ISSN 0005-1144
ATKAFF 51(3), 293–301(2010)

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Performance Analysis of Multiserver ATM Buffers Routing Multimedia Traffics with Geometric Service Time

UDK 621.396.2:004.7
IFAC 5.8.0

Original scientific paper

In this article we analyze the performance of discrete-time multiserver ATM buffers with batch arrivals, geometric service time and priority scheduling. Such systems can be applied to the detailed performance evaluation of ATM multiplexers and ATM switching elements with dedicated buffer output queuing arrangements. Packets belong to two classes, class-1 packets represents real time traffics which is delay sensitive but loss insensitive, and class-2 packets represents non real time traffics which is delay insensitive but loss sensitive. Higher service priority is given to real time traffics (*e.g.* audio and video) to respond to the delay sensitivity in the presence of multiple servers in the system. Expressions for the probability generating functions of the main performance measures of the system and expected values of these expressions are evaluated.

Key words: Performance Analysis, Output buffered ATM switch, Multimedia Traffics, Discrete Priority Multiserver Queues

Analiza učinkovitosti ATM međuspremnika u višeserverskim sustavima kod usmjeravanja širokopojasnog prometa s geometrijskom razdiobom međudolaznih vremena. U ovome članku analiziramo učinkovitost vremenski diskretnih ATM međuspremnika u višeserverskim sustavima s batch dolascima, geometrijskom razdiobom međudolaznih vremena i prioritetnim pristupom. Ovakvi sustavi mogu se koristiti za detaljnu ocjenu učinkovitosti ATM multipleksera i ATM komutatora. Paketi pripadaju dvjema klasama; paketi klase 1 predstavljaju stvarno-vremenski promet koji je osjetljiv na kašnjenje ali nije osjetljiv na gubitak sadržaja, dok paketi klase 2 predstavljaju nestvarno-vremenski promet koji nije osjetljiv na kašnjenje ali je zato osjetljiv na gubitak sadržaja. Zbog osjetljivosti na kašnjenje, u višeserverskim sustavima viši prioritetni red daje se stvarno-vremenskom prometu (npr. audio i video). Ocijenjene su funkcije izvodnice glavnih mjera učinkovitosti, kao i njhove očekivane vrijednosti.

Ključne riječi: analiza učinkovitosti, ATM komutatori, širokopojasni promet, diskretni višeserverski prioritetni redovi

1 INTRODUCTION

In recent years, there has been much interest devoted to incorporating multimedia applications in IP networks. Different types of traffic need different Quality of Service (QoS) standards. For real-time applications, it is important to have bounded mean delay, while for non real time applications, the loss ratio (LR) is the restrictive quantity. Several types of time priority and space priority strategies have been presented by several authors, an overview of both types can be found in [1]. An overview of some basic Head of Line (HOL) priority queueing models can be found in [2], and the references therein. In [3] we presented a discrete time, batch arrival, multiserver queueing system, with infinite buffer and geometric service times with two rates and in [4] we proved that applying the proposed priority scheduling can improve the network performance and solve (or reduce) the problem of receiving disordered packets at the new foreign agent (FA) after the handover process of a mobile node. [5] and [6] have studied discrete-time priority queues with deterministic service times equal to one slot. [5] analyzes the system contents and cell delay in the case of a multiserver queue. In [6], the system contents and the delay for Markov modulated high priority arrivals and geometrically distributed low priority arrivals are presented. [7] analyzes the system contents for the different classes, for a queue fed by a two-state Markov modulated arrival process. [8] studies the system contents and cell delay, in the special case of an output queueing switch with Bernoulli arrivals. Furthermore, non-preemptive HOL priority queues have been considered in [9] and [10]. [9] analyses the interdeparture time distribution in a queue fed by a Poisson process. In [10], a non-preemptive queue in continuous time is presented, with a switched Poisson process arrival process for the high priority packets. [11] studies the mean waiting time, for a queue fed by an Independent and Identical distributed (i.i.d) arrival process. [12] studies a discrete-time MAP/G/1 queue, using matrix-analytic techniques.

At many instances, discrete-time multiserver queues have been successfully used in the performance evaluation of computer and communication systems in various contexts, such as TDM [13], voice-data integration [14], deflection routing [16] and more recently, ATM-switching technology. A lot of applications have in common that, in order to achieve certain QoS standards, some packets should get preferential treatment over others. In the ATM context for instance, traffics of certain service classes can not sustain large delays, while traffics of other classes can not tolerate large packet loss. Implementing a priority structure can resolve these conflicting requirements.

When analyzing systems with priorities, one can approximate the effect of high priority packets on the queueing of low-priority packets as (uncorrelated) server interruptions. However, to capture the effect completely, an analysis taking all classes of packets into account is unavoidable. While both analytical and numerical results have been obtained on many occasions with respect to performance measures related to the buffer contents distribution for the case of single server as well as for the multiserver case, the derivation of delay characteristics has received much less attention in the past.

In most cases, analytic results concerning the delay are limited to the mean value of the packet delay, which can be obtained by means of Little's Theorem [15], although other performance measures related to the delay, such as the variance and the tail distribution, are equally important for a wide range of applications, including system design in ATM-based B-ISDN networks.

In [17] and [18] packet delay in different multiserver systems is investigated. Also, the discrete-time queues with or without server interruptions have received great attention in the scientific literature. One of the earlier papers is the analysis in [19], who investigated a finite multiserver queue without server interruptions. Most authors, however, analyze an infinite system [20]. Some make specific assumptions about the arrival process [21] while general independent arrivals are considered elsewhere [22].

In this article we generalize our work published in [3]. In [3], we modelled a priority, discrete time, batch arrival, multiserver queueing system, with infinite buffer and two geometric service times with two parameters while in this work both classes of packets are having geometric service time with the same service rate.

This article is organized as follows. The model assumptions are presented in the next Section. In Sections 3 and 4 we analyze the system occupancy and the unfinished work. A special case of the study is given in Section 5 and the conclusion is given in the last Section.

2 MODEL ASSUMPTIONS

In this article, a two dimensional traffic model is formulated, and the main performance measures of the system are evaluated. The model can be described as $Geo^{A_1+A_2}/Geo/c$ which means batch arrival of two classes of packets with size A_1 of class-1 packets and size A_2 of class-2 packets, geometric distribution to the interarrival time, geometric service time, and c servers. Let us assume our mathematical model, with the following assumptions:

1. The time axis is divided into slots, each equal to the transmission time of one packet.
2. Packets belong to two classes, class-1 packets represent real time traffics (*e.g.* audio and video) which are delay sensitive but loss insensitive, and class-2 packets represent non real time traffics (data) which are delay insensitive but loss sensitive.
3. The number of servers in the queueing model, is equal to $c > 0$.
4. For every slot a packet will arrive with probability r and will not arrive with probability $\overline{r} = 1 - r$.
5. The packet interarrival time is geometrically distributed with parameter r.
6. A packet is either of class-1 with probability λ or of class-2 with probability $\overline{\lambda} = 1 - \lambda$.
7. Class-1 arrival rate is $r_1 = \lambda r$ and the class-2 arrival rate is $r_2 = \overline{\lambda} r$.
8. The interarrival times of class-1 and class-2 packets are geometrically distributed with parameters r_1 and r_2, respectively.
9. The packet arrival rate r, regardless of class, is related to r_1 and r_2 through the relation
$$r = r_1 + r_2. \tag{1}$$
10. Let A_1^k and A_2^k be Random Variables (RVs) representing the numbers of class-1 and class-2 packets arriving to the system in slot k respectively, with PGF $A_1(z)$ and $A_2(z)$.
11. The A_1^k are independent and identically distributed (iid) RVs, and so are the A_2^k.

12. Let $A^k = A_1^k + A_2^k$ be the total size of the batch that arrives into the system in slot k,with joint PGF $A(z_1, z_2)$.

13. Let D_1^{k+1}be the number of class-1 packets that will leave the system at the end of slot $k+1$. In each slot, given that a server is busy, a class-1 packet leaves the server with probability s or does not leave with probability $\overline{s}$ then the number of class-1 packets departing per slot follows a binomial distribution.

14. Let D_2^{k+1}be the number of class-2 packets that will leave the system at the end of slot $k+1$. The D_2^{k+1} are Bernoulli RVs, with parameter s. That is, a class-2 packet being served in a certain slot will end service by the end of that slot with probability s and will not do so with probability $\overline{s} = 1 - s$. This implies that the service times of class-1 and class-2 packets are geometrically distributed with parameter s.

15. Let $X^k, k = 1, 2, \ldots$, be the service time of class-1 and class-2 packets that arrive into the system in slot k. It is clear that the X^k are iid. Let x_i and $X(z)$ be the common distribution and common PGF of X^k. From the assumptions, it can be shown that $x_i = s\overline{s}^{i-1}$, and that

$$X(z) = \frac{sz}{1 - \overline{s}z}. \tag{2}$$

16. Class-1 packets have high service priority over class-2 packets. Thus we can look at the system as having two logical queues, one of class-1 packets and one of class-2 packets, as shown in Fig. 1. No class-2 packet can enter service unless the number of class-1 packets in the queue is less than the number of the servers.

17. A newly arriving batch of a given class is placed at the end of its appropriate queue to be served after all batches arriving ahead of it have been served.

18. After having been placed in the queue, the packets enter service on a First Come First Serve (FCFS) basis.

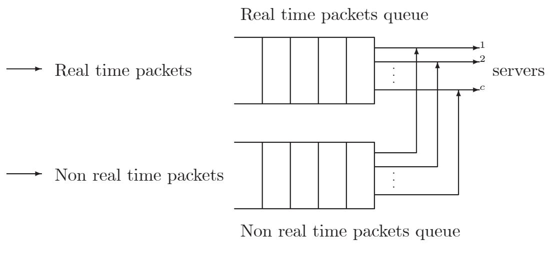

Fig. 1. Two logical queues with c servers

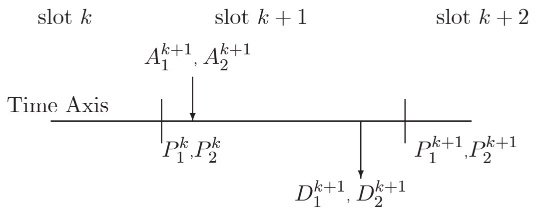

Fig. 2. The system occupancy of class-1 and class-2 packets in two successive slots

3 SYSTEM OCCUPANCY

In this Section we analyze the steady state system occupancy, i.e. the number of packets in the system at the end of an arbitrary slot during the steady state.

Let P_1^k, P_2^k be two RVs denoting the class-1 and class-2 system occupancy in slot k respectively, i.e. the number of class-1 and class-2 packets in the system at the end of slot k. Using the RVs defined above, and referring to the assumptions in Section 2 and to Fig. 1, and Fig. 2, the system occupancy of the two classes at the end of slot $k+1$ is given by the two stochastic equations

$$P_i^{k+1} = P_i^k - D_i^{k+1} + A_i^{k+1}, \qquad i = 1, 2. \tag{3}$$

The pairs (P_1^k, P_2^k), $k = 0, 1, \cdots$, form a two dimensional Markov chain with P_1^k and P_2^k being mutually dependent for each slot k. Let $p_{i,j}^k$ be the joint distribution of the pair (P_1^k, P_2^k). That is, $p_{i,j}^k = \Pr\left[P_1^k = i, P_2^k = j\right]$. And let $P^k(z_1, z_2)$ be the PGF of $p_{i,j}^k$. That is,

$$P^k(z_1, z_2) \triangleq \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} p_{i,j}^k z_1^i z_2^j = E\left[z_1^{P_1^k} z_2^{P_2^k}\right]. \tag{4}$$

Note the use of $\triangleq$ because the generating function represents an entire infinite sequence. Now we will embark on deriving $P^k(z_1, z_2)$, from which the common (stationary) PGF $P(z_1, z_2)$ will be obtained. Using (3) and (4), we get

$$\begin{aligned} P^{k+1}(z_1, z_2) = & E\left[z_1^{A_1^{k+1}} z_2^{A_2^{k+1}}\right] \times \\ & E\left[z_1^{P_1^k - D_1^{k+1}} z_2^{P_2^k - D_2^{k+1}}\right]. \end{aligned} \tag{5}$$

The separation of the expectations in (5) is a consequence of the fact that both A_1^{k+1} and A_2^{k+1} are independent of P_1^k, D_1^{k+1}, P_2^k and D_2^{k+1}. The first factor in (5) is just $A(z_1, z_2)$. The second factor can be evaluated by employing some tedious, but straightforward, first probability principles. After substituting for the first factor and evaluating the second, (5) becomes

$$
\begin{aligned}
&P^{k+1}(z_1, z_2) = \\
&A(z_1, z_2)P^k(z_1, z_2) \sum_{m=0}^{c} z_1^{-m} \binom{c}{m} s^m\, \overline{s}^{c-m} + A(z_1, z_2)\times \\
&\left\{ \sum_{i=0}^{c-1} \left(\sum_{j=0}^{c-i-1} \sum_{m=0}^{i} \sum_{n=0}^{j} z_1^{i-m} z_2^{j-n} p_{i,j}^{k} \binom{i}{m} s^m\, \overline{s}^{i-m} \times \right.\right. \\
&\binom{j}{n} s^n \overline{s}^{j-n} - \sum_{j=0}^{c-i-1} \sum_{m=0}^{i} \sum_{n=0}^{c-i} z_1^{i-m} z_2^{j-n} p_{i,j}^{k} \binom{i}{m} s^m\, \overline{s}^{i-m} \times \\
&\left. \binom{c-i}{n} s^n \overline{s}^{c-i-n} \right) + \sum_{i=0}^{c-1} \left(\sum_{j=0}^{\infty} \sum_{m=0}^{i} \sum_{n=0}^{c-i} z_1^{i-m} z_2^{j} p_{i,j}^{k} \times \right. \\
&\binom{i}{m} s^m\, \overline{s}^{i-m} \binom{c-i}{n} s^n \overline{s}^{c-i-n} - \sum_{j=0}^{\infty} \sum_{m=0}^{c} z_1^{i-m} z_2^{j} p_{i,j}^{k} \times \\
&\left.\left. \binom{c}{m}\ s^m\, \overline{s}^{c-m} \right) \right\}.
\end{aligned}
\tag{6}
$$

For system stability, it is required that the total arrival rate into the system r be strictly less than the class-1 and class-2 service rate s. In other words, it is required that

$$r < s. \tag{7}$$

If this condition is met, the system will reach steady state after a sufficiently large number of slots. That is, as $k \to \infty$, the distributions $p_{i,j}^k$ converge to the common distributions $p_{i,j}$ and the PGF $P^k(\cdot,\cdot)$ converges to the common PGF $P(\cdot,\cdot)$, making (6) yields

$$
\begin{aligned}
P(z_1, z_2) = &\frac{A(z_1, z_2) z_1^c}{z_1^c - A(z_1, z_2)\left(s + \overline{s} z_1\right)^c} \times \\
&\left\{ \sum_{i=0}^{c-1} \sum_{j=0}^{c-i-1} \left[\left(s + \overline{s} z_2\right)^j \left(s + \overline{s} z_1\right)^i \right.\right. \\
&\left. - z_2^{j-c+i} \left(s + \overline{s} z_2\right)^{c-i} \left(s + \overline{s} z_1\right)^i \right] p_{i,j} \\
&+ \sum_{i=0}^{c-1} \sum_{j=0}^{\infty} \left[z_2^{j-c+i} \left(s + \overline{s} z_1\right)^i \left(s + \overline{s} z_2\right)^{c-i} \right. \\
&\left.\left. - z_1^{i-c} z_2^{j} \left(s + \overline{s} z_1\right)^c \right] p_{i,j} \right\}.
\end{aligned}
\tag{8}
$$

Note that, in (8) the summation over i will give a polynomial in z_1 with the coefficients functions of z_2. To determine the unknown probabilities $p_{i,j}$ in (8),we will use Rouche's Theorem [23, p.123]. To apply Rouche's Theorem on the denominator of (8), consider

$$f(z_1) = z_1^c, \quad g(z_1) = -A(z_1, z_2)\left(s + \overline{s} z_1\right)^c.$$

It can be shown that

$$|f(z_1)| > |g(z_1)|.$$

Then Rouche's Theorem can be applied to the denominator of (8) and we conclude that both functions z_1^c and $z_1^c - A(z_1, z_2)\left(s_1 + \overline{s_1} z_1\right)^c$ have the same number of zeros within the unit disk, namely c zeros, all of them are functions of z_2. Let us denote these c zeros by $\xi_m(z_2)$, $m = 0, 1, \ldots, c-1$, then

$$\xi_m^c(z_2) = A(\xi_m(z_2), z_2)\left(s + \overline{s}\xi_m(z_2)\right)^c.$$

For any such function $\xi_m(z_2)$ such that $|\xi_m(z_2)| \leq 1, m = 0, 1, \ldots, c-1$ we must have a simple zero. One of these zeros (functions) equals 1 if $z_2 = 1$ and we denote it by $\xi_0(z_2), \xi_0(1) = 1$ which does not yield any information about the unknown probabilities in the numerator of (8) because at this value the denominator of (8) vanishes ($A(\xi_0(1), 1) = 1$), regardless of these unknowns. Let us denote the zeros other than $\xi_0(z_2)$ by $\xi_1(z_2), \xi_2(z_2), \ldots, \xi_{c-1}(z_2)$. As the function $P(z_1, z_2)$ is bounded, both the numerator and the denominator of (8) must be zero for the same values of z_1, z_2. Substituting with the zeros $\xi_m(z_2)$, $m = 1, 2, \ldots, c-1$ of the denominator of (8) in the numerator of (8), provided that $A(\xi_m(z_2), z_2) \neq 0$, they yield the following $c-1$ simultaneous equations in the unknowns $p_{i,j}$

$$
\begin{aligned}
&\sum_{i=0}^{c-1} \left(\sum_{j=0}^{c-i-1} \xi_m^c(z_2) \left[\left(s + \overline{s} z_2\right)^j \left(s + \overline{s}\xi_m(z_2)\right)^i \right.\right. \\
&\left. - z_2^{j-c+i} \left(s + \overline{s} z_2\right)^{c-i} \left(s + \overline{s}\xi_m(z_2)\right)^i \right] p_{i,j} \\
&+ \sum_{j=0}^{\infty} \left[\xi_m^c(z_2) z_2^{j-c+i} \left(s + \overline{s}\xi_m(z_2)\right)^i \left(s + \overline{s} z_2\right)^{c-i} \right. \\
&\left.\left. - \xi_m^i(z_2) z_2^j \left(s + \overline{s}\xi_m(z_2)\right)^c \right] p_{i,j}, \right. \\
&m = 1, 2, \ldots, c-1.
\end{aligned}
\tag{9}
$$

The equation number c needed to solve for the unknown probabilities comes from applying the normalization condition $P(1,1) = 1$ and using L'Hospital's rule, to get

$$
\begin{aligned}
cs - A'(1,1) = &\sum_{i=0}^{c-1} \left(\sum_{j=0}^{c-i-1} \left(j\overline{s} + c - (i+j)\right) p_{i,j} \right. \\
&\left. + \sum_{j=0}^{\infty} \overline{s}\left(i - c\right) p_{i,j} \right).
\end{aligned}
\tag{10}
$$

Equations (9) and (10) represent the c equations which can be solved to give the unknown probabilities $p_{i,j}$ as soon as the number of servers c is defined.

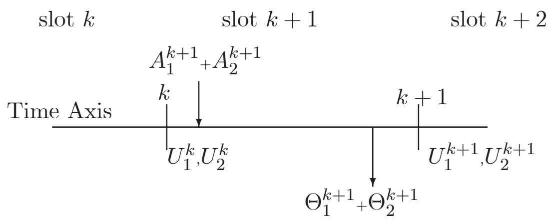

Fig. 3. The unfinished work of class-1 and class-2 packets in two successive slots

4 UNFINISHED WORK

In this Section we analyze the system unfinished work, i.e. the number of slots needed to empty the system of all its contents at the end of an arbitrary slot . We will assume the following:

1. Let U_1^k, U_2^k be two RVs denoting class-1 and class-2 packets unfinished work in slot k, i.e. the number of slots needed to empty the system of all class-1 and class-2 packets existing at the end of slot k.

2. Let $U^k = U_1^k + U_2^k$ be a RV denoting the total unfinished work, regardless of the packet class, at the end of slot k.

3. Let $\Theta_1^{k+1}, \Theta_2^{k+1}$ be the number of slots elapsing at the end of slot $k+1$ from U_1^k, U_2^k, the unfinished work in the preceding slot, respectively.

4. Let $X^{(i)}$ be a RV representing the service time of the i^{th} class-1 or class-2 packet of all class-1 or class-2 packets arriving in the same slot. i.e. it represents the number of slots that the i^{th} class-1 or class-2 packet spends in the server, note that the RV $X^{(i)}$ have the same distribution and the same PGF as the RV X.

5. Let G_1^k, G_2^k be two RVs representing the service times of class-1 and class-2 packets arriving together in slot k with PGF $G_1(z), G_2(z)$ respectively. That is

$$\begin{aligned} G_1^k &= \sum_{i=1}^{A_1^k} X^{(i)}, \\ G_2^k &= \sum_{i=1}^{A_2^k} X^{(i)}. \end{aligned} \tag{11}$$

6. Let $G(z_1, z_2)$ be the joint PGF of G_1^k and G_2^k, i.e. of their joint distribution $g_{i,j}$. That is

$$\begin{aligned} G(z_1, z_2) &\triangleq \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} g_{i,j} z_1^i z_2^j \\ &= E\left[z_1^{G_1^k} z_2^{G_2^k}\right] \qquad (12) \\ &= E\left[z_1^{\sum_{m=1}^{A_1^k} X^{(m)}} z_2^{\sum_{n=1}^{A_2^k} X^{(n)}}\right]. \qquad (13) \end{aligned}$$

Using the fact that $X^{(i)}$are iid RVs with the common PGF $X(z)$ in (2), then

$$G(z_1, z_2) = A\left(X(z_1), X(z_2)\right). \tag{14}$$

Using the RVs defined above, taking into consideration their stated distributions and interdependence, referring to the assumptions and to Fig. 3, the unfinished work of the two classes of packets in an arbitrary slot $k+1$ is given by the following equations

$$U_i^{k+1} = U_i^k - \Theta_i^{k+1} + G_i^{k+1}, \qquad i = 1, 2. \tag{15}$$

Let $u_{i,j}^k$ be the joint distribution of the pair $\left(U_1^k, U_2^k\right)$. That is $u_{i,j}^k = \Pr\left[U_1^k = i, U_2^k = j\right]$. And let $U^k(z_1, z_2)$ be the joint PGF of $u_{i,j}^k$. That is

$$U^k(z_1, z_2) \triangleq \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} u_{i,j}^k z_1^i z_2^j = E\left[z_1^{U_1^k} z_2^{U_2^k}\right]. \tag{16}$$

Then, using (15) in (16), we get

$$\begin{aligned} U^{k+1}(z_1, z_2) =& E\left[z_1^{G_1^{k+1}} z_2^{G_2^{k+1}}\right] \times \\ & E\left[z_1^{U_1^k - \Theta_1^{k+1}} z_2^{U_2^k - \Theta_2^{k+1}}\right]. \end{aligned} \tag{17}$$

The first factor in (17) was obtained in (14). The second factor can be evaluated by employing some tedious, yet straightforward, first probability principles. After substituting for the first factor from (14) and evaluating the second, (17) becomes

$$\begin{aligned} U^{k+1}(z_1, z_2) =& A\left(X(z_1), X(z_2)\right) \times \\ & \left\{\sum_{i=0}^{c-1}\sum_{j=0}^{c-i-1}\left(1 - z_2^{j-(c-i)}\right) u_{i,j}^k \right. \\ & \left. + \sum_{i=0}^{c-1}\sum_{j=0}^{\infty}\left(z_2^{j-(c-i)} - z_1^{i-c} z_2^j\right) u_{i,j}^k \right\} \\ & + A\left(X(z_1), X(z_2)\right) z_1^{-c} U^k(z_1, z_2). \end{aligned} \tag{18}$$

At the steady state, after large number of slots i.e. as $k \to \infty$, the PGF $U^k(z_1, z_2)$ converges to the common PGF $U(z_1, z_2)$ and the probabilities $u_{i,j}^k$ converge to the common distributions $u_{i,j}$. So taking the limit of (18), as $k \to \infty$, and solving for $U(z_1, z_2)$, make it yields

$$U(z_1,z_2) = \frac{A\left(X(z_1),X(z_2)\right)}{z_1^c - A\left(X(z_1),X(z_2)\right)} \times \left\{ \sum_{i=0}^{c-1} \sum_{j=0}^{c-i-1} z_1^c \left(1 - z_2^{j-(c-i)}\right) u_{i,j} + \sum_{i=0}^{c-1} \sum_{j=0}^{\infty} z_1^c \left(z_2^{j-(c-i)} - z_1^{i-c} z_2^j\right) u_{i,j} \right\}. \tag{19}$$

To verify (19), we will use the following cases:

1. Let $c = 1$ in (19), so we turn the system to $Geo^{A_1+A_2}/D/1$ queueing system with priority, to get

$$U(z_1,z_2) = \frac{A\left(X(z_1),X(z_2)\right)\left(\left(z_1 - z_2\right)U\left(0,z_2\right)\right.}{z_2\left(z_1 - A\left(X(z_1),X(z_2)\right)\right)} \frac{\left. + z_1\left(z_2 - 1\right)\ u_{0,0}\right)}{z_2\left(z_1 - A\left(X(z_1),X(z_2)\right)\right)}. \tag{20}$$

   Equation (20) is identical to the joint unfinished work expression obtained in the analysis of the $Geo^{A_1+A_2}/D/1$ queuing system in [24].

2. Let $z_1 = z_2 = z$ in (19), now we turn the system to a system of a single class of packets i.e. removing the priority, and thus we will have after simplifications

$$U(z,z) = \frac{A\left(X(z)\right)}{z^c - A\left(X(z)\right)} \sum_{n=0}^{c-1} \left(z^c - z^{n)}\right) u_n. \tag{21}$$

   Equation (21) is identical after appropriate changes to that of the unfinished work in the analysis of the system $Geo^A/Geo/c$. Note that, in (19) the summation over i will give a polynomial in z_1 with the coefficients are functions of z_2. To determine the unknown probabilities $u_{i,j}$ in (19) we will apply Rouche's Theorem, as was done before in Section 3, to conclude that both functions z_1^c and $z_1^c - A\left(X(z_1),X(z_2)\right)$ has the same number of zeros within the unit disk, namely c zeros. We consider now functions $\xi_m(z_2)$, $m = 0,1,\ldots,c-1$ within the unit disk $|z_1| \le 1, |z_2| \le 1$ for which the denominator of (19) is zero. We then have

$$\xi_m^c(z_2) - A(\xi_m(z_2), z_2) = 0. \tag{22}$$

   For any such function $\xi_m(z_2)$ such that $|\xi_m(z_2)| \le 1, m = 0,1,\ldots,c-1$ we must have a simple zero. One of these zeros (functions) equals 1 if $z_2 = 1$ and we denote it by $\xi_0(z_2), \xi_0(1) = 1$ which does not yield any information about the unknown probabilities $u_{i,j}$ in the numerator because at this value the denominator of (19) vanishes ($A(\xi_0(1),1) = 1$), regardless of these unknowns. let us denote the zeros other than $\xi_0(z_2)$ of the denominator as $\xi_1(z_2), \xi_2(z_2), \ldots, \xi_{c-1}(z_2)$. As the PGF $U(z_1,z_2)$ is bounded on the unit disk $|z_1| \le 1, |z_2| \le 1$, both the numerator and the denominator of (19) must be zero for the same values of z_1, z_2. Substituting with the zeros $\xi_m(z_2)$, $m = 1,2,\ldots,c-1$ of the denominator of (19) in the numerator, they yield the following $c-1$ simultaneous equations in the unknown probabilities $u_{i,j}$

$$\sum_{i=0}^{c-1} \xi_m^c(z_2) \left(\sum_{j=0}^{c-i-1} \left(1 - z_2^{j-(c-i)}\right) u_{i,j} + \sum_{j=0}^{\infty} \left(z_2^{j-(c-i)} - \xi_m^{i-c}(z_2) z_2^j\right) u_{i,j} \right) = 0, \quad m = 1,2,\ldots,c-1, \tag{23}$$

and we will get the equation number c, needed to solve for the unknown probabilities $u_{i,j}$, from applying the normalizing condition $U(1,1) = 1$ and using L'Hospital's rule, to find

$$\sum_{i=0}^{c-1} \sum_{j=0}^{c-i-1} \left(c - (j+i)\right) u_{i,j} = c - \frac{1}{s} A^{'}\left(1,1\right). \tag{24}$$

Equations (23) and (24) can be solved to get the unknown probabilities $u_{i,j}$. However, there is more direct way. We can see that the second factor in (19) is a polynomial in z_1 of degree c with the coefficients functions of z_2, then, it is uniquely determined by c independent conditions. So any function that satisfies these c conditions, is the unique solution we are looking for. Now, let us assume that this factor is written in the form

$$\begin{aligned}\Phi(z_1,z_2) =& B(z_2)(z_1 - \xi_1(z_2))(z_1 - \xi_2(z_2))\ldots \\ & (z_1 - \xi_{c-1}(z_2))(z_1 - \xi_0(z_2)) \\ =& B(z_2) \prod_{i=0}^{c-1} (z_1 - \xi_i(z_2)).\end{aligned} \tag{25}$$

If the value of the function $B(z_2)$ can be chosen such that the function $\Phi(z_1,z_2)$, defined in (25), also satisfies the condition

$$U(z_1,z_2) = \frac{A\left(X(z_1),X(z_2)\right)B(z_2)}{z_1^c - A\left(X(z_1),X(z_2)\right)} \times \prod_{i=0}^{c-1}(z_1-\xi_i(z_2)), \tag{26}$$

where $\xi_i(z_2)$ are the c zeros of $z_1^c - A\left(X(z_1),X(z_2)\right)$ within the unit disk of the complex $z-$plane then the formula of $\Phi(z_1,z_2)$ given by (25) will be the desired formula of $\Phi(z_1,z_2)$, where the function $B(z_2)$ is defined such that (25) is satisfied. Marginal PGF of class-1 packets unfinished work can be derived by putting $z_1 = z, z_2 = 1$ in (19), to get

$$\begin{aligned} U_1(z) &= U(z,1) \\ &= \frac{A_1\left(X(z)\right)}{z^c - A_1\left(X(z)\right)}\sum_{i=0}^{c-1}\left(z^c - z^i\right)u_i, \end{aligned} \tag{27}$$

Now, to find the unknown probabilities $u_{i,j}$, it can be shown, using the same approach followed in Section 3 to find the unknowns, that

$$U_1(z) = \left(c - \frac{A_1'(1)}{s}\right)\frac{A_1\left(X(z)\right)(z-1)}{z^c - A_1\left(X(z)\right)} \times \prod_{i=1}^{c-1}\frac{(z-\xi_i)}{(1-\xi_i)}, \tag{28}$$

where ξ_i, $i = 1, 2, \cdots c-1$ represents the $c-1$ roots of $z^c - A_1\left(X(z)\right)$ inside the unit disk of the complex $z-$plane, excluding the root $\xi_0 = 1$.

5 SPECIAL CASE

As a special case, let us consider a system with one server. Let $c = 1$ in (8), so we turn the system to $Geo^{A_1+A_2}/Geo/1$ with priority, to get after calculations

$$P(z_1,z_2) = \frac{A(z_1,z_2)\left\{(z_2-1)\,z_1 s p_{0,0} +\right.}{z_2\left(z_1 - A(z_1,z_2)\left(z_1\bar{s}+s\right)\right)} \frac{\left. s\left(z_1-z_2\right)P(0,z_2)\right\}}{z_2\left(z_1 - A(z_1,z_2)\left(z_1\bar{s}+s\right)\right)}, \tag{29}$$

To determine $P(z_1,z_2)$ in (29) completely we must find $p_{0,0}$ and $P(0,z_2)$. First, we find $p_{0,0}$ using normalization condition $P(1,1) = 1$ after applying L'Hospital's rule, thus

$$p_{0,0} = \frac{s - A'(1,1)}{s}. \tag{30}$$

Substituting for $p_{0,0}$ from (30) in (29), thus

$$P(z_1,z_2) = \frac{A(z_1,z_2)\left\{sP(0,z_2)(z_1-z_2)\right.}{z_2\left(z_1 - (s+\bar{s}z_1)A(z_1,z_2)\right)} + \frac{\left.\left(s - A'(1,1)\right)z_1(z_2-1)\right\}}{z_2\left(z_1 - (s+\bar{s}z_1)A(z_1,z_2)\right)}. \tag{31}$$

To verify the result in (31), substitute for $z_1 = z_2 = z$ in (31) which reduces the system to a system with one class of packets. So

$$P(z) = \frac{\left(s - A'(1,1)\right)(z-1)A(z)}{z - (s+\bar{s}z)A(z)}. \tag{32}$$

Note that, (32) is similar to that in [25, pp. 87] but with replacing $(s+\bar{s}z)$ with $B(z)$ after simplifying that in [25]. Now, to find an explicit form for the unknown function $P(0,z_2)$ in (31) we will apply Rouche's and Lagrange's Theorems to (31). Applying Rouche's Theorem to the denominator of (31), as done before in Section 3, we conclude that the function $z_1 - (s+\bar{s}z_1)A(z_1,z_2)$ has one zero within the unit disk. Using Lagrange's theorem, it can be shown that this zero is

$$\xi = \sum_{k=1}^{\infty}\frac{1}{k!}\left\{\frac{\partial^{k-1}}{\partial z_1^{k-1}}\left(A(z_1,z_2)(s+\bar{s}z_1)\right)^k\right\}_{z_1=0}. \tag{33}$$

Since $P(z_1,z_2)$ is analytic inside and on the unit disk, the numerator of (31) should have the same zero. Hence ξ as given by (33) is also a zero for the numerator of (31). That is, provided that $A(\xi,z_2) \neq 0$, we get

$$P(0,z_2)\,s(\xi - z_2) + \left(s - A'(1,1)\right)\xi(z_2-1) = 0. \tag{34}$$

Solving (34) for $P(0,z_2)$ and substituting for it in (31), we get

$$P(z_1,z_2) = \frac{A(z_1,z_2)\left(s - A'(1,1)\right)(z_2-1)(\xi - z_1)}{\left(z_1 - (s+\bar{s}z_1)A(z_1,z_2)\right)(\xi - z_2)}. \tag{35}$$

The marginal PGF $P_1(z)$ of class-1 packets existing in the system at the end of an arbitrary slot can be derived from (29) by putting $z_1 = z$, $z_2 = 1$ in (29), as follows

$$P_1(z) = \frac{sA_1(z)P(0,1)(z-1)}{z - (s+\bar{s}z)A_1(z)}. \tag{36}$$

To find $P(0,1)$, we use the normalization condition $P_1(1) = 1$ in (36) after applying L'Hospital's rule, then

$$P(0,1) = 1 - \rho_1, \tag{37}$$

where $\rho_1 = \frac{A_1'(1)}{s}$. Using (37) in (36), we get

$$P_1(z) = \frac{A_1(z)\left(s - A_1'(1)\right)(z-1)}{z - (s+\bar{s}z)A_1(z)}. \tag{38}$$

By comparison with single class queueing theory (see e.g. [25, pp. 87.]), we can first see that this equation possesses a familiar form, which is reassuring regarding the progress of the derivation. Second, we can identify ρ_1 as the class-1 traffic intensity of the system. Finally, we can see from the equation that ρ_1 is the probability that the system is class-1 busy, or the class-1 system utilization (i.e. the fraction of time the system is utilized by class-1 packets). To get an expression for the unfinished work of one server, let $c = 1$ in (19), so

$$U(z_1, z_2) = \frac{A\left(X(z_1), X(z_2)\right)\left\{z_1\left(z_2 - 1\right)u_{0,0} + \right.}{z_2\left(z_1 - A\left(X(z_1), X(z_2)\right)\right)} \\ \frac{\left.\left(z_1 - z_2\right)U(0, z_2)\right\}}{z_2\left(z_1 - A\left(X(z_1), X(z_2)\right)\right)}, \tag{39}$$

where the unknowns probabilities $u_{0,0}$ and $U(0, z_2)$ in (39) can be evaluated using the same approach we followed early in this Section. Marginal PGF of class-1 packets unfinished work is obtained by substituting for $c = 1$ in (28), thus

$$U_1(z) = \left(1 - \frac{A_1^{'}(1)}{s}\right)\frac{A_1\left(X(z)\right)(z-1)}{z - A_1\left(X(z)\right)}. \tag{40}$$

We will get PGF of the total unfinished work of the system when $c = 1$ by substituting for $z_1 = z_2 = z$ in (39), hence

$$U(z, z) = \frac{A\left(X(z)\right)(z-1)\,u_{0,0}}{z - A\left(X(z)\right)}. \tag{41}$$

To find the unknown $u_{0,0}$ in (41) we use the normalization condition $U(z, z) = 1$ in (41), after applying L'Hospital's rule, to get

$$u_{0,0} = 1 - \frac{1}{s}A^{'}\left(1, 1\right). \tag{42}$$

Now, Substituting for $u_{0,0}$ from (42) in (41), we finally find

$$U(z) = U(z, z) = \left(1 - \frac{1}{s}A^{'}\left(1, 1\right)\right)\frac{A\left(X(z)\right)(z-1)}{z - A\left(X(z)\right)}. \tag{43}$$

6 CONCLUSION

In this article, two dimensional traffic models are formulated. The main contributions have been using multiserver case, rather than one server, as has been the case in most previous studies. Two transmission requirements of the two classes has been modeled. We apply our results to an ATM switch which has been modelled as a priority, discrete time, multiserver, batch arrival, infinite buffer queueing system, with geometric service time.

We have obtained PGFs for three performance measures: system occupancy, unfinished work, and total unfinished work. The PGFs have been used to derive the corresponding expectations. The results of the analysis have been verified in many ways. First, they have been used to generate the results of some previous analyses (mainly those of single class) as special cases. Second, they obviously seem to preserve classical queueing relations (e.g. the famous Little's formula).

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Received: 2009-10-19
Accepted: 2010-08-04