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THREE-DIMENSIONAL HYPERBOLIC GEOMETRY WITH PLANES AND PLANE PARALLELISM AS ONLY PRIMITIVE NOTIONS

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Abstract. We show that Euclidean Möbius planes can be axiomatized in terms of circles and circle-tangency, and that 3-dimensional hyperbolic geometry can be axiomatized in terms of planes and plane-parallelism.

Euclidean Möbius planes have been axiomatized in the 1950s, and a particularly simple axiom system, in terms of *points*, *circles*, *incidence* I, and *orthogonality* $\perp$ can be found in [1, III.2]. The axiom system, which will be referred to as Σ, consists of (M I), (M II), (M III) (p. 205), (M) (p. 218), as well as (O I), (O II), (O III) (p. 240), (O IV), (O V), (O VI) (pp. 245, 246, 248).

In 1905 H. Liebmann [5] pointed out, and H. S. M. Coxeter has tirelessly reminded us since the mid-sixties of its existence (in most detail in [2], [3]), the existence of an isomorphism between hyperbolic three-space and the inversive plane. Under this isomorphism planes in hyperbolic three-space correspond to circles in the inversive plane.

We have shown in [6] that circles and circle-orthogonality (or another binary notion related to the inversive distance between two circles) can serve as primitive notions for Euclidean Möbius planes, and that the corresponding notions can axiomatize 3-dimensional hyperbolic geometry.

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We will show in the present note that Benz's axiom system can be rephrased with *circles* as the only variables and *circle tangency*, to be denoted by τ, as the only predicate, and thus that three-dimensional hyperbolic geometry over Euclidean fields can be axiomatized in terms of *planes* and *parallelism*.

To translate the axioms for Euclidean Möbius planes in a language with circles and tangency, we need to show how one can interpret 'points', as well as the relation of incidence, 'point P is on circle h' and orthogonality, 'circle a is orthogonal to circle b', in the language of τ.

(i) A *point* is defined to be a pair of tangent circles (h_1, h_2).

(ii) Point $P = (h_1, h_2)$ is *leaning on* sphere h_3, in symbols $(h_1, h_2)/h_3$, if h_3 is equal to h_1 or h_2, or else if h_3 is tangent to h_i for $i \in \{1, 2\}$, and if any two different circles m_1 and m_2 which are tangent to all the h_i for $i = 1, 2, 3$, are tangent to each other as well, i. e. with mod 3 addition in the indices,

$$(h_1,h_2)/h_3 \quad :\Leftrightarrow \quad (\tau(h_1h_2) \wedge (h_3 = h_1 \vee h_3 = h_2) \vee \{\bigwedge_{i=1}^{3} \tau(h_ih_{i+1})$$
$$\wedge[(\forall m_1m_2)\,(m_1 \neq m_2 \bigwedge_{1\leq i\leq 2, 1\leq j\leq 3} \tau(m_ih_j)) \rightarrow \tau(m_1m_2)]\}.$$

(iii) Point $P = (h_1, h_2)$ is *incident with* (or *on*) circle h_3, in symbols $P \text{ I } h_3$, if P is leaning on h_3 or else if h_3 is not tangent to any circle h on which P is leaning.

(iv) Two points $P = (h_1, h_2)$ and $Q = (l_1, l_2)$ are *equal* if and only if P is on both l_1 and l_2.

(v) $k \perp k' :\Leftrightarrow k \neq k' \wedge (\exists ABCuv)\, A \neq B \wedge (A, B \text{ I } k, k') \wedge C \text{ I } k$
$\wedge \tau(uk') \wedge \tau(vk') \wedge \tau(uv) \wedge (A, C \text{ I } u) \wedge (B, C \text{ I } v)$.

Definitions (i)-(iv) are taken from [7], and (v) from [4]. In definition (v) and below $(A_1, \ldots, A_n \text{ I } h_1, \ldots, h_m)$ stands for $\bigwedge_{1\leq i\leq n, 1\leq j\leq m} A_i \text{ I } h_j$, and points and incidence are to be considered as abbreviations, in other words we consider the definiens to be a sentence expressed in terms of circles and τ. We can now rephrase all the axioms in Σ as axioms in the language with circles as the only variables and τ as the only predicate. Let Σ_0 denote the axiom system obtained in this manner. Let $\Sigma_1 := \Sigma_0 \cup \{(1)\}$, where

$$\tau(ab) \leftrightarrow (\exists P)(\forall P')\,(P \text{ I } a, b) \wedge ((P' \text{ I } a, b) \rightarrow P = P'), \tag{1}$$

in which we consider again the point variables as having been replaced by two circle variables, as well as point-circle incidence and point equality having been replaced by their respective definitions, such that (1) is actually a sentence in terms of circles and τ.

Given that in every model of Σ_0, if we interpret sets of two tangent circles as points, and define the incidence and the orthogonality as done in (i)-(v), points, circles, incidence, and orthogonality have the desired interpretations, the definiens in (1) says precisely that the circles a and b are tangent, so that in every model of Σ_1 circles and τ must have the desired interpretation. We have thus shown that

THEOREM 1. *Euclidean Möbius planes are axiomatizable in terms of circles and circle tangency. Three-dimensional hyperbolic space is axiomatizable in terms of planes and parallelism.*

Given that ascending chains of models of 3-dimensional hyperbolic geometry with planes and parallelism as primitive notions must be models of hyperbolic geometry, by the Łoś-Suszko-Chang theorem there must be a $\forall\exists$-axiom system for it (and for Euclidean Möbius planes as well). Given that the axiom system obtained by the rephrasing described above is however not such an axiom system, it would be of interest to find a $\forall\exists$-axiom system for these two theories.

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