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FINITE p-GROUPS WITH A UNIQUENESS CONDITION FOR NON-NORMAL SUBGROUPS

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Abstract. We determine up to isomorphism all finite p-groups G which possess non-normal subgroups and each non-normal subgroup is contained in exactly one maximal subgroup of G. For $p = 2$ this problem was essentially more difficult and we obtain in that case two new infinite families of finite 2-groups.

We consider here only finite p-groups and our notation is standard. It is easy to see that minimal nonabelian p-groups and 2-groups of maximal class have the property that each non-normal subgroup is contained in exactly one maximal subgroup. It turns out that there are two further infinite families of 2-groups which also have this property. More precisely, we shall prove the following result which gives a complete classification of such p-groups.

Theorem 1. *Let G be a finite p-group which possesses non-normal subgroups and we assume that each non-normal subgroup of G is contained in exactly one maximal subgroup. Then one of the following holds:*

(a) *G is minimal nonabelian;*
(b) *G is a 2-group of maximal class;*
(c) *$G = \langle a, b\rangle$ is a non-metacyclic 2-group, where $a^{2^n} = 1$, $n \geq 3$, $o(b) = 2$ or 4, $a^b = ak$, $k^2 = a^{-4}$, $[k, a] = 1$, $k^b = k^{-1}$ and we have either:*
  (c1) *$b^2 \in \langle a^{2^{n-1}}, a^2k\rangle \cong E_4$, in which case $|G| = 2^{n+2}$, $\Phi(G) = \langle a^2\rangle \times \langle a^2k\rangle \cong C_{2^{n-1}} \times C_2$, $Z(G) = \langle a^{2^{n-1}}\rangle \times \langle a^2k\rangle \cong E_4$, and $\langle a\rangle \times \langle a^2k\rangle \cong C_{2^n} \times C_2$ is the unique abelian maximal subgroup of G, or:*

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(c2) $b^2 \notin \langle a^{2^{n-1}}, a^2k \rangle \cong E_4$, *in which case* $o(b) = 4$, $|G| = 2^{n+3}$, $\Phi(G) = \langle a^2 \rangle \times \langle a^2k \rangle \times \langle b^2 \rangle \cong C_{2^{n-1}} \times C_2 \times C_2$, $Z(G) = \langle a^{2^{n-1}} \rangle \times \langle a^2k \rangle \times \langle b^2 \rangle \cong E_8$, *and* $\langle a \rangle \times \langle a^2k \rangle \times \langle b^2 \rangle \cong C_{2^n} \times C_2 \times C_2$ *is the unique abelian maximal subgroup of* G.

In any case, $G' = \langle k \rangle \cong C_{2^{n-1}}$, *a centralizes* $\Phi(G)$, *and b inverts each element of* $\Phi(G)$, *and so each subgroup of* $\Phi(G)$ *is normal in* G;

(d) $G = \langle a, b \rangle$ *is a splitting metacyclic 2-group, where* $a^{2^n} = b^4 = 1$, $n \geq 3$, $a^b = a^{-1}z^{\epsilon}$, $\epsilon = 0, 1$, $z = a^{2^{n-1}}$. *Here* $|G| = 2^{n+2}$, $\Phi(G) = \langle a^2 \rangle \times \langle b^2 \rangle \cong C_{2^{n-1}} \times C_2$, $Z(G) = \langle z \rangle \times \langle b^2 \rangle \cong E_4$, $G' = \langle a^2 \rangle \cong C_{2^{n-1}}$, *and* $\langle a \rangle \times \langle b^2 \rangle \cong C_{2^n} \times C_2$ *is the unique abelian maximal subgroup of* G. *Since a centralizes* $\Phi(G)$ *and b inverts each element of* $\Phi(G)$, *it follows that each subgroup of* $\Phi(G)$ *is normal in* G.

To facilitate the proof of Theorem 1, we prove the following

Lemma 2 (Y. Berkovich). *Let* G *be a* p*-group,* $p > 2$, *such that all subgroups of* $\Phi(G)$ *are normal in* G. *Then* $\Phi(G) \leq Z(G)$.

Proof. By [1, Satz III, 7.12], $\Phi(G)$ is abelian. Suppose that $\Phi(G)$ is cyclic. Let $U/\Phi(G)$ be a subgroup of order p in $G/\Phi(G)$. Assume that U is nonabelian. Then $U \cong M_{p|\Phi(G)|}$ so $U = \Phi(G)\Omega_1(U)$, where $\Omega_1(U)$ is a normal subgroup of type (p,p) in G. In that case, $\Omega_1(U)$ centralizes $\Phi(G)$ so U is abelian, a contradiction. Let $M = \{U < G \mid \Phi(G) < U, \ |U : \Phi(G)| = p\}$. Then $C_G(\Phi(G)) \geq \langle U \mid U \in M \rangle = G$ so $\Phi(G) \leq Z(G)$.

Now let $\Phi(G)$ be noncyclic. Then $\Phi(G) = Z_1 \times \cdots \times Z_n$, where $Z_1, \ldots, Z_n$ are cyclic and $n > 1$. By induction on n, $\Phi(G/Z_i) \leq Z(G/Z_i)$ for all i. Let $f \in \Phi(G)$ and $x \in G$. Then $[f, x] \in Z_1 \cap \cdots \cap Z_n = \{1\}$ so $f \in Z(G)$. It follows that $\Phi(G) \leq Z(G)$. □

Proof of Theorem 1. Let G be a p-group which possesses non-normal subgroups and we assume that each non-normal subgroup of G is contained in exactly one maximal subgroup. In particular, G is nonabelian with $d(G) \geq 2$ and so each subgroup of $\Phi(G)$ must be normal in G. Suppose that $\Phi(G)$ is nonabelian. Then $p = 2$ and $\Phi(G)$ is Hamiltonian, i.e., $\Phi(G) = Q \times E$, where $Q \cong Q_8$ and $exp(E) \leq 2$. But then E is normal in G and $\Phi(G/E) = \Phi(G)/E \cong Q_8$, contrary to a classical result of Burnside. Thus $\Phi(G)$ is abelian and each subgroup of $\Phi(G)$ is G-invariant.

If every cyclic subgroup of G is normal in G, then every subgroup of G is normal in G, a contradiction. Hence there is a non-normal cyclic subgroup $\langle a \rangle$ of G. In that case $a \notin \Phi(G)$ but $a^p \in \Phi(G)$ so that $\langle a \rangle \Phi(G)$ must be the unique maximal subgroup of G containing $\langle a \rangle$. It follows that $d(G) = 2$.

If $\Phi(G) \leq Z(G)$, then each maximal subgroup of G is abelian and so G is minimal nonabelian which gives the possibility (a) of our theorem.

From now on we assume that $\Phi(G) \not\leq Z(G)$. Set $G = \langle a, b \rangle$. Then $[a, b] \neq 1$ and $[a, b] \in \Phi(G)$. Therefore $\langle [a, b] \rangle$ is normal in G and $G/\langle [a, b] \rangle$

is abelian which implies that $G' = \langle [a,b] \rangle \neq \{1\}$. If $|G'| = p$, then the fact $d(G) = 2$ forces that G would be minimal nonabelian. But then $\Phi(G) \leq Z(G)$, a contradiction. Hence G' is cyclic of order $\geq p^2$.

(i) First assume $p > 2$. By Lemma 2, $\Phi(G) \leq Z(G)$, a contradiction.

(ii) Now assume $p = 2$. If $\Phi(G)$ is cyclic, then (since $\Phi(G) = \mho_1(G)$) G has a cyclic subgroup of index 2. But $|G'| \geq 4$ and so G is not isomorphic to M_{2^s}, $s \geq 4$, and so G is of maximal class, which gives the possibility (b) of our theorem. From now on we shall assume that $\Phi(G)$ is not cyclic.

Set $G = \langle a, b \rangle$, $k = [a,b]$, and $\langle z \rangle = \Omega_1(\langle k \rangle)$ so that $G' = \langle k \rangle$, $o(k) \geq 4$, and $\langle z \rangle \leq Z(G)$. Since $\langle a^2 \rangle$ and $\langle b^2 \rangle$ (being contained in $\Phi(G)$) are normal in G, we have $\Phi(G) = \langle a^2 \rangle \langle b^2 \rangle \langle k \rangle$ and so the abelian subgroup $\Phi(G)$ is a product of three cyclic subgroups which implies $d(\Phi(G)) = 2$ *or* 3.

From $[a,b] = k$ follows $a^{-1}(b^{-1}ab) = k$ and $b^{-1}(a^{-1}ba) = k^{-1}$ and so

(1) $$a^b = ak,$$

(2) $$b^a = bk^{-1}.$$

From (1) follows $(a^2)^b = (a^b)^2 = (ak)^2 = akak = a^2k^ak$ and so

(3) $$(a^2)^b = a^2(k^ak).$$

From (2) follows $(b^2)^a = (b^a)^2 = (bk^{-1})^2 = bk^{-1}bk^{-1} = b^2(k^{-1})^bk^{-1}$ and so

(4) $$(b^2)^a = b^2(k^bk)^{-1}.$$

We also have

$$a^2 = (a^2)^{b^2} = (a^2k^ak)^b = a^2k^akk^{ab}k^b$$

and so

(5) $$kk^ak^bk^{ab} = 1.$$

Finally, we compute (using (4))

$$\begin{aligned} (ab)^2 &= abab = a^2a^{-1}b^{-1}b^2ab = a^2(a^{-1}b^{-1}ab)(b^2)^{ab} \\ &= a^2kb^2(kk^b)^{-1} = a^2b^2(k^{-1})^b \end{aligned}$$

and so

(6) $$(ab)^2 = a^2b^2(k^{-1})^b.$$

Suppose that $G/\Phi(G)$ acts faithfully on $\langle k \rangle$. In that case $o(k) \geq 2^3$ and we may choose the generators $a, b \in G - \Phi(G)$ so that $k^a = k^{-1}$, $k^b = kz$ (where $\langle z \rangle = \Omega_1(\langle k \rangle)$). Using (3) and (4) we get $(a^2)^b = a^2$ (and so $a^2 \in Z(G)$) and $(b^2)^a = b^2k^{-2}z$. Since $k^a = k^{-1}$, we have $\langle k \rangle \cap \langle a \rangle \leq \langle z \rangle$. The subgroup $\langle b^2 \rangle$ (being contained in $\Phi(G)$) is normal in G and so $k^{-2}z \in \langle b^2 \rangle$ and $k^2 \in \langle b^2 \rangle$ (since $z \in \langle k^2 \rangle$). We have $\langle b \rangle \cap \langle k \rangle = \langle k^2 \rangle$ since $k^b = kz \neq k$ and so $k \notin \langle b \rangle$. If $b^2 \in \langle k^2 \rangle$, then $(b^2)^a = b^{-2}$ and on the other hand $(b^2)^a = b^2k^{-2}z$ and so $b^4 = k^2z$. But $b^2 \in \langle k^2 \rangle$ implies $b^4 \in \langle k^4 \rangle$, a contradiction. Hence $b^2 \notin \langle k^2 \rangle$ and so we can find an element $s \in \langle b^2 \rangle - \langle k \rangle$ such that $s^2 = k^{-2}$. Then

$(sk)^2 = s^2k^2 = 1$ and so sk is an involution in $\Phi(G)$ which is not contained in $\langle k \rangle$ and therefore $sk \neq z$. But $(sk)^b = sk^b = (sk)z$ and so $\langle sk \rangle$ is not normal in G, a contradiction.

We have proved that $G/\Phi(G)$ does not act faithfully on $\langle k \rangle$. Then we can choose our generator $a \in G - \Phi(G)$ so that $k^a = k$. Using (3) we get $(a^2)^b = a^2k^2$ and so $1 \neq k^2 \in \langle a^2 \rangle$ since $\langle a^2 \rangle$ is normal in G. From (5) we get $(k^2)^b = k^{-2}$. Suppose that $\langle k^2 \rangle = \langle a^2 \rangle$. Then we get $a^{-2} = (a^2)^b = a^2k^2$ and so $k^2 = a^{-4}$, a contradiction. We have obtained:

(7)
$$k^a = k,\ (a^2)^b = a^2k^2,\ (k^2)^b = k^{-2},\ \{1\} \neq \langle k^2 \rangle < \langle a^2 \rangle,\ o(a) = 2^n,\ n \geq 3.$$

Suppose that $k^b = kz$. Then (5) and (7) imply $k^4 = 1$ and so $k^b = kz = k^{-1}$. It follows that we have to analyze the following three possibilities for the action of b on $\langle k \rangle$: $k^b = k^{-1}z$ with $o(k) \geq 2^3$, $k^b = k$, and $k^b = k^{-1}$.

(ii1) Suppose $k^b = k^{-1}z$ with $o(k) \geq 2^3$. Then (4) gives $(b^2)^a = b^2z$ and so $z \in \langle b^2 \rangle$ (since $\langle b^2 \rangle$ is normal in G) and $\langle z \rangle < \langle b^2 \rangle$ because $b^2 \notin Z(G)$. Since (by (7)) $\langle k^2 \rangle < \langle a^2 \rangle$ and $o(k^2) \geq 4$, it follows $o(a^2) \geq 2^3$ and

$$\langle z \rangle = \Omega_1(\langle k \rangle) = \Omega_1(\langle a \rangle) = \Omega_1(\langle b \rangle) \leq Z(G).$$

From $o(a^2) \geq 2^3$, $k^2 \in \langle a^4 \rangle$, $o(k^2) \geq 4$, and $(k^2)^b = k^{-2}$ follows $(a^2)^b = a^{-2}z^{\epsilon}$ $(\epsilon = 0, 1)$ and $C_{\langle a^2 \rangle}(b) = \langle z \rangle$ so that $\langle a^2 \rangle \cap \langle b^2 \rangle = \langle z \rangle$. Let v be an element of order 4 in $\langle a^2 \rangle$ so that $v^2 = z$ and $v^b = v^{-1} = vz$. Let s be an element of order 4 in $\langle b^2 \rangle$ so that $s^2 = z$. We have $(vs)^2 = v^2s^2 = 1$ and so vs is an involution in $\Phi(G) - \langle a \rangle$ but $(vs)^b = v^{-1}s = (vs)z$, a contradiction.

(ii2) Suppose $k^b = k$ so that (5) and (7) imply $k^4 = 1$ and $k^2 = z$. Then (4) and (7) imply $(b^2)^a = b^2z$ and $(a^2)^b = a^2z$. Also, $\langle z \rangle < \langle a^2 \rangle$ and $\langle z \rangle < \langle b^2 \rangle$ since $\langle a^2 \rangle$ and $\langle b^2 \rangle$ are normal in G, $a^2 \notin Z(G)$ and $b^2 \notin Z(G)$. If $a^2 \in \langle b^2 \rangle$, then $a^2 \in Z(G)$ and if $b^2 \in \langle a^2 \rangle$, then $b^2 \in Z(G)$. This is a contradiction. Hence $D = \langle a^2 \rangle \cap \langle b^2 \rangle \geq \langle z \rangle$ and D is a proper subgroup of $\langle a^2 \rangle$ and $\langle b^2 \rangle$. Because of the symmetry, we may assume $o(a) \geq o(b)$ so that $|\langle a^2 \rangle/D| \geq |\langle b^2 \rangle/D| = 2^u$, $u \geq 1$. We set $(b^2)^{2^u} = d$ so that $D = \langle d \rangle$. We may choose an element $a' \in \langle a^2 \rangle - D$ such that $(a')^{2^u} = d^{-1}$. Then $(a'b^2)^{2^u} = 1$ and $\langle a'b^2 \rangle \cong C_{2^u}$ with $\langle a'b^2 \rangle \cap D = \{1\}$. On the other hand, $(a'b^2)^a = a'(b^2)^a = (a'b^2)z$, where $z \in D$, a contradiction.

(ii3) Finally, suppose $k^b = k^{-1}$. From (4) follows $(b^2)^a = b^2$ and so $b^2 \in Z(G)$. By (7), $(a^2)^b = a^2k^2$, $\langle k^2 \rangle < \langle a^2 \rangle$, and so $o(a^2) \geq 4$. Also, $(a^2k)^a = a^2k$, $(a^2k)^b = (a^2k^2)k^{-1} = a^2k$, and so $a^2k \in Z(G)$.

(ii3a) First assume $k \notin \langle a^2 \rangle$. We investigate for a moment the special case $o(k) = 4$, where $k^2 = z$, $\langle z \rangle = \Omega_1(\langle k \rangle) = \Omega_1(\langle a \rangle)$ and $(a^2)^b = a^2z$. If $o(a^2) > 4$, then take an element v of order 4 in $\langle a^4 \rangle$ so that $v^2 = z$ and $v^b = v$. In that case $(vk)^2 = v^2k^2 = 1$ and so vk is an involution in $\Phi(G) - \langle a^2 \rangle$ and $(vk)^b = vk^{-1} = (vk)z$, a contradiction. Hence $o(a^2) = 4$, $a^4 = z$, $k^2 = z = a^{-4}$, $(a^2)^b = a^2z = a^{-2}$, $\langle a^2, k \rangle$ is an abelian group of type

$(4,2)$ acted upon invertingly by b, and a^2k is a central involution in G. Now suppose $o(k) \geq 8$. In that case $o(k^2) \geq 4$, $k^2 \in \langle a^4 \rangle$, $o(a^2) \geq 8$, and b inverts $\langle k^2 \rangle$, which implies $(a^2)^b = a^{-2}z^\epsilon$, $\epsilon = 0, 1$. On the other hand, $(a^2)^b = a^2k^2$ and so $k^2 = a^{-4}z^\epsilon$. Let v be an element of order 4 in $\langle a^4 \rangle$ so that $v^2 = z$ and $v^b = v^{-1} = vz$. Then we compute:

$$(a^2vk)^2 = a^4zk^2 = z^{\epsilon+1}, \ \ (a^2vk)^b = a^2k^2v^{-1}k^{-1} = (a^2vk)z.$$

If $\epsilon = 1$, then a^2vk is an involution in $\Phi(G) - \langle a^2 \rangle$ and $\langle a^2vk \rangle$ is not normal in G. Thus, $\epsilon = 0$, $(a^2)^b = a^{-2}$, $k^2 = a^{-4}$, a^2k is an involution in $\Phi(G) - \langle a^2 \rangle$ and b inverts each element of $\langle a^2, k \rangle = \langle a^2 \rangle \times \langle a^2k \rangle$, where $a^2k \in Z(G)$.

We have proved that in any case $k^2 = a^{-4}$, $o(a^2) \geq 4$, $o(k) \geq 4$, and b inverts each element of the abelian group $\langle a^2, k \rangle = \langle a^2 \rangle \times \langle a^2k \rangle$, where a^2k is an involution contained in $Z(G)$.

It remains to determine $b^2 \in Z(G)$. Suppose $o(b^2) \geq 4$ and let $\langle s \rangle$ be a cyclic subgroup of order 4 in $\langle b^2 \rangle$ so that $s \in Z(G)$. Obviously, $s \notin \langle a^2, k \rangle$ since $Z(G) \cap \langle a^2, k \rangle = \langle z \rangle \times \langle a^2k \rangle \cong E_4$. Let v be an element of order 4 in $\langle a^2 \rangle$ so that $v^2 = z$ and $v^b = v^{-1} = vz$. We have:

$$(vs)^b = v^{-1}s = (vs)z \ \text{ and } \ (vs)^2 = v^2s^2 = zs^2.$$

If $s^2 = z$, then vs is an involution in $\Phi(G) - \langle a^2, k \rangle$ and $vs \notin Z(G)$, a contradiction. Hence $s^2 \neq z$ so that $\langle v, s \rangle = \langle v \rangle \times \langle s \rangle \cong C_4 \times C_4$. But $(vs)^b = (vs)z$, $(vs)^2 = zs^2 \neq z$, and so $\langle vs \rangle$ is not normal in G, a contradiction. It follows that $o(b^2) \leq 2$. Hence we have either $b^2 \in \langle z, a^2k \rangle$, $\Phi(G) = \langle a^2, k \rangle = \langle a^2 \rangle \times \langle a^2k \rangle$, and we have obtained the possibility (c1) of our theorem or b^2 is an involution in $\Phi(G) - \langle a^2, k \rangle$, $\Phi(G) = \langle a^2 \rangle \times \langle a^2k \rangle \times \langle b^2 \rangle$, and we have obtained the possibility (c2) of our theorem. Note that in both cases a centralizes $\Phi(G)$ and b inverts each element of $\Phi(G)$.

(ii3b) We assume $k \in \langle a^2 \rangle$. Since $o(k) \geq 4$, $k^b = k^{-1}$, $\langle a \rangle$ is normal in G, $o(a) \geq 8$, and b induces on $\langle a \rangle$ an automorphism of order 2, we get $a^b = a^{-1}z^\epsilon$, $\epsilon = 0, 1$, where $\langle z \rangle = \Omega_1(\langle a \rangle) = \Omega_1(\langle k \rangle)$. On the other hand, (1) gives $a^b = ak$ and so $k = a^{-2}z^\epsilon$ which gives $G' = \langle k \rangle = \langle a^2 \rangle \cong C_{2^{n-1}}$, where $o(a) = 2^n$, $n \geq 3$, and $z = a^{2^{n-1}}$.

Since $\Phi(G) = \langle a^2, b^2 \rangle$ and $\Phi(G)$ is noncyclic, we have $b^2 \notin \langle a^2 \rangle$ and we know that $b^2 \in Z(G)$. Suppose $o(b^2) \geq 4$ and let s be an element of order 4 in $\langle b^2 \rangle$. Let v be an element of order 4 in $\langle a^2 \rangle$ so that $v^2 = z$ and $v^b = v^{-1} = vz$. Then

$$(vs)^b = v^{-1}s = (vs)z \ \text{ and } \ (vs)^2 = v^2s^2 = zs^2.$$

If $s^2 = z$, then vs is an involution in $\Phi(G) - \langle a^2 \rangle$ and $vs \notin Z(G)$, a contradiction. Hence $s^2 \neq z$ so that $\langle v, s \rangle = \langle v \rangle \times \langle s \rangle \cong C_4 \times C_4$. But $\langle vs \rangle$ is not normal in G, a contradiction. Hence b^2 is an involution in $\Phi(G) - \langle a^2 \rangle$ and so $\Phi(G) = \langle a^2 \rangle \times \langle b^2 \rangle \cong C_{2^{n-1}} \times C_2$ and $Z(G) = \langle z \rangle \times \langle b^2 \rangle \cong E_4$. Also note that a centralizes $\Phi(G)$ and b inverts each element of $\Phi(G)$. We have obtained the possibility (d) of our theorem. □

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