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A LOCAL TO GLOBAL SELECTION THEOREM FOR SIMPLEX-VALUED FUNCTIONS

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Abstract. Suppose we are given a function $\sigma : X \to K$ where X is a paracompact space and K is a simplicial complex, and an open cover $\{U_\alpha \,|\, \alpha \in \Gamma\}$ of X, so that for each $\alpha \in \Gamma$, $f_\alpha : U_\alpha \to |K|$ is a map that is a selection of σ on its domain. We shall prove that there is a map $f : X \to |K|$ which is a selection of σ. We shall also show that under certain conditions on such a set of maps or on the complex K, there exists a $\sigma : X \to K$ with the property that each f_α is a selection of σ on its domain and that there is a selection $f : X \to |K|$ of σ. The term selection, as used herein, will always refer to a map f, i.e., continuous function, having the property that $f(x) \in \sigma(x)$ for each x in the domain.

1. Introduction

The purpose of this paper is to prove certain selection theorems of the type "local to global" where the target is a polyhedron. We will apply an enhanced version from [6] of a method due to E. Michael [5] to pass from local selections to global ones.

Throughout this paper, map will mean continuous function and if K is a simplicial complex, then its polyhedron $|K|$ will always have the weak topology induced by the triangulation K. As in [1] and [5], paracompact spaces are assumed to be Hausdorff.

Let us recall that if K is a simplicial complex and $f : X \to |K|$ and $g : X \to |K|$ are functions, then f is said to be *contiguous* to g if for each $x \in X$ there exists a simplex $\tau \in K$ such that $f(x)$, $g(x) \in \tau$; one says that g is a K-*modification* of f if for each $x \in X$, whenever $f(x) \in \tau \in K$, then

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$g(x) \in \tau$. When f is contiguous to g, it need not be true that either one of them is a K-modification of the other. Also contiguity is not an equivalence relation although it is reflexive and symmetric.

Let $\sigma : X \to K$ be a function where X is a space and K is a simplicial complex. If $\{U_\alpha \mid \alpha \in \Gamma\}$ is an open cover of X and for each $\alpha \in \Gamma$, $f_\alpha : U_\alpha \to |K|$ is a map, then the statement that $\{f_\alpha \mid \alpha \in \Gamma\}$ is a *selection* of σ means that for each $\alpha \in \Gamma$ and $x \in U_\alpha$, $f_\alpha(x) \in \sigma(x)$. This is a generalization of the usual notion of a selection because if $\Gamma = \{\alpha\}$ is singleton and $U_\alpha = X$, then f_α is a selection in the usual sense.

If we consider a family $\{f_\alpha \mid \alpha \in \Gamma\}$ of maps f_α of subsets U_α of X to $|K|$ (e.g., as above), one may say that the maps f_α are "pointwise contiguous in pairs" if for each $x \in X$ and $\alpha, \beta \in \Gamma$ such that $x \in U_\alpha \cap U_\beta$, $f_\alpha(x)$ and $f_\beta(x)$ lie in a simplex of K.

We now state the main results of this paper, the first of which strengthens Theorem 1.2 of [2] by removing the requirement that the space X be hereditarily normal.

Theorem 1.1. *Let X be a paracompact space, K a simplicial complex, and $\sigma : X \to K$ a function. Suppose that $\{U_\alpha \mid \alpha \in \Gamma\}$ is an open cover of X and $\{f_\alpha : U_\alpha \to |K| \mid \alpha \in \Gamma\}$ is a selection of σ. Then there exists a selection $f : X \to |K|$ of σ.*

The term "infinite simplex" is probably not in common use.

Definition 1.2. *Let K be a simplicial complex and $\{\tau_i \mid i \in \mathbb{N}\}$ a collection of simplexes of K such that for each $i \in \mathbb{N}$, $\dim \tau_i = i - 1$ and τ_i is a face of τ_{i+1}. Then we shall refer to $\{\tau_i \mid i \in \mathbb{N}\}$ as an* **infinite simplex** *of K.*

A simplicial complex K contains an infinite simplex if and only if there is a full subcomplex L of K which has a countably infinite set of vertices. Although infinite simplexes are not simplicial complexes, the concept is useful as follows. If we are given a collection $\{f_\alpha \mid \alpha \in \Gamma\}$ of maps f_α of subsets U_α of $X = \{0\}$ that is pointwise contiguous in pairs to the polyhedron $|K|$ of the full complex $K(\mathbb{N})$ whose vertex set is $\mathbb{N}$, there need not be a $\sigma : X \to K(\mathbb{N})$ such that this collection is a selection for σ. For example, fix an infinite simplex $\{\tau_i \mid i \in \mathbb{N}\}$ in $K(\mathbb{N})$, and for each $i \in \mathbb{N}$, let $f_i(0)$ be the barycenter of τ_i.

The statement that a simplicial complex K contains no infinite simplexes is equivalent to the statement that each simplex of K is contained in a *principal simplex*, that is, one which is contained in no other simplex of K.

Theorem 1.3. *Let X be a paracompact space, K a simplicial complex, and $\{U_\alpha \mid \alpha \in \Gamma\}$ an open cover of X. Suppose that for each $\alpha \in \Gamma$, there exists a map $f_\alpha : U_\alpha \to |K|$ such that if $\alpha, \beta \in \Gamma$, then one of the maps $f_\alpha|U_\alpha \cap U_\beta$ and $f_\beta|U_\alpha \cap U_\beta$ is a K-modification of the other. If either,*

(A) : *K contains no infinite simplex or,*

(B) : $\{U_\alpha \mid \alpha \in \Gamma\}$ *is a point-finite open cover of* X,

then there exist,

1. $\sigma : X \to K$ *such that* $\{f_\alpha \mid \alpha \in \Gamma\}$ *is a selection of* σ, *and*
2. *a map* $f : X \to |K|$ *such that* f *is a selection of* σ.

The following lemma is an easy consequence of the definitions.

Lemma 1.4. *Let* K *be a simplicial complex,* X *a space,* $\sigma : X \to K$ *a function,* $\{U_\alpha \mid \alpha \in \Gamma\}$ *an open cover of* X, *and for each* $\alpha \in \Gamma$, $f_\alpha : U_\alpha \to |K|$ *a map. If* $\{f_\alpha \mid \alpha \in \Gamma\}$ *and* $f : X \to |K|$ *are selections of* σ, *then* $f|U_\alpha$ *is contiguous to* f_α *for each* $\alpha \in \Gamma$. □

Lemma 1.4 shows that in both Theorem 1.1 and Theorem 1.3, for each $\alpha \in \Gamma$, $f|U_\alpha$ is contiguous to f_α. It is worth noting (for example, Lemma 4′ of [4]) that this implies that $f|U_\alpha \simeq f_\alpha$.

2. Lemmas

Let us state Lemma 1.1 of [2] which, as indicated in the latter, follows from the proof of Lemma 4 of [4].

Lemma 2.1. *Let* X *be a normal space and* K *a simplicial complex. Let* $A \subset X$ *be a closed set and* V, $U \subset X$ *open sets such that* $A \subset V \subset \overline{V} \subset U$. *If* $h : U \to |K|$ *and* $g : V \to |K|$ *are maps such that* $h|V$ *is contiguous to* g, *then there exists a map* $k : U \to |K|$ *such that:*

1. k *is contiguous to* h,
2. $k|\,A = g|\,A$,
3. $k|(U \setminus V) = h|(U \setminus V)$, *and*
4. *if* $x \in V$ *and* $h(x)$, $g(x) \in \sigma \in K$, *then* $k(x) \in \sigma$. □

Lemma 2.2. *Let* K *be a simplicial complex that has no infinite simplexes and* $\{x_\alpha \mid \alpha \in \Gamma\}$ *be a nonempty subset of* $|K|$ *such that for each* α, $\beta \in \Gamma$, *at least one of the following is true:*

1. *if* $x_\alpha \in \tau \in K$, *then* $x_\beta \in \tau$, *or*
2. *if* $x_\beta \in \tau \in K$, *then* $x_\alpha \in \tau$.

Then there exists a principal simplex σ *of* K *such that* $\{x_\alpha \mid \alpha \in \Gamma\} \subset \sigma$.

Proof. It is sufficient to find a simplex σ of K such that $x_\alpha \in \sigma$ for all $\alpha \in \Gamma$. Fix $\alpha_1 \in \Gamma$ and let σ_1 be the simplex of K such that $x_{\alpha_1} \in \operatorname{int}\sigma_1$. Define $\Gamma_1 = \{\gamma \in \Gamma \mid x_\gamma \in \sigma_1\}$. If $\Gamma_1 = \Gamma$, then put $\sigma = \sigma_1$, and we are done. Otherwise, choose $\alpha_2 \in \Gamma \setminus \Gamma_1$. Let σ_2 be the simplex of K such that $x_{\alpha_2} \in \operatorname{int}\sigma_2$. Since $x_{\alpha_2} \notin \sigma_1$, then (1) and (2) of the hypothesis show that we must have $x_{\alpha_1} \in \sigma_2$. But then σ_2 intersects the interior of σ_1, showing that σ_1 is a face of σ_2. Put $\Gamma_2 = \{\gamma \in \Gamma \mid x_\gamma \in \sigma_2\}$. Clearly, $\Gamma_1 \subset \Gamma_2$. If $\Gamma_2 = \Gamma$, then put $\sigma = \sigma_2$, and we are done. If one continues this process, it will end after a finite number, say n, of steps since K has no infinite simplexes. This means that $\Gamma_n = \Gamma$. Designate $\sigma = \sigma_n$. Our proof is complete. □

In case the complex K in Lemma 2.2 contains infinite simplexes, then the following result can be useful. Its proof can be modelled on the preceding one.

LEMMA 2.3. *Let K be a simplicial complex and Σ be a nonempty finite subset of $|K|$ such that for each x, $y \in \Sigma$, at least one of the following is true:*

1. *if $x \in \tau \in K$, then $y \in \tau$, or*
2. *if $y \in \tau \in K$, then $x \in \tau$.*

Then there exists a simplex σ of K such that $\Sigma \subset \sigma$. □

Recall that a simplicial complex K is called *locally finite-dimensional* if for each vertex v of K there exists a nonnegative integer n such that $\dim \tau \leq n$ for each simplex τ of K having v as a vertex.

COROLLARY 2.4. *If K is a locally finite-dimensional simplicial complex, then K has no infinite simplexes.* □

The next statement follows from Corollary 2.4 and (A) of Theorem 1.3.

COROLLARY 2.5. *Let X be a paracompact space, K a locally finite-dimensional simplicial complex, and $\{U_\alpha \,|\, \alpha \in \Gamma\}$ an open cover of X. Suppose that for each $\alpha \in \Gamma$, there exists a map $f_\alpha : U_\alpha \to |K|$ such that if α, $\beta \in \Gamma$, then one of the maps $f_\alpha|U_\alpha \cap U_\beta$ and $f_\beta|U_\alpha \cap U_\beta$ is a K-modification of the other. Then there exist,*

1. *$\sigma : X \to K$ such that $\{f_\alpha \,|\, \alpha \in \Gamma\}$ is a selection of σ, and*
2. *a map $f : X \to |K|$ such that f is a selection of σ.* □

3. PROOFS OF THEOREMS

Recall that a collection $\mathcal{K}$ of subsets of a space is called *discrete* if each point of the space has a neighborhood that intersects at most one element of the collection. An application of Theorem 1.1.13 of [1] shows that this is true if and only if the collection of the closures of the elements of $\mathcal{K}$ is discrete.

In [6] we observed that Theorem 3.6(a) of E. Michael [5] could be improved by applying Michael's Proposition 3.3 in his proof of that theorem. This was stated as Lemma 1 in [6]. Here it is in a slightly different form (taking into account the citation from [1] mentioned in the preceding paragraph).

PROPOSITION 3.1. *Let X be a paracompact space and $\mathcal{G}$ a collection of subsets of X. Suppose that the following are true:*

1. *$\mathcal{G}$ contains an open cover of X,*

2. *if* $U \in \mathcal{G}$ *and* W *is open in* U*, then* $W \in \mathcal{G}$*,*

3. *if* U*,* Q *are open elements of* $\mathcal{G}$*, then* $U \cup Q \in \mathcal{G}$*, and*

4. *if* $\mathcal{K} \subset \mathcal{G}$ *is a discrete collection of open subsets of* X*, then* $\bigcup \mathcal{K} \in \mathcal{G}$*.*

Then the entire space X *is in* $\mathcal{G}$*.* □

We now give our proof of Theorem 1.1.

Proof. Let $\mathcal{G}$ be the collection of all open subsets G of X such that there exists an open neighborhood U of $\overline{G}$ and a selection $h : U \to |K|$ of $\sigma|U$. Our proof will be complete if we show that $X \in \mathcal{G}$, and we shall apply Proposition 3.1 to do this. To obtain (1) of Proposition 3.1, just apply the shrinking theorem for normal spaces to the open cover $\{U_\alpha \,|\, \alpha \in \Gamma\}$. Item (2) is obviously so.

To obtain (3) of Proposition 3.1, suppose that U_0 and Q_0 are in $\mathcal{G}$. Choose open sets U, Q of X such that $\overline{U}_0 \subset U$, $\overline{Q}_0 \subset Q$ and maps $h : U \to |K|$, $g : Q \to |K|$ so that the pair $\{h, g\}$ is a selection of the restriction of σ to $U \cup Q$. Also let U_1, U_2, Q_1, Q_2 be sets open in X such that, $\overline{U}_0 \subset U_1 \subset \overline{U}_1 \subset U_2 \subset \overline{U}_2 \subset U$, and $\overline{Q}_0 \subset Q_1 \subset \overline{Q}_1 \subset Q_2 \subset \overline{Q}_2 \subset Q$.

Put $A = \overline{U}_1 \cap \overline{Q}_1$ and $V = U_2 \cap Q_2$. Apply Lemma 2.1 to the preceding data and let $k : U \to |K|$ be as indicated there. Define a map $l : \overline{U}_1 \cup \overline{Q}_1 \to |K|$ by,

$$l(x) = \begin{cases} g(x) & \text{if } x \in \overline{Q}_1, \\ k(x) & \text{if } x \in \overline{U}_1. \end{cases}$$

An application of (2) of Lemma 2.1 shows that l is a well-defined map. Let us show that l is a selection of $\sigma|\overline{U}_1 \cup \overline{Q}_1$. Since $x \in \overline{Q}_1$ implies that $l(x) = g(x) \in \sigma(x)$, then we only have to consider $x \in \overline{U}_1 \setminus \overline{Q}_1$. Hence $l(x) = k(x)$ and $x \in U$. There are two cases.

Case 1. $x \in U \setminus V$. By (3) of Lemma 2.1, $k(x) = h(x) \in \sigma(x)$.

Case 2. $x \in V = U_2 \cap Q_2 \subset U \cap Q$. Then because of the selection assumption, $h(x)$, $g(x) \in \sigma(x)$, so by (4) of Lemma 2.1, $l(x) = k(x) \in \sigma(x)$.

Of course this implies that $l|U_1 \cup Q_1 : U_1 \cup Q_1 \to |K|$ is a selection of $\sigma|U_1 \cup Q_1$. Surely, $U_1 \cup Q_1$ is an open neighborhood of $\overline{U_0 \cup Q_0} = \overline{U}_0 \cup \overline{Q}_0$, so we have demonstrated that (3) holds true.

At last, (4) of Proposition 3.1 is obtained by a simple application of the fact that X is collectionwise normal (see, e.g., Theorem 5.1.18 of [1]). □

Here is our proof of Theorem 1.3.

Proof. Consider first (A). Let $x \in X$; put $\Gamma_x = \{\alpha \in \Gamma \,|\, x \in U_\alpha\}$. Suppose that α, $\beta \in \Gamma_x$. Since one of the maps $f_\alpha|U_\alpha \cap U_\beta$ and $f_\beta|U_\alpha \cap U_\beta$ is a K-modification of the other, then at least one of the conditions (1), (2) of Lemma 2.2 is true. In order to obtain $\sigma : X \to K$, apply Lemma 2.2 to Γ_x, obtaining a principal simplex $\sigma(x) \in K$ such that $x_\alpha = f_\alpha(x) \in \sigma(x)$ for all $\alpha \in \Gamma_x$. Certainly $\{f_\alpha \mid \alpha \in \Gamma\}$ is a selection of $\sigma : X \to K$. The existence of a map $f : X \to |K|$ such that f is a selection of σ follows from Theorem 1.1.

A proof of (B) can be done by the same technique as the preceding, using Lemma 2.3 in place of Lemma 2.2. ◻

Let us close the paper with a Proposition showing that the infinite simplex hypothesis in part (A) of Theorem 1.3 cannot be dismissed.

PROPOSITION 3.2. *Part* (A) *of Theorem 1.3 may fail to be true if* K *contains infinite simplexes.*

PROOF. As before, let $K(\mathbb{N})$ be the full complex whose vertex set is $\mathbb{N}$. Define X to be $\mathbb{N}$ with the discrete topology. Then X is a paracompact space. For each $n \in \mathbb{N}$, let τ_n be the simplex of $K(\mathbb{N})$ whose vertex set is $\{k \in \mathbb{N} \mid k \leq n\}$, and put b_n equal the barycenter of τ_n.

For each $n \in \mathbb{N}$, let $U_n = \{1, n\}$ and $f_n : U_n \to |K(\mathbb{N})|$ be given by $f_n(t) = b_n$ for all $t \in U_n$. Surely, $\{U_n \mid n \in \mathbb{N}\}$ is an open cover of X and for each $n \in \mathbb{N}$, $f_n : U_n \to |K(\mathbb{N})|$ is a map. Also, if $m, n \in \mathbb{N}$, then $U_m \cap U_n = \{1\}$. If $m \leq n$ and $f_n(1) = b_n$ lies in a simplex τ of $K(\mathbb{N})$, then τ_n is a face of τ and $f_m(1) = b_m$ lies in a face of τ_n, so it lies in a face of τ. Hence, one of $f_m|U_m \cap U_n$ and $f_n|U_m \cap U_n$ is a $K(\mathbb{N})$-modification of the other. But there is no $\sigma : X \to K(\mathbb{N})$ such that $\{f_n \mid n \in \mathbb{N}\}$ is a selection of σ because no simplex from $K(\mathbb{N})$ contains $\{f_n(1) \mid n \in \mathbb{N}\}$. ◻

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