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D-CONTINUUM X ADMITS A WHITNEY MAP FOR $C(X)$ IF AND ONLY IF IT IS METRIZABLE

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Abstract. The main purpose of this paper is to prove: a) a D-continuum X admits a Whitney map for $C(X)$ if and only if it is metrizable, b) a continuum X admits a Whitney map for $C^2(X)$ if and only if it is metrizable.

1. Introduction

All spaces in this paper are compact Hausdorff and all mappings are continuous. The weight of a space X is denoted by $w(X)$.

A *generalized arc* is a Hausdorff continuum with exactly two non-separating points (end points) x, y. Each separable arc is homeomorphic to the closed interval $I = [0, 1]$.

We say that a space X is *arcwise connected* if for every pair x, y of points of X there exists a generalized arc L with end points x, y.

Let X be a space. We define its hyperspaces as the following sets:

$$\begin{aligned} 2^X &= \{F \subseteq X : F \text{ is closed and nonempty}\}, \\ C(X) &= \{F \in 2^X : F \text{ is connected}\}, \\ C^2(X) &= C(C(X)), \\ X(n) &= \{F \in 2^X : F \text{ has at most } n \text{ points}\},\ n \in \mathbb{N}. \end{aligned}$$

For any finitely many subsets S_1, ..., S_n, let

$$\langle S_1, ..., S_n \rangle = \left\{ F \in 2^X : F \subset \bigcup_{i=1}^{n} S_i,\ \text{and } F \cap S_i \neq \emptyset,\ \text{for each } i \right\}.$$

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The topology on 2^X is the Vietoris topology, i.e., the topology with a base $\{< U_1, ..., U_n >: U_i$ is an open subset of X for each i and each $n < \infty$ }, and $C(X), X(n)$ are subspaces of 2^X. Moreover, $X(1)$ is homeomorphic to X.

Let X and Y be the spaces and let $f : X \to Y$ be a mapping. Define $2^f : 2^X \to 2^Y$ by $2^f(F) = f(F)$ for $F \in 2^X$. By [8, p. 170, Theorem 5.10] 2^f is continuous and $2^f(C(X)) \subset C(Y)$, $2^f(X(n)) \subset Y)$. The restriction $2^f|C(X)$ is denoted by $C(f)$.

A continuum X is called a *D-continuum* if for every pair C, D of its disjoint non-degenerate subcontinua there exists a subcontinuum $E \subset X$ such that $C \cap E \neq \emptyset \neq D \cap E$ and $(C \cup D) \setminus E \neq \emptyset$.

Lemma 1.1. [6, Lemma 2.3]. *If X is an arcwise connected continuum, then X is a D-continuum.*

Lemma 1.2. [6, Lemma 2.4]. *If X is a locally connected continuum, then X is D-continuum.*

Let Λ be a subspace of 2^X. By a *Whitney map* for Λ [9, p. 24, (0.50)] we will mean any mapping $g : \Lambda \to [0, +\infty)$ satisfying

a) if $A, B \in \Lambda$ such that $A \subset B$ and $A \neq B$, then $g(A) < g(B)$ and
b) $g(\{x\}) = 0$ for each $x \in X$ such that $\{x\} \in \Lambda$.

If X is a metric continuum, then there exists a Whitney map for 2^X and $C(X)$ ([9, pp. 24-26], [3, p. 106]). On the other hand, if X is non-metrizable, then it admits no Whitney map for 2^X [1]. It is known that there exist non-metrizable continua which admit and ones which do not admit a Whitney map for $C(X)$ [1].

In the sequel we shall use the following theorem.

Theorem 1.3. [6, Theorem 3.3]. *If a D-continuum X admits a Whitney map for $C(X)$, then $C(X) \setminus X(1)$ is metrizable and $w(C(X) \setminus X(1)) \leq \aleph_0$.*

It is known that if X is a continuum, then $C(X)$ is arcwise connected [7, p. 1209, Theorem]. Hence, using Lemma 1.1 and Theorem 1.3, we obtain the following corollary.

Corollary 1.4. *If X is a continuum which admits a Whitney map for the hyperspace $C^2(X)$, then $C^2(X) \setminus C(X)(1)$ is metrizable and*

$$w(C^2(X) \setminus C(X)(1)) \leq \aleph_0.$$

2. Main theorems

In this section we shall prove the main theorems of the paper, Theorems 2.2 and 2.6.

For this purpose we shall use the notion of a network of a topological space.

A family $\mathcal{N} = \{M_s : s \in S\}$ of a subsets of a topological space X is a *network* for X if for every point $x \in X$ and any neighbourhood U of x there exists an $s \in S$ such that $x \in M_s \subset U$ [2, p. 170]. The *network weight* of a space X is defined as the smallest cardinal number of the form card$(\mathcal{N})$, where $\mathcal{N}$ is a network for X; this cardinal number is denoted by $nw(X)$.

THEOREM 2.1. [2, p. 171, Theorem 3.1.19]. *For every compact space X we have $nw(X) = w(X)$.*

Now we shall prove the main theorem of this paper.

THEOREM 2.2. *A D-continuum X admits a Whitney map for $C(X)$ if and only if it is metrizable.*

PROOF. If X is metrizable, then X admits a Whitney map ([3, p. 106], [9, pp. 24-26]). Conversely, suppose that X admits a Whitney map for $C(X)$. By Theorem 1.3 we have that $C(X)\setminus X(1)$ is metrizable and $w(C(X)\setminus X(1)) \leq \aleph_0$. This means that there exists a countable base $\mathcal{B} = \{B_i : i \in \mathbb{N}\}$ of $C(X)\setminus X(1)$. For each B_i let $C_i = \{x \in X : x \in B,\ B \in B_i\}$, i.e., the union of all continua B contained in B_i.

CLAIM 1. *The family $\{C_i : i \in N\}$ is a network of X.* Let X be a point of X and let U be an open subsets of X such that $x \in U$. There exists and open set V such that $x \in V \subset \mathrm{Cl}V \subset U$. Let K be a component of $\mathrm{Cl}V$ containing x. By Boundary Bumping Theorem [10, p. 73, Theorem 5.4] K is non-degenerate and, consequently, $K \in C(X)\setminus X(1)$. Now, $\langle U\rangle \cap (C(X)\setminus X(1))$ is a neighbourhood of K in $C(X)\setminus X(1)$. It follows that there exists a $B_i \in \mathcal{B}$ such that $K \in B_i \subset \langle U\rangle \cap (C(X)\setminus X(1))$. It is clear that $C_i \subset U$ and $x \in C_i$ since $x \in K$. Hence, the family $\{C_i : i \in N\}$ is a network of X.

CLAIM 2. $nw(X) = \aleph_0$. Apply Claim 1 and the fact that $\mathcal{B}$ is countable.

CLAIM 3. $w(X) = \aleph_0$. By Claim 2 we have $nw(X) = \aleph_0$. Moreover, by Theorem 2.1 $w(X) = \aleph_0$.

CLAIM 4. Finally, X is metrizable. $\square$

Since each arcwise connected continuum is a D-continuum (Lemma 1.1) we have the following corollary which generalize Theorem 3.4 of the paper [5, p. 19].

COROLLARY 2.3. *An arcwise connected continuum X admits a Whitney map for $C(X)$ if and only if X is metrizable.*

An *arboroid* is a hereditarily unicoherent arcwise connected continuum. A metrizable arboroid is a *dendroid*. If X is an arboroid and $x, y \in X$, then there exists a unique arc $[x, y]$ in X with endpoints x and y.

A point t of an arboroid X is said to be a *ramification point* of X if t is the only common point of some three arcs such that it is the only common point of any two, and an end point of each of them.

If an arboroid X has only one ramification point t, it is called a *generalized fan* with the top t. A metrizable generalized fan is called a *fan*.

The following corollary is a stronger result than Theorem 4.20 in [4] which states that a generalized fan X admits a Whitney map for $C(X)$ if and only if it is metrizable.

COROLLARY 2.4. *Let X be an arboroid. Then X admits a Whitney map for $C(X)$ if and only if it is metrizable.*

PROOF. Apply Corollary 2.3. $\square$

From Lemma 1.2 it follows that each locally connected continuum is a D-continuum. Thus, we have the following corollary of Theorem 2.2.

COROLLARY 2.5. *A locally connected X continuum admits a Whitney map for $C(X)$ if and only if it is metrizable.*

The following theorem shows that the existence of a Whitney map for $C^2(X)$ is equivalent to metrizability of X.

THEOREM 2.6. *A continuum X admits a Whitney map for $C^2(X)$ if and only if X is metrizable.*

PROOF. From Corollary 1.4 it follows that if X a continuum which admits a Whitney map for $C^2(X)$, then $C^2(X) \setminus C(X)(1)$ is metrizable and $w(C^2(X) \setminus C(X)(1)) \leq \aleph_0$. By Theorem 2.2 $w(C(X)) = \aleph_0$ since $C(X)$ is arcwise connected. This means that $w(X) = \aleph_0$ since X is homeomorhic to $X(1) \subset C(X)$. Hence, X is metrizable. $\square$

It is known [2, p. 171, Corollary 3.1.20] that if a compact space X is the countable union of its subspaces $X_n, n \in \mathbb{N}$, such that $w(X_n) \leq \aleph_0$, then $w(X) \leq \aleph_0$. Using this fact and theorems proved in the previous section we obtain the following theorems.

THEOREM 2.7. *If a continuum X is the countable union either of its D-subcontinua or of its arcwise connected subcontinua, then X admits a Whitney map for $C(X)$ if and only if it is metrizable.*

THEOREM 2.8. *If a compact space X is the countable union of its subcontinua and admits a Whitney map for $C^2(X)$, then X is metrizable.*

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