GLASNIK MATEMATIČKI
Vol. 41(61)(2006), 329 – 334

LORENTZIAN MATRIX MULTIPLICATION AND THE MOTIONS ON LORENTZIAN PLANE

Halİt Gündoğan and Osman Keçİlİoğlu
Kırıkkale University, Turkey

Abstract. In this paper, a new matrix multiplication is defined in $\mathbb{R}_n^m \times \mathbb{R}_p^n$ by using Lorentzian inner product in $\mathbb{R}^n$, where $\mathbb{R}_n^m$ is set of matrices of m rows and n columns. With this multiplication it has been shown that $\mathbb{R}_n^n$ is an algebra with unit. By means of orthogonal matrices with respect to this multiplication, coordinate transformations are defined on n-dimensional Lorentz space L^n. As a special case, motions on L^2 Lorentz plane are obtained.

1. Introduction

Let $\mathbb{R}_n^m$ be the set of all $m \times n$ matrices. $\mathbb{R}_n^m$ with the matrix addition and the scalar-matrix multiplication is a real vector space. More properties of the ordinary matrix multiplication can be found in [4].

Let L^n be the vector space $\mathbb{R}^n$ provided with the Lorentzian inner product

$$\langle x, y\rangle_L = -x_1y_1 + \sum_{i=2}^{n} x_iy_i \text{ for } x = (x_1, x_2, \ldots, x_n),\ y = (y_1, y_2, \ldots, y_n).$$

In L^n, a vector x is said to be time-like if $\langle x, x\rangle_L < 0$, space-like if $\langle x, x\rangle_L > 0$ and light-like if $\langle x, x\rangle_L = 0$ [6].

For $r \in \mathbb{R}^+$, the Lorentzian circle of radius r in L^2 is defined in [1] by

$$\left\{(x, y) \in L^2 \,\middle|\, -x^2 + y^2 = r^2\right\} = \{(r\sinh\theta, r\cosh\theta) \mid \theta \in \mathbb{R}\}$$

See [7] for the hyperbolic angles defined on the Lorentz plane L^2 .

2000 *Mathematics Subject Classification.* 51M05, 53A17.
Key words and phrases. Lorentz space, rotation, reflection.

2. Lorentzian matrix multiplication and properties

Let $A = [a_{ij}] \in \mathbb{R}^m_n$ and $B = [b_{jk}] \in \mathbb{R}^n_p$. Then we define a new matrix multiplication denoted by " $\cdot_L$ " as

$$A \cdot_L B = \left[-a_{i1}b_{1k} + \sum_{j=2}^{n} a_{ij}b_{jk} \right].$$

We call this multiplication as Lorentzian matrix multiplication and we simply say L-multiplication. Note that $A \cdot_L B$ is an $m \times p$ matrix. Also note that if we let A_i to be ith row of A and B^j to be jth column of B then (i, j) entry of $A \cdot_L B$ is $\left\langle A_i, B^j \right\rangle_L$. We denote $\mathbb{R}^m_n$ with L-multiplication by L^m_n.

Theorem 2.1. *The following statements are satisfied.*

i) *For every* $A \in L^m_n, B \in L^n_p, C \in L^p_r,\ A \cdot_L (B \cdot_L C) = (A \cdot_L B) \cdot_L C$.
ii) *For every* $A \in L^m_n, B, C \in L^n_p,\ A \cdot_L (B + C) = A \cdot_L B + A \cdot_L C$.
iii) *For every* $A, B \in L^m_n, C \in L^n_p,\ (A + B) \cdot_L C = A \cdot_L C + B \cdot_L C$.
iv) *For every* $k \in \mathbb{R},\ A \in L^m_n, B \in L^n_p,\ k(A \cdot_L B) = (kA) \cdot_L B = A \cdot_L (kB)$.

Definition 2.2. *An* $n \times n$ *L-identity matrix according to L-multiplication, denoted by* I_n, *is defined by*

$$I_n = \begin{bmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{n \times n}.$$

Note that for every $A \in L^m_n,\ I_m \cdot_L A = A \cdot_L I_n = A$.

Corollary 2.3. L^n_n *with L-multiplication is an algebra with unit.*

Definition 2.4. *An* $n \times n$ *matrix* A *is called L-invertible if there exists an* $n \times n$ *matrix* B *such that* $A \cdot_L B = B \cdot_L A = I_n$. *Then* B *is called the L-inverse of* A *and is denoted by* A^{-1}.

Definition 2.5. *The transpose of a matrix* $A = [a_{ij}] \in L^m_n$ *is denoted by* A^T *and defined as* $A^T = [a_{ji}] \in L^n_m$.

Definition 2.6. *A matrix* $A \in L^n_n$ *is called L-orthogonal matrix if* $A^{-1} = A^T$.

Definition 2.7. *L-determinant of a matrix* $A = [a_{ij}] \in \mathbb{R}^n_n$ *is denoted by* $\det A$ *and defined as*

$$\det A = \sum_{\sigma \in S_n} s(\sigma) a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(n)n},$$

where S_n *is the set of all permutations of the set* $\{1, 2, \ldots, n\}$ *and* $s(\sigma)$ *is the sign of the permutation* σ.

THEOREM 2.8. *For every $A, B \in L_n^n$,* $\det(A \cdot_L B) = -\det A \cdot \det B$.

PROOF. Let $A = [a_{ij}]$, $B = [b_{jk}]$ and $A \cdot_L B = C$. Let us denote ith column of matrix A by A^i and kth column of matrix C by C^k. Then

$$\begin{aligned}
C^k &= -b_{1k}A^1 + b_{2k}A^2 + \cdots + b_{nk}A^n, \qquad 1 \le k \le n. \\
\det(A \cdot_L B) &= \det[-b_{11}A^1 + b_{21}A^2 + \cdots + b_{n1}A^n, \\
&\qquad -b_{12}A^1 + b_{22}A^2 + \cdots + b_{n2}A^n, \\
&\qquad \ldots, \\
&\qquad -b_{1n}A^1 + b_{2n}A^2 + \cdots + b_{nn}A^n] \\
&= \sum_{\sigma \in S_n} -b_{\sigma(1)1}b_{\sigma(2)2}\cdots b_{\sigma(n)n} \det[A^{\sigma(1)}, A^{\sigma(2)}, \ldots, A^{\sigma(n)}] \\
&= -\sum_{\sigma \in S_n} s(\sigma) b_{\sigma(1)1}b_{\sigma(2)2}\cdots b_{\sigma(n)n} \det[A^1, A^2, \ldots, A^n] \\
&= -\det A \sum_{\sigma \in S_n} s(\sigma) b_{\sigma(1)1}b_{\sigma(2)2}\cdots b_{\sigma(n)n} \\
&= -\det A \cdot \det B.
\end{aligned}$$

□

3. Coordinate transformations in n-dimensional Lorentz space

Recall that the transformation which preserves the distance between two points is called a rigid transformation [5]. Let A be an $n \times n$ matrix and d be a vector in L^n. Let us consider the function

$$f : L^n \longrightarrow L^n, \qquad f(x) = A \cdot_L x + d.$$

The function f must preserve distance in L^n to be a rigid transformation. Let p, q be any two points in L^n. Recall the distance between the points p and q is

$$d(p,q) = \|p - q\| = \sqrt{\langle p - q, p - q \rangle_L} = \sqrt{(p - q)^T \cdot_L (p - q)}.$$

Then

$$\begin{aligned}
d(f(p), f(q)) &= \|f(p) - f(q)\| \\
&= \sqrt{\langle f(p) - f(q), f(p) - f(q) \rangle_L} \\
&= \sqrt{(f(p) - f(q))^T \cdot_L (f(p) - f(q))} \\
&= \sqrt{((Ap + d) - (Aq + d))^T \cdot_L ((Ap + d) - (Aq + d))} \\
&= \sqrt{(p - q)^T \cdot_L A^T \cdot_L A \cdot_L (p - q)}.
\end{aligned}$$

Thus,

$$d(f(p), f(q)) = d(p,q) \Longleftrightarrow A^T \cdot_L A = I_n \Longleftrightarrow A \text{ is } L\text{-orthogonal matrix.}$$

We summarize this result in the following theorem.

THEOREM 3.1. *Let A be an $n \times n$ matrix and d be a vector in L^n. Let us consider the function $f : L^n \longrightarrow L^n$, $f(x) = A \cdot_L x + d$. If the matrix A is an L-orthogonal matrix, then the function f is a rigid transformation.*

Observe that $\det A = \mp 1$ for L-orthogonal matrix A. We call L-orthogonal matrices with $\det A = -1$ rotations and L-orthogonal matrices with $\det A = 1$ reflections.

Here we consider the special case $n = 2$. In [2, 3], the rotations on the Lorentz plane L^2 have been discussed by means of the ordinary matrix multiplication. We now develop the motions on the Lorentz plane L^2 by using L-multiplication.

Let $p(x, y)$ be a point on the Lorentzian circle centered at the origin with radius r on the Lorentz plane L^2. Then

$$\begin{cases} x = r \sinh \varphi \\ y = r \cosh \varphi \end{cases} .$$

If the point p is revolved about the origin with the rotation angle θ to the point p', then the coordinates of $p'(x', y')$ can be written as

$$\begin{cases} x' = r \sinh(\varphi + \theta) \\ y' = r \cosh(\varphi + \theta) \end{cases} .$$

Thus, we have the following rotation equations

$$\begin{cases} x' = x \cosh \theta + y \sinh \theta \\ y' = x \sinh \theta + y \cosh \theta \end{cases} .$$

These equations can be re-written in the form of a matrix equation as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -\cosh \theta & \sinh \theta \\ -\sinh \theta & \cosh \theta \end{bmatrix} \cdot_L \begin{bmatrix} x \\ y \end{bmatrix}.$$

If

$$A = \begin{bmatrix} -\cosh \theta & \sinh \theta \\ -\sinh \theta & \cosh \theta \end{bmatrix},$$

then the matrix A is L-orthogonal and $\det A = -1$. Consequently, the function

$$f : L^2 \longrightarrow L^2, \quad f(x) = A \cdot_L x$$

corresponds to a rotation about the origin with the hyperbolic angle θ in L^2 and denoted by $R(\theta)$.

THEOREM 3.2. *The function $R(\theta)$ transforms the time-like vectors to time-like vectors, the space-like vectors to space-like vectors and the light-like vectors to light-like vectors in L^2.*

PROOF. Let $x = (x_1, x_2) \in L^2$. Then

$$(R(\theta))(x) = A \cdot_L x = (x_1 \cosh\theta + x_2 \sinh\theta, x_1 \sinh\theta + x_2 \cosh\theta).$$

Thus $\langle A \cdot_L x, A \cdot_L x\rangle_L = -x_1^2 + x_2^2 = \langle x, x\rangle_L$. □

If

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

then the matrix B is L-orthogonal. The function

$$g : L^2 \longrightarrow L^2, \quad g(x) = B \cdot_L x$$

is a reflection according to y-axis in L^2. Notice that $\det B = 1$.

Let matrix S be $A \cdot_L B \cdot_L A^T$. Then

$$S = \begin{bmatrix} \cosh 2\theta & \sinh 2\theta \\ \sinh 2\theta & \cosh 2\theta \end{bmatrix}.$$

Note that S is L-orthogonal and $\det S = +1$. Thus we can give the following theorem.

THEOREM 3.3. *The function $h : L^2 \longrightarrow L^2$, $h(x) = S \cdot_L x$ corresponds to the reflection according to a space-like line which has the hyperbolic angle θ with respect to y-axis at origin in L^2. Also, the function $-h$ corresponds to the reflection according to a time-like line which has the hyperbolic angle θ with respect to x-axis at origin in L^2.*

PROOF. Let l be a space-like line which has a hyperbolic angle θ with respect to y-axis at the origin on the Lorentzian plane L^2. The reflection of the point $p(x, y)$ according to the line l is

$$h(p) = q = (-x \cosh 2\theta + y \sinh 2\theta, -x \sinh 2\theta + y \cosh 2\theta).$$

Let space-like vector $\overrightarrow{n} = (a, b)$ be directrix of the line l. Then

$$\cosh 2\theta = \frac{a^2 + b^2}{-a^2 + b^2}, \quad \sinh 2\theta = \frac{2ab}{-a^2 + b^2}.$$

Thus,

$$h(p) = q = \frac{1}{-a^2 + b^2}(-a^2x - b^2x + 2aby, -2abx + a^2y + b^2y).$$

The vector $\overrightarrow{pq}$ is perpendicular to the vector $\overrightarrow{n}$ and $d(p, l) = d(q, l)$.

Analogously, the reflection according to the time-like line can be proven. □

COROLLARY 3.4. 1) *If the functions $h_i : L^2 \longrightarrow L^2$, $i = 1, 2$, $h_i(x) = S_i \cdot_L x$ are reflections with respect to space-like lines (or time-like lines) which have the hyperbolic angles θ_i at origin in L^2, then the function $h : L^2 \longrightarrow L^2$, $h(x) = (S_1 \cdot_L S_2)x$ corresponds to rotation about the origin with the hyperbolic angle $2(\theta_1 - \theta_2)$.*

2) *If one of the* h_1, h_2 *is reflection according to space-like line with hyperbolic angle* θ_1 *and the other one is reflection according to time-like line with hyperbolic angle* θ_2*, then* $h(x) = -(S_1 \cdot_L S_2) \cdot_L x$ *for each* $x \in L^2$*.*

PROOF. 1) Since

$$S_1 \cdot_L S_2 = \begin{bmatrix} -\cosh 2(\theta_1 - \theta_2) & \sinh 2(\theta_1 - \theta_2) \\ -\sinh 2(\theta_1 - \theta_2) & \cosh 2(\theta_1 - \theta_2) \end{bmatrix},$$

we have $h(x) = (R(2(\theta_1 - \theta_2)))(x)$.

2) Analogously, by Theorem 3.3. □

COROLLARY 3.5. *Let*

$$A = \begin{bmatrix} -\cosh\theta & \sinh\theta \\ -\sinh\theta & \cosh\theta \end{bmatrix}$$

and p *be a vector in* L^2*. Then the motion in* L^2 *can be given by the function*

$$f : L^2 \longrightarrow L^2, \quad f(x) = A \cdot_L x + p.$$

The function f *corresponds to the pure translation when* $A = I_2$ *and the pure rotation when* $p = 0$*.*

References

[1] G. S. Birman and K. Nomizu, *Trigonometry in Lorentzian geometry*, Amer. Math. Monthly **91** (1984), 543-549.

[2] G. S. Birman and K. Nomizu, *The Gauss-Bonnet theorem for 2-dimensional spacetimes*, Michigan Math. J. **31** (1984), 77-81.

[3] A. A. Ergin, *On the 1-parameter Lorentzian motions*, Comm. Fac. Sci. Univ. Ankara Ser. A_1 Math. Statist. **40** (1991), 59-66.

[4] S. Lang, Linear Algebra, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1971.

[5] J. M. McCarthy, An Introduction to Theoretical Kinematics, MIT Press, Cambridge, 1990.

[6] B. O'Neill, Semi-Riemannian Geometry. With Applications to Relativity, Academic Press, Inc., New York, 1983.

[7] J. G. Ratcliffe, Foundations of Hyperbolic Manifolds, Springer-Verlag, New York, 1994.

H. Gündoğan
Department of Mathematics, Kırıkkale University
Kırıkkale
Turkey
E-mail: hagundogan@hotmail.com

O. Keçilioğlu
Department of Mathematics, Kırıkkale University
Kırıkkale
Turkey
E-mail: kecilioglu@kku.edu.tr

Received: 16.12.2005.
Revised: 1.3.2006.