_i$,

$$V_i(x) = C \int_{\alpha_{_N} |x|^N - |\omega_i'|}^{|\Omega_i|} (\sigma + |\omega_i'|)^{\frac{p'}{N} - p'} \, (a_i^*)^{\frac{-p'}{2}} \, (\sigma) \, d\sigma + D$$

where $C$ and $D$ are constants.

Since $V_i = 0$ on $\widetilde{\gamma}_{i+1}$ and $V_i = 1$ on $\widetilde{\gamma}_i'$, we deduce that $D = 0$, $C = \dfrac{1}{I_i}$ and then

$$V_i(x) = \frac{1}{I_i} \int_{\alpha_{_N} |x|^N - |\omega_i'|}^{|\Omega_i|} (\sigma + |\omega_i'|)^{\frac{p'}{N} - p'} \, (a_i^*)^{\frac{-p'}{2}} \, (\sigma) \, d\sigma \quad \text{for all } x \in \Lambda_i.$$

b) We have $\dfrac{\partial V_i}{\partial \mu^{\mathcal{B}}} = \displaystyle\sum_{j=1}^{N} \widetilde{a}^{\frac{p}{2}} \, |\nabla V_i|^{p-2} \, \frac{\partial V_i}{\partial x_j} \mu^j$ where $\mu^j$ is the $j$th component of $\mu$, then from a) we deduce

$$\frac{\partial V_i}{\partial \mu^{\mathcal{B}}} = -N^{p-1} (\alpha_{_N})^{\frac{p}{N} - 1} |x|^{-N} (I_i)^{1-p} \sum_{j=1}^{N} x_j \mu^j$$

and

$$-\int_{\widetilde{\gamma}_i'} \frac{\partial V_i}{\partial \mu^{\mathcal{B}}} \, d\gamma = N^p (\alpha_{_N})^{\frac{p}{N}} (I_i)^{1-p}$$

(because $\int_{\widetilde{\gamma}_i'} x_j \mu^j \, d\gamma = |\widetilde{\omega}_i'| = |\omega_i'|$ and if $x \in \widetilde{\gamma}_i'$ we have $\alpha_{_N} |x|^N = |\widetilde{\omega}_i'| = |\omega_i'|$). $\square$

THEOREM 4.4 (Explicit resolution of symmetrized problem). *Let us set* $\widetilde{c}_p = \displaystyle\int_\Lambda \widetilde{a}^{\frac{p}{2}} \, |\nabla U|^{\frac{p}{2}} \, dx$ *where* $U$ *being the solution of the symmetrized problem (4.1),* $K_i$ *be the value of* $U$ *on* $\widetilde{\gamma}_i \cup \widetilde{\gamma}_i'$ $(i = 2, \ldots, n)$ *and* $U_i = U|\Lambda_i$ $(i = 1, \ldots, n)$*, then we have:*

a) $K_i = 1 - \left( \displaystyle\sum_{j=1}^{i-1} I_j \right) \left( \displaystyle\sum_{j=1}^{n} I_j \right)^{-1}, \qquad i \in \{2, \ldots, n\}\,;$

b) $(\widetilde{c}_p)^{\frac{-p'}{p}} = N^{-p'} (\alpha_{_N})^{\frac{-p'}{N}} \displaystyle\sum_{j=1}^{n} I_j\,;$

c) *for* $i \in \{1, \ldots, n\}$ *and* $x \in \Lambda_i$,

$$U_i(x) = K_i - N^{-p'} (\alpha_{_N})^{\frac{-p'}{N}} (\widetilde{c}_p)^{\frac{p'}{p}} \int_0^{\alpha_{_N} |x|^N - |\omega_i'|} f_i(\sigma) d\sigma\,.$$

Proof. Using Theorem 2.5, Proposition 4.2 and Remark 4.3 we have:

a) $(\widetilde{c}_p)^{\frac{-1}{1-p}} = \displaystyle\sum_{j=1}^{n}\left(-\int_{\widetilde{\gamma}'_j}\frac{\partial V_j}{\partial \mu^{\mathcal{B}}}d\gamma\right)^{\frac{-1}{1-p}} = N^{-p'}(\alpha_{_N})^{\frac{-p'}{N}}\sum_{j=1}^{n} I_j\,;$

b) $K_i = 1-(\widetilde{c}_p)^{\frac{1}{1-p}}\displaystyle\sum_{j=1}^{i-1}\left(-\int_{\widetilde{\gamma}_j}\frac{\partial V_j}{\partial \mu^{\mathcal{B}}}d\gamma\right)^{\frac{-1}{1-p}} = 1-\left(\sum_{j=1}^{i-1} I_j\right)\left(\sum_{j=1}^{n} I_j\right)^{-1},$

$i\in\{2,\dots,n\}\,;$

c) for $i\in\{1,\dots,n\}$ and $x\in\Lambda_i$ we have:

$$\begin{aligned}
U_i(x) &= (K_i-K_{i+1})V_i(x)+K_{i+1}\\
&= K_i-(K_i-K_{i+1})(1-V_i(x))\\
&= K_i-N^{-p'}(\alpha_{_N})^{\frac{-p'}{N}}(\widetilde{c}_p)^{\frac{1}{1-p}}\left(I_i-\int_{\alpha_{_N}|x|^N-|\omega'_i|}^{|\Omega_i|} f_i(\sigma)d\sigma\right)\\
&= K_i-N^{-p'}(\alpha_{_N})^{\frac{-p'}{N}}(\widetilde{c}_p)^{\frac{p'}{p}}\int_0^{\alpha_{_N}|x|^N-|\omega'_i|} f_i(\sigma)d\sigma.
\end{aligned}$$

□

Remark 4.5. For the symmetrized problem the inequality (3.1) becomes an equality when $t'=K_i$ and $t=U_i(x)$. Actually, since $a$ is strictly positive, also is $a_i^*$ and it follows from the expression of $U_i$ given in Theorem 4.4 that it is strictly decreasing along radii.

Theorem 4.6 (isoperimetric inequalities). *Let* $\overline{u}$ *(resp.* $\overline{U}$*) be the extension of* $u$ *(resp.* $U$*) by* $1$ *on* $\omega_1$ *(resp.* $\widetilde{\omega}_1$*),* $0$ *on* $\omega'_{n+1}\backslash\overline{\omega_{n+1}}$ *(resp.* $\widetilde{\omega}'_{n+1}\backslash\overline{\widetilde{\omega}_{n+1}}$*) and* $k_i$ *(resp.* $K_i$*) on* $\omega'_i\backslash\overline{\omega_i}$ *(resp.* $\widetilde{\omega}'_i\backslash\,\overline{\widetilde{\omega}_i}$*). For* $i\in\{1,\dots,n\}$*,* $j\in\{1,\dots,i\}$ *and* $x\in\overline{\Lambda}_i$ *we have*

a) $(c_p)^{\frac{-p'}{p}}(k_j-\overline{u}_*(\alpha_{_N}|x|^N))\leqslant(\widetilde{c}_p)^{\frac{-p'}{p}}(K_j-\overline{U}(x))\,,$

b) $(c_p)^{\frac{-p'}{p}}(1-\overline{u}_*(\alpha_{_N}|x|^N))\leqslant(\widetilde{c}_p)^{\frac{-p'}{p}}(1-\overline{U}(x))\,,$

*in particular for* $1\leqslant j\leqslant i\leqslant n+1$ *we have*

c) $(c_p)^{\frac{-p'}{p}}(k_j-k_i)\leqslant(\widetilde{c}_p)^{\frac{-p'}{p}}(K_j-K_i)\,,$

d) $(c_p)^{\frac{-p'}{p}}(1-k_i)\leqslant(\widetilde{c}_p)^{\frac{-p'}{p}}(1-K_i)\,,$

*and more particularly,*

e) $c_p\geqslant\widetilde{c}_p.$

Proof. a) Let $i\in\{1,\dots,n\}$ and $x\in\overline{\Lambda}_i$. Then

$$|\omega'_i|\leqslant\alpha_{_N}|x|^N\leqslant|\omega_{i+1}|\ \text{ and }\overline{u}_*(\alpha_{_N}|x|^N)=u_{i_*}(\alpha_{_N}|x|^N-|\omega'_i|)$$

hence

$$k_{i+1}\leqslant\overline{u}_*(\alpha_{_N}|x|^N)\leqslant k_i.$$

We also have

$$K_{i+1} \leqslant \overline{U}(x) \leqslant K_i.$$

Let $j \in \{1,\dots,n\}$. If $j = i$ :

From (3.1) with $t' = k_i$ and $t = \overline{u}_*(\alpha_{_N}|x|^N)$ we have

$$\begin{aligned}&(c_p)^{\frac{-p'}{p}}(k_i - \overline{u}_*(\alpha_{_N}|x|^N))\\ &\quad\leqslant N^{-p'}(\alpha_{_N})^{\frac{-p'}{N}} \int_0^{|u_i > \overline{u}_*(\alpha_{_N}|x|^N)|} (|\omega_i'| + \sigma)^{\frac{p'}{N} - p'} (a_i^*)^{\frac{-p'}{2}}(\sigma) d\sigma.\end{aligned}$$

Since

$$\begin{aligned}|u_i > \overline{u}_*(\alpha_{_N}|x|^N)| &= |u_i > u_{i_*}(\alpha_{_N}|x|^N - |\omega_i'|)|\\ &= |u_{i_*} > u_{i_*}(\alpha_{_N}|x|^N - |\omega_i'|)|\\ &\leqslant \alpha_{_N}|x|^N - |\omega_i'|,\end{aligned}$$

we deduce from the expression of $U$ given in Theorem 4.4 that

$$(c_p)^{\frac{-p'}{p}}(k_i - \overline{u}_*(\alpha_{_N}|x|^N)) \leqslant (\widetilde{c}_p)^{\frac{-p'}{p}}(K_i - \overline{U}(x)).$$

If $j < i$ :

By using the above, Corollary 3.3 and Theorem 4.4 we have

$$\begin{aligned}&(c_p)^{\frac{-p'}{p}}(k_j - \overline{u}_*(\alpha_{_N}|x|^N))\\ &\quad= \sum_{m=j}^{i-1}(c_p)^{\frac{-p'}{p}}(k_m - k_{m+1}) + (c_p)^{\frac{-p'}{p}}(k_i - \overline{u}_*(\alpha_{_N}|x|^N))\\ &\quad\leqslant \sum_{m=j}^{i-1}(\widetilde{c}_p)^{\frac{-p'}{p}}(K_m - K_{m+1}) + (\widetilde{c}_p)^{\frac{-p'}{p}}(K_i - \overline{U}(x))\\ &\quad= (\widetilde{c}_p)^{\frac{-p'}{p}}(K_j - \overline{U}(x)).\end{aligned}$$

b) It's enough to take $j = 1$ in a).

With $x \in \widetilde{\gamma}_i'$ we deduce c) from a) and d) from b) because $\alpha_{_N}|x|^N = |\omega_i'|$ and $\overline{u}_*(\alpha_{_N}|x|^N) = u_{i_*}(0) = k_i$, we also have $\overline{U}(x) = K_i$. ☐

## 5. Comparison with Ferone's result

In this section we consider the particular case where the matrix $A = aI$, $I$ is the unit matrix. In this isotropic case, problem (1.4) reads

$$\inf\left\{\int_\Omega a^{\frac{p}{2}}\,|\nabla v|^p\,dx\,,\,v \in H\right\}. \tag{5.1}$$

Up to a change in the definition of $a$, this is exactly the problem studied by Ferone (see [6]) who considers the following symmetrized problem:

$$\inf\left\{\int_{\Lambda^*}(a^*)^{\frac{p}{2}}(\alpha_{_N}|x|^N - |\omega_i'|)\,|\nabla V|^p\,dx\,,\,V \in H^*\right\} \tag{5.2}$$

where $\Lambda^* = \widetilde{D} \backslash \widetilde{\omega}_1'$, $\widetilde{D}$ is the N-dimensional ball with center at the origin and measure $|D|$ with $D = \omega_{n+1} \backslash (C_2 \cup \cdots \cup C_n)$ and

$$H^* = \left\{ V \in W^{1,p}_{a^*}(\Lambda^*) \, : \, V = 1 \text{ on } \widetilde{\gamma}_1', \, V = 0 \text{ on } \partial \widetilde{D} \right\}.$$

Note that $\Lambda^*$ has same measure as $\Omega$, it is an annulus bounded by two spheres with center zero and inner sphere $\widetilde{\gamma}_1'$. Also note that (5.2) is a capacity problem but, unlike (5.1) or (4.1), it has no inner perfect conductor.

Let us denote by $c_p$ the infimum in (5.1) and by $c_p^*$ the infimum in (5.2). The result obtained by Ferone (see [6]) is:

$$c_p \geqslant c_p^*. \tag{5.3}$$

As it is clear that $(\widetilde{a})^* = a^*$, (5.2) is also the symmetrized problem (in Ferone's sense) of (4.1) and the comparison (5.3) applied to problems (4.1) and (5.2) gives:

Theorem 5.1.

$$\widetilde{c}_p \geqslant c_p^*.$$

Remark 5.2. From theorems 4.6 and 5.1, we deduce $c_p \geqslant \widetilde{c}_p \geqslant c_p^*$. Thus, in the particular isotropic case, we obtain a better comparison than Ferone's one, in addition we also obtain comparison for the unknown potentials $k_i$. Moreover, our symmetrized problem (4.1) is more natural than Ferone's one.

Remark 5.3. The proofs given in this paper differ from those given in [6] since we apply technics of relative rearrangement.

## 6. The particular case $a_i(x) = \beta_i$ (constant>0)

We still consider problem (1.4). In this section we suppose that the functions $a_i$ $(i = 1, \ldots, n)$ are constant: $a_i(x) = \beta_i > 0$ in $\Omega_i$. In this case the symmetrized problem (4.1) becomes completely explicit and, in addition to the results already obtained in Theorem 4.6, we are able to get other isoperimetric inequalities. This is done below.

Theorem 6.1 (Explicit resolution of the symmetrized problem when $a_i(x) = \beta_i$). *When $a_i(x) = \beta_i$, the explicit expression of $K_i$, $\widetilde{c}_p$ given in Theorem 4.4 are available with:*

$$I_j = \begin{cases} \left(\frac{p'}{N} - p' + 1\right)^{-1} (\beta_j)^{\frac{-p'}{2}} \left[ |\omega_{j+1}|^{\frac{p'}{N} - p' + 1} - |\omega_j'|^{\frac{p'}{N} - p' + 1} \right] & \textit{if } N \neq p \\ (\beta_j)^{\frac{-p'}{2}} \left[ \log |\omega_{j+1}| - \log |\omega_j'| \right] & \textit{otherwise}, \end{cases}$$

*moreover, the expression of $U_i(x)$ given in Theorem 4.4 is available with*

$$\int_0^{\alpha_N |x|^N - |\omega_i'|} f_i(\sigma) d\sigma$$

*replaced by:*

$$\begin{cases} \left(\frac{p'}{N}-p'+1\right)^{-1}(\beta_i)^{\frac{-p'}{2}}\left[(\alpha_N|x|^N)^{\frac{p'}{N}-p'+1}-|\omega_i'|^{\frac{p'}{N}-p'+1}\right] & \textit{if } N\neq p\\ (\beta_i)^{\frac{-p'}{2}}\left[\log(\alpha_N|x|^N)-\log|\omega_i'|\right] & \textit{otherwise.}\end{cases}$$

Proof. The proof is just computation. ☐

Theorem 6.2 (Isoperimetric inequalities when $a_i(x) = \beta_i$). *When $a_i(x) = \beta_i$, in addition to a), b), c), d) and e) in Theorem 4.6 we have:*

f) *for almost every $x \in \omega'_{n+1}$,*

$$\begin{aligned} 0 \;&\leqslant\; -(c_p)^{\frac{-p'}{p}}\frac{d\overline{u}_*}{ds}(\alpha_N|x|^N) \leqslant -(\widetilde{c}_p)^{\frac{-p'}{p}}\frac{d\overline{U}_*}{ds}(\alpha_N|x|^N)\\ &= \begin{cases} N^{-p'}(\alpha_N)^{\frac{-p'}{N}}(\beta_i)^{\frac{-p'}{2}}(\alpha_N|x|^N)^{\frac{p'}{N}-p'} & \textit{if } x\in\Omega\\ 0 & \textit{if } x\in \omega_1'\cup(\omega_2'\backslash\overline{\omega}_2)\cup\cdots\cup(\omega_{n+1}'\backslash\overline{\omega}_{n+1})\end{cases}\end{aligned}$$

*and it follows*

g) *for every $x, y$ in $\overline{\widetilde{\omega}_{n+1}}$ with $|x| \leqslant |y|$*

$$(c_p)^{\frac{-p'}{p}}\left(\overline{u}_*(\alpha_N|x|^N)-\overline{u}_*(\alpha_N|y|^N)\right) \leqslant (\widetilde{c}_p)^{\frac{-p'}{p}}\left(\overline{U}(x)-\overline{U}(y)\right)$$

*in particular,*

h) *for $i \in \{1,\dots,n\}$, $j \in \{i,\dots,n+1\}$ and $x \in \overline{\Lambda}_i$,*

$$\begin{aligned} (c_p)^{\frac{-p'}{p}}\left(\overline{u}_*(\alpha_N|x|^N)-k_j\right) &\leqslant (\widetilde{c}_p)^{\frac{-p'}{p}}\left(\overline{U}(x)-K_j\right),\\ (c_p)^{\frac{-p'}{p}}\overline{u}_*(\alpha_N|x|^N) &\leqslant (\widetilde{c}_p)^{\frac{-p'}{p}}\overline{U}(x),\end{aligned}$$

i) *for $i \in \{1,\dots,n+1\}$,*

$$(c_p)^{\frac{-p'}{p}}k_i \leqslant (\widetilde{c}_p)^{\frac{-p'}{p}}K_i.$$

Proof. f) Let $s$ and $s'$ be such that $|\omega_i'| \leqslant s' \leqslant s \leqslant |\omega_{i+1}|$ and $\overline{u}_*(s) < \overline{u}_*(s')$. Let $\varepsilon$ be such that $0 < \varepsilon \leqslant \overline{u}_*(s') - \overline{u}_*(s)$.

With $t = \overline{u}_*(s)$ and $t' = \overline{u}_*(s') - \varepsilon$ in (3.1) and tending $\varepsilon$ to 0 we obtain

$$\overline{u}_*(s')-\overline{u}_*(s) \leqslant N^{-p'}(\alpha_N)^{\frac{-p'}{N}}(c_p)^{\frac{p'}{p}}(\beta_i)^{\frac{-p'}{2}}\int_{s'}^{s}\sigma^{\frac{p'}{N}-p'}d\sigma$$

(because $|u_i > t'| \geqslant s' - |\omega_i'|$ and $|u_i > t| \leqslant s - |\omega_i'|$) with $s' = s - \theta$, one gets by letting $\theta$ decrease to zero and by using Theorem 6.1,

$$-\frac{d\overline{u}_*}{ds} \leqslant N^{-p'}(\alpha_N)^{\frac{-p'}{N}}(c_p)^{\frac{p'}{p}}(\beta_i)^{\frac{-p'}{2}}s^{\frac{p'}{N}-p'} = -\frac{d\overline{U}_*}{ds}$$

and this for almost every $s$ in $]|\omega_i'|\,,\,|\omega_{i+1}|[$ $(i = 1,\dots,n)$.

As $\overline{u}_*$ and $\overline{U}_*$ are constant in each connected component of the complementary of $]|\omega_1'|,|\omega_2|[\cup\cdots\cup]|\omega_n'|,|\omega_{n+1}|[$, the proof of f) is complete.

We obtain g) by integration and h), i) easily follow. ☐

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