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MORPHISMS OUT OF A SPLIT EXTENSION OF A HILBERT C^*-MODULE

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Abstract. Let us have a split extension W of a Hilbert C^*-module V by a Hilbert C^*-module Z. Like in the case of C^*-algebras (well known theorem of T. A. Loring), every morphism out of W, more precisely from W to an arbitrary Hilbert C^*-module U, can be described as a pair of morphisms from V and Z, respectively, into U that satisfies certain conditions. It turns out that besides the generalization of the Loring's condition, an additional condition has to be posed.

1. Preliminaries

A Hilbert C^*-module V over a C^*-algebra A (a Hilbert A-module) is a generalization of a Hilbert space in the sense that the "inner product" $(\cdot \mid \cdot) : V \times V \to A$ defined on it takes values in a C^*-algebra A instead of the field of complex numbers $\mathbb{C}$ (see [3, 6]). V is said to be full if the (closed) ideal in A generated by elements $(v_1 \mid v_2)$, $v_1, v_2 \in V$ is A.

When making quotients of a Hilbert C^*-module by its submodule, only the quotient of a Hilbert C^*-module by its ideal submodule is again a Hilbert C^*-module. What is an ideal submodule? The ideal submodule V_I of V associated to an ideal $I \subseteq A$ is $V_I = \{vb : v \in V,\ b \in I\} = \{v \in V : (v \mid x) \in I, \forall x \in V\}$. Denote by $\Pi : V \to V|_{V_I}$, $\pi : A \to A|_I$ canonical quotient maps. $V|_{V_I}$ has a natural Hilbert $A|_I$-module structure with the operation of right multiplication and the inner product given by: $\Pi(v)\pi(a) = \Pi(va)$, $(\Pi(v_1) \mid \Pi(v_2)) = \pi((v_1 \mid v_2))$.

A sum of submodules V_1, V_2 in V is $V_1 + V_2 = \{v_1 + v_2 : v_1 \in V_1, v_2 \in V_2\}$; a sum is called direct and denoted by $V_1 \dot{+} V_2$ if $V_1 \cap V_2 = \{0\}$. Further, a sum

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is orthogonal and denoted by $V_1 \oplus V_2$ if elements of V_1 and V_2 are mutually "orthogonal" i.e. if $(v_1 \mid v_2) = 0$ for all $v_1 \in V_1$, $v_2 \in V_2$.

A definition of an extension (see [1, 2]) of a Hilbert C^*-module is given inside of the category which objects are Hilbert C^*-modules and morphisms are given as follows: for a Hilbert A-module V and a Hilbert B-module W a map $\Phi : V \to W$ is a morphism (or a φ-morphism) of Hilbert C^*-modules if there is a morphism (a $*$-homomorphism) $\varphi : A \to B$ of C^*-algebras such that $(\Phi(v_1) \mid \Phi(v_2)) = \varphi((v_1 \mid v_2))$ for all $v_1, v_2 \in V$. An extension of a Hilbert A-module V is a triple (W, B, Φ) such that: W is a Hilbert B-module, $\Phi : V \to W$ is a φ-morphism for a morphism $\varphi : A \to B$ of C^*-algebras, $\varphi(A)$ is an ideal in a C^*-algebra B and $\Phi(V)$ is the ideal submodule of W associated to $\varphi(A)$ i.e. $\Phi(V) = W\varphi(A)$. So we have an exact sequence of Hilbert C^*-modules and morphisms of modules

$$0 \longrightarrow V \xrightarrow{\ \Phi\ } W \xrightarrow{\ \Pi\ } Z \longrightarrow 0$$

and the corresponding exact sequence of underlying C^*-algebras

$$0 \longrightarrow A \xrightarrow{\ \varphi\ } B \xrightarrow{\ \pi\ } C \longrightarrow 0.$$

The latter is called split if there is a morphism $\sigma : C \to B$ such that $\pi\sigma = id$. Then a C^*-algebra B actually splits i.e. $B = A \dot{+} \sigma(C)$ ([6, Proposition 3.1.3]). This sum is orthogonal if $\sigma(C)$ also sits as an ideal in B.

In the case of Hilbert C^*-modules an extension (W, B, Φ) (or W for short) of a Hilbert C^*-module V is said to be split if there is a morphism $\Sigma : Z \to W$ (that is, a σ-morphism for a morphism of C^*-algebras $\sigma : C \to B$) such that $\Pi\Sigma = id$. We are going to deal with full Hilbert C^*-modules, because then the underlying exact sequence of C^*-algebras is split too.

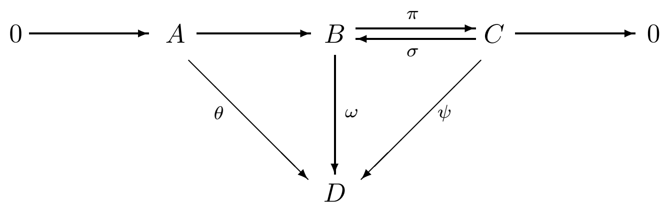
Quite as it should be, a full split extension W of a full Hilbert C^*-module V really splits into the direct sum of $\Phi(V)$ and $\Sigma(Z)$ (a straightforward computation imitating the C^*-algebra case shows it). This sum is orthogonal if $\Sigma(Z)$ is the ideal submodule of W associated to the ideal $\sigma(C) \subseteq B$ when $B = \varphi(A) \oplus \sigma(C)$ ($\Phi(V)$ is already the ideal submodule of W associated to the ideal $\varphi(A) \subseteq B$, assured by the fact that W is an extension of V):

$$\begin{aligned}(\Phi(V) \mid \Sigma(Z)) &= (W\varphi(A) \mid W\sigma(C)) = \varphi(A)\,(W \mid W)\,\sigma(C) \\ &= \varphi(A)B\sigma(C) \subseteq \varphi(A)\sigma(C) = 0.\end{aligned}$$

It was T. A. Loring who described morphisms out of a split extension of a C^*-algebra by pairs of morphisms out of the ideal and the quotient algebra:

THEOREM 1.1 ([4, 7.3.8]). *Let us have a split extension (C) of a C^*-algebra A and an arbitrary C^*-algebra D:*

(C)



A morphism $\omega : B \to D$ is in a bijective correspondence with a pair of morphisms (θ, ψ) ($\theta : A \to D$, $\psi : C \to D$) such that for all $a \in A$, $c \in C$ we have

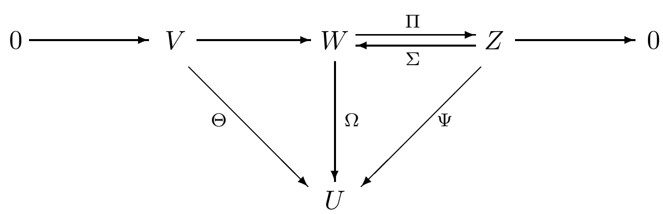
$$\psi(c)\theta(a) = \theta(\sigma(c)a). \tag{C*}$$

Here we solve the problem of describing morphisms out of a full extension of a full Hilbert C^*-module by pairs of morphisms out of an ideal submodule and a quotient module.

2. MAIN THEOREM

We want to generalize Theorem 1.1 to full Hilbert C^*-modules and morphisms of modules; there we have a diagram:

(H)



So, we have a full split extension (W, B, Φ) of a full Hilbert A-module V in which $\Phi : V \to W$ is an inclusion map and an arbitrary Hilbert D-module U. We want to establish a bijection between a morphism $\Omega : W \to U$ (an ω-morphism for a morphism of C^*-algebras $\omega : B \to D$) and a pair of morphisms (Θ, Ψ) where $\Theta : V \to U$ is a θ-morphism for a morphism of C^*-algebras $\theta : A \to D$ and $\Psi : Z \to U$ is a ψ-morphism for a morphism $\psi : C \to D$. Here, the condition (C*) naturally generalizes to:

$$(\Psi(z) \mid \Theta(v)) = \theta((\Sigma(z) \mid v)), \quad v \in V,\ z \in Z. \tag{1}$$

(As V is an ideal submodule of W associated to an ideal $A \subseteq B$, $(\Sigma(z) \mid v) \in A$ for all $v \in V, z \in Z$.) It turns out that the condition (1) is not enough for a statement similar to the one in Theorem 1.1. Namely, "under" a diagram

(H) of Hilbert C^*-modules, we have a diagram (C) of C^*-algebras, but the condition (1) does not imply the condition (C*) necessary for establishing a bijection between a morphism ω and a pair of morphisms (θ, ψ) on this C^*-algebra level.

Example 2.1. Let us rewrite (1) in an equivalent form $(\Theta(v) \mid \Psi(z)) = \theta((v \mid \Sigma(z)))$. We have $\theta((v \mid \Sigma(z)))\psi(c) = \theta((v \mid \Sigma(z))\,\sigma(c))$; indeed

$$\begin{aligned}\theta((v \mid \Sigma(z)))\psi(c) &= (\Theta(v) \mid \Psi(z))\,\psi(c) = (\Theta(v) \mid \Psi(zc)) \\ &= \theta((v \mid \Sigma(zc))) = \theta((v \mid \Sigma(z))\,\sigma(c)).\end{aligned}$$

However, elements of the form $(v \mid \Sigma(z))$, $v \in V$, $z \in Z$ do not generate a C^*-algebra A (so that we can conclude that (C*) is valid): if we take that $\Sigma(Z)$ is an ideal submodule of W associated to an ideal $\sigma(C)$ in B and that $B = A \oplus \sigma(C)$, then the sum $V + \Sigma(Z)$ in a decomposition of W is orthogonal, i.e. $\{(v \mid \Sigma(z)) : v \in V, z \in Z\} = \{0\}$.

If we have a diagram (H) of Hilbert C^*-modules and morphisms of modules with V full and underlying diagram (C) of C^*-algebras and morphisms of C^*-algebras, the condition (C*) on morphisms of C^*-algebras is equivalent to the condition $\Theta(v)\psi(c) = \Theta(v\sigma(c)), v \in V,\ c \in C$ on morphisms of modules: we know that every $v \in V$ can be written as $v = v'a,\ v' \in V,\ a \in A$ (Hewit-Cohen factorization [5, Proposition 2.31]) and therefore $\Theta(v)\psi(c) = \Theta(v\sigma(c))$ is equivalent to $\Theta(va)\psi(c) = \Theta(va\sigma(c))$. If we suppose that (C*) is true, then $\Theta(va)\psi(c) = \Theta(va\sigma(c))$ follows immediately. On the other hand, if we have $\Theta(v)\psi(c) = \Theta(v\sigma(c))$ and V is a full Hilbert C^*-module, then

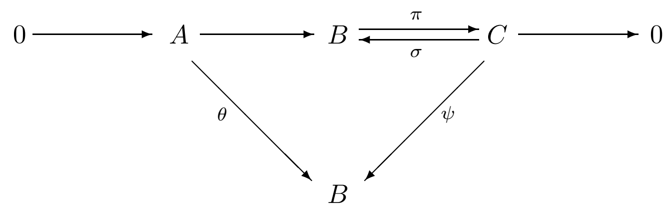
$$\begin{aligned}\theta((v \mid v))\psi(c) &= (\Theta(v) \mid \Theta(v))\,\psi(c) = (\Theta(v) \mid \Theta(v)\psi(c)) \\ &= (\Theta(v) \mid \Theta(v\sigma(c))) = \theta((v \mid v\sigma(c))) \\ &= \theta((v \mid v)\,\sigma(c))\end{aligned}$$

and so $\theta(a)\psi(c) = \theta(a\sigma(c))$ for all $a \in A, c \in C$.

Example 2.2. Let us describe morphisms of modules that would be useful as counterexamples in what follows. As well as a C^*-algebra A itself can be reorganized to become a Hilbert A-module (with the inner product $(a_1 \mid a_2) = {a_1}^* a_2$; the corresponding norm is exactly a C^*-norm), a morphism of C^*-algebras $\varphi : A \to B$ can be realized of as a morphism of modules over itself. More generally, if we take $v \in B$ such that $v^*v = 1 \in B$, a morphism $\psi : A \to B$ given by $\psi(a) = v\varphi(a)$ (obviously not a morphism of C^*-algebras: not multiplicative nor a $*$-map) is a morphism of Hilbert C^*-modules.

Here comes an example of a full split extension of a full Hilbert C^*-module where (C*) is satisfied, but (1) is not.

Example 2.3. Let us take a diagram



of C^*-algebras with $\theta : A \to B$ an identity map. Suppose that the condition (C^*) for morphisms of C^*-algebras is valid. Now understand all C^*-algebras as Hilbert C^*-modules and let morphisms of modules be given: $\Theta : A \to B$ by $\Theta(a) = va$ for $v \in B$ such that $v^*v = 1$ and $\Psi = \psi = \sigma$, $\Sigma = \sigma$. Now $\Theta(a)\psi(c) = \Theta(a\sigma(c))$ is obviously valid but (1) (it transforms to $\sigma(c)^*va = \sigma(c)^*a$) is not.

We conclude that for a case of Hilbert C^*-modules besides the condition (1) we ought to have a condition that would, on underlying C^*-algebras, ensure (C^*). It is the condition

$$\Psi(z)\theta(a) = \Theta(\Sigma(z)a), \qquad z \in Z, a \in A. \tag{2}$$

Indeed, $\Sigma(z)a \in V$ for all $z \in Z, a \in A$ thanks to the fact that V is an ideal submodule of W associated to an ideal $A \subseteq B$. Having conditions (1) and (2) at hand, as well as supposing that a Hilbert C^*-module Z is full, we have (C^*): for all $z \in Z, a \in A$

$$\begin{aligned} \psi((z \mid z))\theta(a) &= (\Psi(z) \mid \Psi(z))\,\theta(a) = (\Psi(z) \mid \Psi(z)\theta(a)) \\ &\overset{(2)}{=} (\Psi(z) \mid \Theta(\Sigma(z)a)) \overset{(1)}{=} \theta((\Sigma(z) \mid \Sigma(z)a) \\ &= \theta((\Sigma(z) \mid \Sigma(z))\,a) = \theta(\sigma((z \mid z))a). \end{aligned}$$

One has to take care that conditions (2) and (C^*) are not equivalent to each other: the setting of the previous example is the appropriate one to show this. Namely, (C^*) is true, but (2) is not.

THEOREM 2.4. *Let V be a full Hilbert A-module, (W, B, Φ) a full split extension of V in which $\Phi : V \to W$ is an inclusion map, U an arbitrary Hilbert D-module. Then a morphism $\Omega : W \to U$ is in a bijective correspondence with a pair of morphisms (Θ, Ψ) where $\Theta : V \to U$ (a θ-morphism for a morphism of C^*-algebras $\theta : A \to D$) and $\Psi : Z \to U$ (a ψ-morphism for a morphism $\psi : C \to D$) are such that for all $v \in V, z \in Z, a \in A$ we have*

$$(\Psi(z) \mid \Theta(v)) = \theta((\Sigma(z) \mid v)),$$

$$\Psi(z)\theta(a) = \Theta(\Sigma(z)a).$$

PROOF. We have the decomposition of W of the form $W = V \dot{+} \Sigma(Z)$. Suppose we have a pair of morphisms (Θ, Ψ) with given properties. Our modules are taken to be full and so the underlying extension of a C^*-algebra A is split too. From the discussion preceding this theorem we know that given conditions ensure the condition (C*) for morphisms of C^*-algebras. Therefore we can (by Theorem 1.1) associate a morphism $\omega : B \to D$ of C^*-algebras to the pair of morphisms (θ, ψ) (it is given by $\omega(a + \sigma(c)) = \theta(a) + \psi(c)$). Let us define a map $\Omega : W \to U$ by $\Omega(w) = \Omega(v + \Sigma(z)) = \Theta(v) + \Psi(z)$. Ω is an ω-morphism: if we take $v, v' \in V$, $z, z' \in Z$, then

$$\begin{aligned}
&(\Omega(v + \Sigma(z)) \mid \Omega(v' + \Sigma(z'))) = (\Theta(v) + \Psi(z) \mid \Theta(v') + \Psi(z')) \\
&= (\Theta(v) \mid \Theta(v')) + (\Theta(v) \mid \Psi(z')) + (\Psi(z) \mid \Theta(v')) + (\Psi(z) \mid \Psi(z')) \\
&= \theta((v \mid v')) + \theta((v \mid \Sigma(z'))) + \theta((\Sigma(z) \mid v')) + \psi((z \mid z')) \\
&= \theta((v \mid v') + (v \mid \Sigma(z')) + (\Sigma(z) \mid v')) + \psi((z \mid z')) \\
&= \omega(((v \mid v') + (v \mid \Sigma(z')) + (\Sigma(z) \mid v')) + \sigma((z \mid z'))) \\
&= \omega((v + \Sigma(z) \mid v' + \Sigma(z'))).
\end{aligned}$$

Now take an ω-morphism (for a morphism of C^*-algebras $\omega : B \to D$) $\Omega : W \to U$. We know that ω is in a bijective correspondence with a pair (θ, ψ) of morphisms of C^*-algebras where $\theta = \omega|_A$, $\psi = \omega\sigma$. Let us define morphisms of modules in a similar way: $\Theta = \Omega|_V$, $\Psi = \Omega\Sigma$. Obviously Θ is a θ-morphism, Ψ is a ψ-morphism. The pair (Θ, Ψ) satisfies the required conditions: for all $v \in V, z \in Z, a \in A$

$$(\Psi(z) \mid \Theta(v)) = (\Omega(\Sigma(z)) \mid \Omega(v)) = \omega((\Sigma(z) \mid v)) = \theta((\Sigma(z) \mid v)),$$

$$\Psi(z)\theta(a) = \Omega(\Sigma(z))\omega(a) = \Omega(\Sigma(z)a) = \Theta(\Sigma(z)a).$$

□

The significance of this theorem comes to the sight when one assumes that a Hilbert C^*-module U is an extension of an arbitrarily taken Hilbert C^*-module. Then a morphism Ω becomes a morphism between extensions of Hilbert C^*-modules, they are in general not easy to describe.

References

[1] D. Bakić and B. Guljaš, *Extensions of Hilbert C^*-modules*, Houston J. of Math. **30** (2004), 537-558.

[2] D. Bakić and B. Guljaš, *Extensions of Hilbert C^*-modules. II*, Glas. Mat. Ser. III **38(58)** (2003), 341-357.

[3] E. C. Lance, Hilbert C^*-modules. A Toolkit for Operator Algebraists, Cambridge University Press, Cambridge, 1995.

[4] T. A. Loring, Lifting Solutions to Perturbing Problems in C^*-algebras, American Mathematical Society, Providence, 1997.

[5] I. Raeburn and D. P. Williams, Morita Equivalence and Continuous-Trace C^*-algebras, American Mathematical Society, Providence, 1998.

[6] N. E. Wegge-Olsen, K-theory and C^*-algebras. A Friendly Approach, The Clarendon Press, Oxford University Press, New York, 1993.

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