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Fuzzy bases and the fuzzy dimension of fuzzy vector spaces*

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Abstract. In this paper, new definitions of a fuzzy basis and a fuzzy dimension for a fuzzy vector space are presented. A fuzzy basis for a fuzzy vector space (E, μ) is a fuzzy set β on E. The cardinality of a fuzzy basis β is called the fuzzy dimension of (E, μ). The fuzzy dimension of a finite dimensional fuzzy vector space is a fuzzy natural number. For a fuzzy vector space, any two fuzzy bases have the same cardinality. If $\widetilde{E}_1$ and $\widetilde{E}_2$ are two fuzzy vector spaces, then $\dim(\widetilde{E}_1 + \widetilde{E}_2) + \dim(\widetilde{E}_1 \cap \widetilde{E}_2) = \dim(\widetilde{E}_1) + \dim(\widetilde{E}_2)$ and $\dim(\widetilde{\ker} f) + \dim(\widetilde{\mathrm{im}} f) = \dim(\widetilde{E})$ hold without any restricted conditions.

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1. Introduction

After Katsaras and Liu [4] introduced the concept of fuzzy vector spaces, many scholars investigated its properties and characteristics (see [1, 2, 5, 6, 7], etc). For a fuzzy vector space, Lowen [5] gave the definition of fuzzy linear independence which is a direct generalization of normal linear independence. Subsequently P. Lubczonok [7] introduced another alternative concept of fuzzy linear independence, and introduced a definition of fuzzy bases which are the bases of corresponding vector space and satisfy fuzzy linear independence. In this case, the fuzzy bases of fuzzy vector space are still crisp. At the same time, a concept of dimension for a fuzzy vector space was introduced. Lowen [5] defined a dimension of the n-dimensional Euclidean space as an n-tuple. P. Lubczonok [7] defined a fuzzy dimension for all fuzzy vector spaces as a non-negative real number or infinity, and proved that for finite dimensional vector spaces $\widetilde{E}_1$ and $\widetilde{E}_2$ the statement

$$\dim(\widetilde{E}_1 + \widetilde{E}_2) + \dim(\widetilde{E}_1 \cap \widetilde{E}_2) = \dim(\widetilde{E}_1) + \dim(\widetilde{E}_2) \tag{1}$$

is true under certain conditions.

In this paper, we redefine fuzzy basis of a fuzzy vector space (E, μ) as a fuzzy set β on E and discuss its properties. Subsequently, by the definition of fuzzy basis,

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we redefine fuzzy dimension of (E, μ), it is the cardinality of a fuzzy basis β. The fuzzy dimension of a finite dimensional fuzzy vector space is a fuzzy natural number. For a fuzzy vector space, any two fuzzy bases have the same cardinality. We also prove that the formulas (1) and $\dim(\widetilde{\mathrm{im} f}) + \dim(\widetilde{\ker f}) = \dim(\widetilde{E})$ hold without any restricted conditions.

2. Preliminaries

For a set X, a fuzzy set A on X is a mapping $A : X \to [0,1]$. The set of all fuzzy sets on X are denoted by $[0,1]^X$. For $A \in [0,1]^X$ and $a \in [0,1]$, define

$$A_{[a]} = \{x \in X : A(x) \geqslant a\}, \ \ A_{(a)} = \{x \in X : A(x) > a\}.$$

We often do not distinguish a crisp subset A of X and its characteristic function χ_A.

The cardinality of a fuzzy set was introduced in [11] and the notion of fuzzy natural integers was also investigated and developed in [3, 8]. In [10], they were generalized to an L-fuzzy setting as follows:

Definition 1 (see [10]). *Let $\mathbb{N}$ denote the set of all natural numbers and let L be a complete lattice. An L-fuzzy natural integer is an antitone mapping $\lambda : \mathbb{N} \to L$ satisfying*

$$\lambda(0) = \top_L, \quad \bigwedge_{n \in \mathbb{N}} \lambda(n) = \bot_L,$$

where $\top_L$ and $\bot_L$ are the largest element and the smallest element in L, respectively. The set of all L-fuzzy natural integers is denoted by $\mathbb{N}(L)$.

Definition 2 (see [11]). *Let A be a fuzzy set, and define a map $|A| : \mathbb{N} \to [0,1]$ such that $\forall n \in \mathbb{N}$, $|A|(n) = \bigvee \left\{a \in (0,1] : |A_{[a]}| \geqslant n\right\}$. Then $|A| \in \mathbb{N}([0,1])$, which is called the cardinality of A.*

Definition 3 (see [10]). *For any $\lambda, \mu \in \mathbb{N}([0,1])$, the addition $\lambda + \mu$ of λ and μ is defined as follows: for any $n \in \mathbb{N}$,*

$$(\lambda + \mu)(n) = \bigvee_{k+l=n} (\lambda(k) \wedge \mu(l)).$$

Theorem 1 (see [10]). *For any $\lambda, \mu \in \mathbb{N}([0,1])$ and for any $a \in [0,1)$, one has $(\lambda + \mu)_{(a)} = \lambda_{(a)} + \mu_{(a)}$.*

Definition 4 (see [4]). *A fuzzy vector space is a pair $\widetilde{E} = (E, \mu)$, where E is a vector space over a field F, and $\mu : E \to [0,1]$ is a mapping satisfying $\mu(kx + ly) \geq \mu(x) \wedge \mu(y)$ for any $x, y \in E, k, l \in F$.*

If $\widetilde{E} = (E, \mu)$ is a fuzzy vector space, define

$$\widetilde{E}_{[a]} = \mu_{[a]} = \{x \in E : \mu(x) \geqslant a\}, \quad \widetilde{E}_{(a)} = \mu_{(a)} = \{x \in E : \mu(x) > a\},$$

then both $\widetilde{E}_{[a]}$ and $\widetilde{E}_{(a)}$ are the subspaces of E [4].

Definition 5 (see [7]). *Let $\widetilde{E}_1 = (E, \mu_1)$ and $\widetilde{E}_2 = (E, \mu_2)$ be two fuzzy vector spaces on E. The intersection and the sum of $\widetilde{E}_1$ and $\widetilde{E}_2$ are respectively defined to be $\widetilde{E}_1 \cap \widetilde{E}_2 = (E, \mu_1 \wedge \mu_2)$ and $\widetilde{E}_1 + \widetilde{E}_2 = (E, \mu_1 + \mu_2)$, where $\mu_1 \wedge \mu_2$ and $\mu_1 + \mu_2$ are respectively defined by*

$$\begin{aligned}(\mu_1 \wedge \mu_2)(x) &= \mu_1(x) \wedge \mu_2(x) \\ (\mu_1 + \mu_2)(x) &= \bigvee_{x = x_1 + x_2} (\mu_1(x_1) \wedge \mu_2(x_2)) \\ &= \bigvee_{x_1 \in E} (\mu_1(x_1) \wedge \mu_2(x - x_1)).\end{aligned}$$

From [7] we know that both $\widetilde{E}_1 \cap \widetilde{E}_2$ and $\widetilde{E}_1 + \widetilde{E}_2$ are fuzzy vector spaces.

Theorem 2. *For two fuzzy vector spaces $\widetilde{E}_1 = (E, \mu_1)$ and $\widetilde{E}_2 = (E, \mu_2)$, we have the following results:*

(i) For all $a \in [0,1]$, $(\widetilde{E}_1 \cap \widetilde{E}_2)_{[a]} = (\widetilde{E}_1)_{[a]} \cap (\widetilde{E}_2)_{[a]}$;

(ii) For all $a \in [0,1)$, $(\widetilde{E}_1 \cap \widetilde{E}_2)_{(a)} = (\widetilde{E}_1)_{(a)} \cap (\widetilde{E}_2)_{(a)}$;

(iii) For any $a \in [0,1)$, $(\widetilde{E}_1 + \widetilde{E}_2)_{(a)} = (\widetilde{E}_1)_{(a)} + (\widetilde{E}_2)_{(a)}$.

Proof. By Definition 5, we can easily obtain (i) and (ii). Statement (iii) can be proved as follows. For any $a \in [0,1)$, we have

$$\begin{aligned} x \in (\widetilde{E}_1 + \widetilde{E}_2)_{(a)} &\Leftrightarrow \bigvee_{x_1 + x_2 = x} (\mu_1(x_1) \wedge \mu_2(x_2)) > a \\ &\Leftrightarrow \exists x_1, x_2 \text{ such that } \ x_1 + x_2 = x \text{ and } \ \mu_1(x_1) \wedge \mu_2(x_2) > a \\ &\Leftrightarrow \exists x_1, x_2 \text{ such that } \ x_1 + x_2 = x, x_1 \in (\mu_1)_{(a)} \text{ and } x_2 \in (\mu_2)_{(a)} \\ &\Leftrightarrow x \in (\widetilde{E}_1)_{(a)} + (\widetilde{E}_2)_{(a)}. \end{aligned}$$

□

In this paper, we suppose that E is a finite dimensional vector space.

3. A new definition of fuzzy bases

In this section, we shall present a new definition of fuzzy bases for fuzzy vector spaces.

Let $\widetilde{E} = (E, \mu)$ be a fuzzy vector space and $\dim E = n$. Lowen [5] proved that $\mu(E)$ is a finite subset of $[0,1]$. It is easy to prove the following lemma.

Lemma 1. *If $\widetilde{E} = (E, \mu)$ is a fuzzy vector space, then there exists a finite sequence $1 = \alpha_0 \geq \alpha_1 > \alpha_2 > \cdots > \alpha_r \geq 0$ such that*

(i) If $a, b \in (\alpha_{i+1}, \alpha_i]$, then $\mu_{[a]} = \mu_{[b]}$;

(ii) If $a \in (\alpha_{i+1}, \alpha_i]$ and $b \in (\alpha_i, \alpha_{i-1}]$, then $\mu_{[a]} \supsetneq \mu_{[b]}$;

(iii) If $a, b \in [\alpha_{i+1}, \alpha_i)$, then $\mu_{(a)} = \mu_{(b)}$;

(iv) If $a \in [\alpha_{i+1}, \alpha_i)$ and $b \in [\alpha_i, \alpha_{i-1})$, then $\mu_{(a)} \supsetneq \mu_{(b)}$.

For a fuzzy vector space $\widetilde{E} = (E, \mu)$, by Lemma 1 we can obtain a family of vector spaces as follows:

$$\{0\} \subseteq \mu_{[\alpha_1]} \subsetneqq \mu_{[\alpha_2]} \subsetneqq \cdots \subsetneqq \mu_{[\alpha_r]} \subseteq E. \tag{2}$$

We shall call (2) the family of irreducible level subspaces of $\widetilde{E} = (E, \mu)$. Suppose that $\mu_{[\alpha_1]} \neq \{0\}$, otherwise we can choose $\mu_{[\alpha_2]}$. Let B_{α_1} be a basis of $\mu_{[\alpha_1]}$. We can obtain a basis B_{α_2} of $\mu_{[\alpha_2]}$ by extending B_{α_1}. Further, we can obtain a basis B_{α_3} of $\mu_{[\alpha_3]}$ by extending B_{α_2}. Analogously, we can obtain a basis B_{α_r} of $\mu_{[\alpha_r]}$ by extending $B_{\alpha_{r-1}}$. Thus we obtain a sequence

$$B_{\alpha_1} \subsetneqq B_{\alpha_2} \subsetneqq B_{\alpha_3} \subsetneqq \cdots \subsetneqq B_{\alpha_r}, \tag{3}$$

where B_{α_i} is a basis of $\mu_{[\alpha_i]}(1 \leq i \leq r)$. Therefore, we can define a fuzzy subset β of E as follows.

$$\beta(x) = \bigvee \{\alpha_i : x \in B_{\alpha_i}\}. \tag{4}$$

Then β is called a fuzzy basis of $\widetilde{E}$ corresponding to (3).

The proof of the following theorem is trivial.

Theorem 3. *Let β be a fuzzy basis of $\widetilde{E} = (E, \mu)$ obtained by (3). Then the following statements hold:*

(i) If $a, b \in (\alpha_{i+1}, \alpha_i]$, then $\beta_{[a]} = \beta_{[b]} = B_{\alpha_i}$;

(ii) If $a \in (\alpha_{i+1}, \alpha_i]$ and $b \in (\alpha_i, \alpha_{i-1}]$, then $\beta_{[a]} \supsetneqq \beta_{[b]}$;

(iii) If $a, b \in [\alpha_{i+1}, \alpha_i)$, then $\beta_{(a)} = \beta_{(b)} = B_{\alpha_{i+1}}$;

(iv) If $a \in [\alpha_{i+1}, \alpha_i)$ and $b \in [\alpha_i, \alpha_{i-1})$, then $\beta_{(a)} \supsetneqq \beta_{(b)}$.

Corollary 1. *Let β be a fuzzy basis of $\widetilde{E} = (E, \mu)$ obtained by (3). Then the following statements hold:*

(i) $a \in (0, 1]$, $\beta_{[a]}$ is a basis of $\mu_{[a]}$;

(ii) $a \in [0, 1)$, $\beta_{(a)}$ is a basis of $\mu_{(a)}$.

From the above corollary, we can easily obtain the following.

Corollary 2. *Let $\widetilde{E} = (E, \mu)$ be a fuzzy vector space and let β_1 and β_2 be two fuzzy bases of $\widetilde{E}$. Then the following statements hold:*

(i) For any $a \in (0, 1]$, $|(\beta_1)_{[a]}| = |(\beta_2)_{[a]}|$;

(ii) For any $a \in [0, 1)$, $|(\beta_1)_{(a)}| = |(\beta_2)_{(a)}|$;

(iii) $|\beta_1| = |\beta_2|$.

4. A new definition of fuzzy dimension

In this section, we redefine the fuzzy dimension of fuzzy vector spaces.

The cardinality of a crisp set A can be regarded as an increasing set of integers $\{0, 1, \cdots, n\}$. Such a set is also mathematically equivalent to the integer n. For a crisp vector space, its dimension was defined by the cardinality of its bases. We can define analogously the fuzzy dimension of fuzzy vector spaces as follows.

Definition 6. *Let $\widetilde{E} = (E, \mu)$ be a fuzzy vector space with a fuzzy basis β. Define $\dim(\widetilde{E}) = |\beta|$. Then $\dim(\widetilde{E})$ is called the fuzzy dimension of $\widetilde{E} = (E, \mu)$.*

Theorem 4. *Let $\widetilde{E} = (E, \mu)$ be a fuzzy vector space with a fuzzy basis β. Then*

$$\begin{aligned}\dim(\widetilde{E})(n) &= \bigvee \left\{a \in [0, 1) : |\beta_{(a)}| \geq n\right\} \\ &= \bigvee \left\{a \in [0, 1) : \dim\left(\widetilde{E}_{(a)}\right) \geq n\right\}\end{aligned}$$

Proof. By Corollary 1, we know that $\dim\left(\widetilde{E}_{(a)}\right) = |\beta_{(a)}|$. For any $n \in \mathbb{N}$, let

$$\lambda = \bigvee\{a \in [0, 1) : \dim\left(\widetilde{E}_{(a)}\right) \geq n\}.$$

Obviously, we have

$$\lambda \leq \dim(\widetilde{E})(n) = \bigvee \left\{a \in (0, 1] : |\beta_{[a]}| \geq n\right\}.$$

In order to show that $\lambda \geq \dim(\widetilde{E})(n)$, suppose that $\dim(\widetilde{E})(n) \neq 0$ and $\dim(\widetilde{E})(n) > b$. Then there exists $a > b$ such that $|\beta_{[a]}| \geq n$. In this case, $n \leq |\beta_{[a]}| \leq |\beta_{(b)}| \leq |\beta_{[b]}|$. This implies $\lambda = \bigvee \left\{a \in [0, 1) : \dim(\widetilde{E})_{(a)} \geq n\right\} \geq b$. Thus we have

$$\lambda \geq \bigvee \left\{b : 0 \leq b < \dim(\widetilde{E})(n)\right\} = \dim(\widetilde{E})(n).$$

This completes the proof. □

Theorem 5. *Let $\widetilde{E} = (E, \mu)$ be a fuzzy vector space. Then*

(i) For any $a \in (0, 1]$, $(\dim(\widetilde{E}))_{[a]} = \dim\left(\widetilde{E}_{[a]}\right)$;

(ii) For any $a \in [0, 1)$, $(\dim(\widetilde{E}))_{(a)} = \dim\left(\widetilde{E}_{(a)}\right)$.

Proof. Let

$$\{0\} \subseteq \mu_{[\alpha_1]} \subsetneq \mu_{[\alpha_2]} \subsetneq \cdots \subsetneq \mu_{[\alpha_r]} \subseteq E$$

be the family of irreducible level subspaces of $\widetilde{E} = (E, \mu)$.

(i) We know from definition of $\dim(\widetilde{E})$ that $\dim\left(\widetilde{E}_{[a]}\right) \leq \left(\dim(\widetilde{E})\right)_{[a]}$. Now we need to show that $\left(\dim(\widetilde{E})\right)_{[a]} \leq \dim\left(\widetilde{E}_{[a]}\right)$. From the definition of fuzzy dimension, we have

$$\begin{aligned} n \le \left(\dim(\widetilde{E})\right)_{[a]} &\Rightarrow \dim(\widetilde{E})(n) \ge a \\ &\Rightarrow \bigvee\{\alpha_i : \dim(\widetilde{E}_{[\alpha_i]}) \ge n\} \ge a \\ &\Rightarrow \exists \alpha_i \ge a \text{ such that } \ n \le \dim\left(\widetilde{E}_{[\alpha_i]}\right) \\ &\Rightarrow n \le \dim\left(\widetilde{E}_{[\alpha_i]}\right) \le \dim\left(\widetilde{E}_{[a]}\right). \end{aligned}$$

Therefore, $\left(\dim(\widetilde{E})\right)_{[a]} = \dim\left(\widetilde{E}_{[a]}\right)$ for any $a \in (0,1]$.

(ii) In order to prove $\left(\dim(\widetilde{E})\right)_{(a)} \le \dim\left(\widetilde{E}_{(a)}\right)$. We suppose that

$$n = \left(\dim(\widetilde{E})\right)_{(a)}.$$

Then $\dim(\widetilde{E})(n) > a$, i.e., $\bigvee\{b \in (0,1] : \dim\left(\widetilde{E}_{[b]}\right) \ge n\} > a$. Hence, there exists $b \in (0,1]$ such that $a < b$ and $n \le \dim\left(\widetilde{E}_{[b]}\right)$. Since $\widetilde{E}_{[b]} \subseteq \widetilde{E}_{(a)}$, thus $n \le \dim\left(\widetilde{E}_{(a)}\right)$ Therefore, $\left(\dim(\widetilde{E})\right)_{(a)} \le \dim\left(\widetilde{E}_{(a)}\right)$.

In order to show $\dim\left(\widetilde{E}_{(a)}\right) \le \left(\dim(\widetilde{E})\right)_{(a)}$, take $\alpha_i > a$ such that $\widetilde{E}_{(a)} = \mu_{[\alpha_i]}$. Then it is easy to see that

$$\dim\left(\widetilde{E}_{(a)}\right) = \dim\left(\mu_{[\alpha_i]}\right) \le \left(\dim(\widetilde{E})\right)_{[\alpha_i]} \le \left(\dim(\widetilde{E})\right)_{(a)}.$$

□

Theorem 6. *Let $\widetilde{E}_1 = (E, \mu_1)$ and $\widetilde{E}_2 = (E, \mu_2)$ be two fuzzy vector spaces, then it holds*

$$\dim(\widetilde{E}_1 + \widetilde{E}_2) + \dim(\widetilde{E}_1 \cap \widetilde{E}_2) = \dim(\widetilde{E}_1) + \dim(\widetilde{E}_2).$$

Proof. For any $a \in [0,1)$, by Theorem 1, Theorem 2 and Theorem 5 we have

$$\begin{aligned} \left(\dim(\widetilde{E}_1 + \widetilde{E}_2) + \dim(\widetilde{E}_1 \cap \widetilde{E}_2)\right)_{(a)} &= \left(\dim(\widetilde{E}_1 + \widetilde{E}_2)\right)_{(a)} + \left(\dim(\widetilde{E}_1 \cap \widetilde{E}_2)\right)_{(a)} \\ &= \dim\left(\left(\widetilde{E}_1 + \widetilde{E}_2\right)_{(a)}\right) + \dim\left((\widetilde{E}_1 \cap \widetilde{E}_2)_{(a)}\right) \\ &= \dim\left((\widetilde{E}_1)_{(a)} + (\widetilde{E}_2)_{(a)}\right) \\ &\quad + \dim\left((\widetilde{E}_1)_{(a)} \cap (\widetilde{E}_2)_{(a)}\right) \\ &= \dim\left((\widetilde{E}_1)_{(a)}\right) + \dim\left((\widetilde{E}_2)_{(a)}\right) \\ &= \left(\dim(\widetilde{E}_1)\right)_{(a)} + \left(\dim(\widetilde{E}_2)\right)_{(a)} \\ &= \left(\dim(\widetilde{E}_1) + \dim(\widetilde{E}_2)\right)_{(a)} \end{aligned}$$

Therefore, $\dim(\widetilde{E}_1 + \widetilde{E}_2) + \dim(\widetilde{E}_1 \cap \widetilde{E}_2) = \dim(\widetilde{E}_1) + \dim(\widetilde{E}_2)$. □

Definition 7 (see [2])**.** *Let $\widetilde{E} = (E, \mu)$ be a fuzzy vector space and $f : E \to E$ a linear transformation. If for all $x \in E$, $\mu(f(x)) \geq \mu(x)$, then f is called a fuzzy linear transformation on $\widetilde{E} = (E, \mu)$.*

Lemma 2. *Let $\widetilde{E} = (E, \mu)$ be a fuzzy vector space and let f be a fuzzy linear transformation on $\widetilde{E}$. Then $\widetilde{\ker f} = (\ker f, \mu|_{\ker f})$ and $\widetilde{\mathrm{im} f} = (\mathrm{im} f, \mu|_{\mathrm{im} f})$ are two fuzzy vector spaces.*

Proof. It is trivial and it is omitted. □

Theorem 7. *Let $\widetilde{E} = (E, \mu)$ be a fuzzy vector space and let $f : E \to E$ be a fuzzy linear transformation on $\widetilde{E}$. Then*

$$\dim(\widetilde{\ker f}) + \dim(\widetilde{\mathrm{im} f}) = \dim(\widetilde{E})$$

Proof. By Theorem 1 and Theorem 5 it holds that for all $a \in [0, 1)$,

$$\begin{aligned}\Big(\dim(\widetilde{\mathrm{im} f}) + \dim(\widetilde{\ker f})\Big)_{(a)} &= \Big(\dim(\widetilde{\mathrm{im} f})\Big)_{(a)} + \Big(\dim(\widetilde{\ker f})\Big)_{(a)} \\ &= \dim\Big((\widetilde{\mathrm{im} f})_{(a)}\Big) + \dim\Big((\widetilde{\ker f})_{(a)}\Big) \\ &= \dim\Big(\widetilde{E}_{(a)} \cap \mathrm{im} f\Big) + \dim\Big(\widetilde{E}_{(a)} \cap \ker f\Big).\end{aligned}$$

It is easy to check that $f|_{\widetilde{E}_{(a)}}$ is a linear transformation on $\widetilde{E}_{(a)}$. Thus we have

$$\begin{aligned}\Big(\dim(\widetilde{\mathrm{im} f}) + \dim(\widetilde{\ker f})\Big)_{(a)} &= \dim(\mathrm{im} f|_{\widetilde{E}_{(a)}}) + \dim(\ker f|_{\widetilde{E}_{(a)}}) \\ &= \dim\Big(\widetilde{E}_{(a)}\Big) = \Big(\dim(\widetilde{E})\Big)_{(a)}.\end{aligned}$$

Therefore, $\dim(\widetilde{\ker f}) + \dim(\widetilde{\mathrm{im} f}) = \dim(\widetilde{E})$. □

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