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Approximation by Jackson-type operator on the sphere*

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Abstract. This paper discusses the approximation by a Jackson-type operator on the sphere. By using a spherical translation operator, a modulus of smoothness of high order, which is used to bound the rate of approximation of the Jackson-type operator, is introduced. Furthermore, the method of multipliers is applied to characterize the saturation order and saturation class of the operator. In particular, the function of saturation class is expressed by an apparent formula. The results obtained in this paper contain the corresponding ones of the Jackson operator.

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Key words: approximation, Jackson-type operator, saturation order, saturation class, sphere

1. Introduction

A volume of work has been carried out for studying Jackson and Bernstein type theorems on the sphere during the past decades by Lizorkin and Nikol'skiĭ [13], Ditzian [8], Dai [7], Dai and Ditzian [6], Li and Yang [12], Rustamov [16] and many other mathematicians. In 1983, to prove direct and inverse theorems on a more special Banach space $H_p^r(\mathbb{S}^{n-1})$, Lizorkin and Nikol'skiĭ [13] constructed an iterated Boolean sum of Jackson operator. They introduced a Jackson-type operator $(Q_{k,s}^d f)$ (which will be defined in the next section), which is a linear combination of an iterated Jackson operator and can be reduced to the known Jackson operator when $d = 1$. Here we are interested in the approximation properties of $(Q_{k,s}^d f)$. After introducing a modulus of smoothness of high order by using the translation operator on the sphere, we will give an estimation of the rate of approximation for the operator $(Q_{k,s}^d f)$. We also use the method of multipliers to obtain saturation theorems of $(Q_{k,s}^d f)$, and the expression of the function of saturation class. The obtained results show that the operator $(Q_{k,s}^d f)$ has a higher degree of approximation and order of saturation. When $d = 1$, our results coincide with those of the known Jackson operator.

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2. Definitions and auxiliary notations

A function on the sphere $\mathbb{S}^{n-1}$ is in $L^p(\mathbb{S}^{n-1})$, $1 \le p \le \infty$, if

$$\|f\|_p := \|f\|_{L^p(\mathbb{S}^{n-1})} := \left\{ \int_{\mathbb{S}^{n-1}} |f(x)|^p d\sigma \right\}^{1/p} < \infty, \quad 1 \le p < \infty$$

and

$$\|f\|_\infty := \|f\|_{L^\infty(\mathbb{S}^{n-1})} := \operatorname{ess}\sup_{x \in \mathbb{S}^{n-1}} |f(x)|, \quad p = \infty,$$

where $d\sigma$ is the surface measure on the sphere $\mathbb{S}^{n-1}$ given by

$$\mathbb{S}^{n-1} := \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = 1\}.$$

Let $C(\mathbb{S}^{n-1})$ be the space of continuous functions defined on $\mathbb{S}^{n-1}$, which is endowed with norm

$$\|f\|_{C(\mathbb{S}^{n-1})} := \max_{x \in \mathbb{S}^{n-1}} |f(x)|.$$

We denote by $\hat{n}$ the north pole of $\mathbb{S}^{n-1}$. For a fixed $\theta \in [0, \pi]$ and all $x \in \varphi_\theta := \{x : x \in \mathbb{S}^{n-1}, (\hat{n} \cdot x) = \cos\theta\}$, if $h(x)$ is unchanged over φ_θ, then it is called a zonal function on the sphere. Clearly, $h(x)$ is isomorphic to a univariate function $f(\theta)$, $0 \le \theta \le \pi$. Let $\hat{n} = (1, 0, \dots, 0)$, $x' = (\cos\theta, \sin\theta, 0, \dots, 0)$, then for $x \in \varphi_\theta$, $h(x) = h(x') = h(\cos\theta, \sin\theta, 0, \dots, 0) = f(\theta)$, $\theta \in [0, \pi]$. It is obvious that the space of all continuous zonal functions $C_\lambda \subset C(\mathbb{S}^{n-1})$ is a Banach space the norm of which is

$$\|f\|_{C_\lambda} := \sup_{0 \le \theta \le \pi} |f(\theta)|.$$

Correspondingly, the space of all p-th Lebesgue integrable zonal functions is a subspace of $L^p(\mathbb{S}^{n-1})$, i.e. $L^p_\lambda \subset L^p(\mathbb{S}^{n-1})$, $1 \le p < \infty$, $2\lambda = n - 2$, and the norm of this space is defined by

$$\|f\|_{p,\lambda} := \left\{ \frac{\Omega_{n-2}}{\Omega_{n-1}} \int_0^\pi |f(\theta)|^p dm_\lambda(\theta) \right\}^{\frac{1}{p}},$$

where $dm_\lambda(\theta) = \sin^{2\lambda}\theta d\theta$, and Ω_{n-1} means the $(n-1)$-dimensional surface area of the unit sphere of $\mathbb{R}^n$. For $p = \infty$, $L^\infty_\lambda \subset L^\infty(\mathbb{S}^{n-1})$,

$$\|f\|_{\infty,\lambda} := \operatorname{ess}\sup_{0 \le \theta \le \pi} |f(\theta)|.$$

Here and elsewhere, we always use X to denote the space $L^p(\mathbb{S}^{n-1})$, for $1 \le p \le \infty$ or $C(\mathbb{S}^{n-1})$.

For $f \in L^1(\mathbb{S}^{n-1})$, the translation operator on the sphere is given by (see [1, 18])

$$S_\theta(f)(\mu) := \frac{1}{\Omega_{n-2} \sin^{2\lambda}\theta} \int_{\mu \cdot \mu' = \cos\theta} f(\mu') d\mu', \quad 0 < \theta < \pi,$$

where $2\lambda = n - 2$. Here we integrate over the family of points $\mu' \in \mathbb{S}^{n-1}$ whose spherical distance from the given points $\mu \in \mathbb{S}^{n-1}$ (i.e. the arc length between μ and μ' on the great circle passing through them) is equal to θ.

Set

$$S_\theta^{(0)}(f)(\mu) := f(\mu),\ S_\theta^{(j)}(f)(\mu) := S_\theta(S_\theta^{(j-1)}(f)(\mu)),\ j = 1, 2, 3, \ldots.$$

We introduce the sphere difference (see [13]):

$$\Delta_\theta^1(f) := \Delta_\theta(f) := (S_\theta^{(1)} f - f)(\mu),$$

$$\Delta_\theta^r(f) := \Delta_\theta(\Delta_\theta^{r-1}(f)) = \sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j} S_\theta^{(j)} f(\mu) = (S_\theta - I)^r f(\mu),\ r = 1, 2, \ldots$$

at the point $\mu \in \mathbb{S}^{n-1}$, where I is the identity operator on $L^p(\mathbb{S}^{n-1})$. Then the $2r$-th modulus of smoothness of $f \in L^p(\mathbb{S}^{n-1})$ is defined by (see [16])

$$\omega^{2r}(f, t)_p := \sup_{0<\theta\le t} \parallel \Delta_\theta^r f \parallel_p,\ 0 < t \le \pi,\ r = 1, 2, \ldots.$$

Obviously, $\omega^{2r}(f, t)_p$ can be rewritten as

$$\omega^{2r}(f, t)_p := \sup_{0<\theta\le t} \parallel (S_\theta - I)^r f \parallel_p .$$

Now we state some properties of spherical harmonics (see [18], [10] and [15]). For integer $k \ge 0$, the restriction to $\mathbb{S}^{n-1}$ of a homogeneous harmonic polynomial of degree k is called a spherical harmonic of degree k. The class of all spherical harmonics of degree k will be denoted by $\mathbb{H}_k^n$, and the class of all spherical harmonics of degree $k \le m$ will be denoted by Π_m^n. Of course, $\Pi_m^n = \bigoplus_{k=0}^m \mathbb{H}_k^n$, and it comprises the restriction to $\mathbb{S}^{n-1}$ of all algebraic polynomials in n variables of total degree not exceeding m.

For $f \in L^p(\mathbb{S}^{n-1})$, the known Jackson operator on $\mathbb{S}^{n-1}$ is defined by (see [13])

$$J_{k,s}(f, \mu) := \frac{1}{\Omega_{n-2}} \int_{\mathbb{S}^{n-1}} f(\mu')\mathcal{D}_{k,s}(\arccos \mu \cdot \mu')d\mu', \quad k = 1, 2, \ldots,$$

where k and s are positive integers,

$$\mathcal{D}_{k,s}(\theta) := A_{k,s}^{-1} \left(\frac{\sin \frac{k\theta}{2}}{\sin \frac{\theta}{2}} \right)^{2s}$$

is the classical Jackson kernel, and $A_{k,s}$ is a constant connected with k and s such that

$$\int_0^\pi \mathcal{D}_{k,s}(\theta) \sin^{2\lambda} \theta d\theta = 1, \quad \lambda = \frac{n-2}{2}.$$

We observe that

$$\begin{aligned} J_{k,s}(f, \mu) &= \frac{1}{\Omega_{n-2}} \int_{\mathbb{S}^{n-1}} f(\mu')\mathcal{D}_{k,s}(\arccos \mu \cdot \mu')d\mu' \\ &= \int_0^\pi \left(\frac{1}{\Omega_{n-2} \sin^{2\lambda} \theta} \int_{\mu\cdot\mu'=\cos\theta} f(\mu')d\mathbb{S}^{n-2}(\mu') \right) \mathcal{D}_{k,s}(\theta) \sin^{2\lambda} \theta d\theta \\ &= \int_0^\pi S_\theta(f, \mu)\mathcal{D}_{k,s}(\theta)\ \sin^{2\lambda} \theta\ d\theta. \end{aligned} \tag{1}$$

We now introduce the definition of the Jackson-type operator as follows (see [13]).

Definition 1. *Let $f \in L^p(\mathbb{S}^{n-1}), 1 \le p \le \infty$, and let d be a positive integer. The Jackson-type operator is given by*

$$(Q^d_{k,s}f)(\mu) := f(\mu) - \int_0^\pi \mathcal{D}_{k,s}(\theta)(I - S_\theta)^d f(\mu) \sin^{n-2}\theta d\theta. \tag{2}$$

It is clear that for $d = 1$, the Jackson-type operator $(Q^d_{k,s}f)(u)$ coincides with the known Jackson operator $J_{k,s}(f, \mu)$ and it is a spherical harmonic of total degree not exceeding $(k-1)s$. But for $d \ge 2$, $(Q^d_{k,s}f)(u)$ is not a spherical harmonic unless f is not so (see [13]).

We briefly introduce the projection $Y_j(\cdot)$ by Gegenbauer polynomials $\{G^\lambda_j\}^\infty_{j=1}$ $(\lambda = \frac{n-2}{2})$ for discussion of saturation property of $(Q^d_{k,s}f)(u)$.

Gegenbauer polynomials $\{G^\lambda_j\}^\infty_{j=1}$ are defined in terms of the generating function (see [17]):

$$\frac{1}{(1-2rt+r^2)^\lambda} = \sum_{j=0}^\infty G^\lambda_j(t) r^j, \tag{3}$$

where $|r| < 1, |t| \le 1$. For any $\lambda > 0$, we have (see [17])

$$G^\lambda_1(t) = 2\lambda t \tag{4}$$

and

$$\frac{d}{dt} G^\lambda_j(t) = 2\lambda G^{\lambda+1}_{j-1}(t). \tag{5}$$

When $\lambda = \frac{n-2}{2}$ (see [18]), there holds

$$G^\lambda_j(t) = \frac{\Gamma(j+2\lambda)}{\Gamma(j+1)\Gamma(2\lambda)} P^n_j(t), \ j = 0, 1, 2 \ldots,$$

where $P^n_j(t)$ is the Legendre polynomial of degree j (see [15]).

Particularly,

$$G^\lambda_j(1) = \frac{\Gamma(j+2\lambda)}{\Gamma(2\lambda)\Gamma(j+1)}, \ j = 0, 1, 2 \ldots,$$

therefore,

$$P^n_j(t) = \frac{G^\lambda_j(t)}{G^\lambda_j(1)}. \tag{6}$$

Besides, for any $j = 0, 1, 2, \ldots$ and $|t| \le 1, |P^n_j(t)| \le P^n_j(1) = 1$ (see [15]). In addition, we use the estimate (see [1])

$$\begin{aligned} \|G^\lambda_j(\cos\theta)\|_{C_\lambda} &= \sup_{0\le\theta\le\pi} |G^\lambda_j(\cos\theta)| \le |G^\lambda_j(1)| = \binom{j+2\lambda-1}{j} \\ &= O(j^{2\lambda-1}), \quad j \to \infty. \end{aligned} \tag{7}$$

We shall rely on the classical statement that every function $f(\mu) \in L^p(\mathbb{S}^{n-1})$ admits a unique Laplace expansion of the form

$$f(\mu) \ = \ \sum_{j=0}^{\infty} Y_j(f;\mu)$$

with respect to the spherical harmonics, where the convergence of this series is meant in a weak sense (see [13]). Here

$$Y_j(f;\mu) \ := \ \frac{\Gamma(\lambda)(j+\lambda)}{2\pi^{\lambda+1}} \int_{\mathbb{S}^{n-1}} G_j^{\lambda}(\mu\cdot\mu')f(\mu')d\mu', \ \ j=0,1,2,\ldots.$$

It follows from (4), (5) and (6) that

$$\lim_{\theta\to 0} \frac{1-P_j^n(\cos\theta)}{1-P_1^n(\cos\theta)} = \frac{j(j+2\lambda)}{2\lambda+1}, \ j=0,1,2\ldots. \tag{8}$$

Next we give the definition of saturation order of operators (see [1]).

Definition 2. *Let $\varphi(\rho)$ be a positive function with respect to ρ, $0<\rho<\infty$, tending monotonely to zero as $\rho\to\infty$. For a sequence of operators $\{I_\rho\}_{\rho>0}$ if there exists $\mathcal{K} \subseteqq L^p(\mathbb{S}^{n-1})$ such that*

(i) If $\|I_\rho(f)-f\|_p = o(\varphi(\rho))$, then $I_\rho(f)=f$;
(ii) $\|I_\rho(f)-f\|_p = O(\varphi(\rho))$ if and only if $f\in\mathcal{K}$,

then I_ρ is said to be saturated on $L^p(\mathbb{S}^{n-1})$ with order $O(\varphi(\rho))$ and $\mathcal{K}$ is called its saturation class.

In order to get the saturated class of $(Q_{k,s}^d f)(\mu)$, we introduce a function class $\mathcal{H}$, which will be proved to be a saturated class of $(Q_{k,s}^d f)(\mu)$ later.

Definition 3. *Let $\mathbb{N}_+$ be a set of non-negative integers, $\psi:\mathbb{N}_+\to\mathbb{R}$, and $\psi(0)=0$. For all $j\in\mathbb{N}_+$, $\mathcal{H}(X,\psi(j))$ is defined as follows:*

$$\mathcal{H}(X,\psi(j)) = \begin{cases} \{f\in C(\mathbb{S}^{n-1}): & \textit{there exists } g\in L^\infty(\mathbb{S}^{n-1}) \\ & \textit{such that } \psi(j)Y_j(f;x)=Y_j(g;x)\} \\ \{f\in L^1(\mathbb{S}^{n-1}): & \textit{there exists } \mu\in M(\mathbb{S}^{n-1}) \\ & \textit{such that } \psi(j)Y_j(f;x)=Y_j(d\mu;x)\} \\ \{f\in L^p(\mathbb{S}^{n-1}): & \textit{there exists } g\in L^p(\mathbb{S}^{n-1}) \\ & \textit{such that } \psi(j)Y_j(f;x)=Y_j(g;x)\} \end{cases}$$

where $M(\mathbb{S}^{n-1})$ is the collection of all Borel measures on $\mathbb{S}^{n-1}$.

Finally, we introduce spherical convolution. In 1955 Calderón and Zygmund (see [5, p.218]) defined the convolution on $\mathbb{S}^{n-1}$ with the help of a zonal function in $[0,\pi]$. Later Bochner [2] made some introduction, and Hirschman, Jr. [11] gave some detailed discussion.

Definition 4. *Let $f \in L^1(\mathbb{S}^{n-1})$ and $g \in L^1_\lambda$, $2\lambda = n-2$. Then*

$$h(x) \;=\; \frac{1}{\Omega_{n-1}} \int_{\mathbb{S}^{n-1}} f(y) g(x \cdot y) dy \tag{9}$$

is called a spherical convolution of f and g, denoted by $h = f * g$. And $f * g$ can be expanded by the spherical function:

$$Y_n(f * g; x) \;=\; \frac{\lambda}{n+\lambda} \hat{g}(n) Y_n(f; x), \tag{10}$$

where $\hat{g}(n)$ is the Gegenbauer coefficient (see [1, p.207]).

3. Some Lemmas

In this section, we show some lemmas as a preparation for the proof of the main results.

Lemma 1. *For $\beta \geq 0, 2s > \beta + n - 1, 0 < \theta \leq \pi$, and $n \geq 3$, we have*

$$\int_0^\pi \theta^\beta \mathcal{D}_{k,s}(\theta) \sin^{2\lambda} \theta d\theta \;\leq\; C_{\beta,s} k^{-\beta} \tag{11}$$

where $\lambda = \frac{n-2}{2}$, and s, n, k are positive integers.

The proof of Lemma 1 is simple (see also [13]).

The following Lemma 2 gives the description of $Q^d_{k,s} f(\mu)$ by multiplies.

Lemma 2. *For $f \in L^p(\mathbb{S}^{n-1}), 1 \leq p \leq \infty$, there holds*

$$(Q^d_{k,s} f)(\mu) = \sum_{j=0}^{\infty} \xi^d_{k,s}(j) Y_j(f; \mu) \tag{12}$$

where

$$\xi^d_{k,s}(j) = 1 - \int_0^\pi \mathcal{D}_{k,s}(\theta) \left(1 - \frac{G^\lambda_j(\cos\theta)}{G^\lambda_j(1)}\right)^d \sin^{n-2}\theta d\theta,\ j = 0, 1, 2 \ldots \tag{13}$$

and the convergence of the series is meant in a weak sense.

Proof. We know $f(\mu) = \sum_{j=0}^{\infty} Y_j(f; \mu)$. Then from Definition 1,

$$(Q^d_{k,s} f)(\mu) = \sum_{j=0}^{\infty} Y_j(f; \mu) - \int_0^\pi \mathcal{D}_{k,s}(\theta)(I - S_\theta)^d f(\mu) \sin^{n-2}\theta d\theta.$$

From [13], it follows that

$$S_\theta(f) = \sum_{j=0}^{\infty} \frac{G^\lambda_j(\cos\theta)}{G^\lambda_j(1)} Y_j(f; \mu).$$

So

$$(Q_{k,s}^d f)(\mu) = \sum_{j=0}^{\infty} \left[1 - \int_0^{\pi} \mathcal{D}_{k,s}(\theta) \left(1 - \frac{G_j^{\lambda}(\cos\theta)}{G_j^{\lambda}(1)} \right)^d \sin^{n-2}\theta d\theta \right] Y_j(f;\mu)$$
$$= \sum_{j=0}^{\infty} \xi_{k,s}^d(j) Y_j(f;\mu).$$

This finishes the proof of Lemma 2. □

The next lemma is useful for determining the saturation order. It can be deduced by the methods in [1] and [4].

Lemma 3. *Suppose that $\{I_\rho\}_{\rho>0}$ is a sequence of operators on $L^p(\mathbb{S}^{n-1})$, and there exists series $\{\lambda_\rho(j)\}_{j=1}^{\infty}$ with respect to ρ, such that*

$$I_\rho(f)(x) = \sum_{j=0}^{\infty} \lambda_\rho(j) Y_j(f)(x)$$

for every $f \in L^p(\mathbb{S}^{n-1})$. If there exists $\varphi(\rho) \to 0+$ $(\rho \to \infty)$ such that for any $j = 0, 1, 2, \ldots,$

$$\lim_{\rho\to\rho_0} \frac{1-\lambda_\rho(j)}{\varphi(\rho)} = \tau_j \neq 0,$$

then $\{I_\rho\}_{\rho>0}$ is saturated on $L^p(\mathbb{S}^{n-1})$ with the order $O(\varphi(\rho))$ and the collection of all constants is the invariant class for $\{I_\rho\}_{\rho>0}$ on $L^p(\mathbb{S}^{n-1})$.

In [9] Dunkl introduced the spherical convolution with respect to the measure and zonal function. Normally M_λ is denoted by the set of Borel measures μ, and it satisfies

$$\|\mu\|_{M_\lambda} := \frac{\Omega_{n-2}}{\Omega_{n-1}} \int_0^{\pi} |d\mu(\theta)| < \infty.$$

Now, let $M(\mathbb{S}^{n-1})$ be the space of all the limited regular Borel measures on $\mathbb{S}^{n-1}$ endowed with the norm

$$\|\mu\|_M := \frac{1}{\Omega_{n-1}} \int_{\mathbb{S}^{n-1}} |d\mu(x)|.$$

Then it is a Banach space and every $\mu \in M(\mathbb{S}^{n-1})$ can be expanded as the following spherical series:

$$S(d\mu; x) \sim \sum_{j=0}^{\infty} Y_j(d\mu; x),$$

where

$$Y_j(d\mu; x) = \frac{\Gamma(\lambda)(j+\lambda)}{2\pi^{\lambda+1}} \int_{\mathbb{S}^{n-1}} G_j^{\lambda}(x\cdot y) d\mu(y), \quad j = 0, 1 \ldots.$$

From the above we get Lemma 4 (see [9]).

Lemma 4. *Let* $f \in L^p_\lambda$, $1 \le p \le \infty$ *or* C_λ, $\mu \in M(\mathbb{S}^{n-1})$. *Then*

$$(f * d\mu)(x) \;=\; \frac{1}{\Omega_{n-1}} \int_{\mathbb{S}^{n-1}} f(x \cdot y) d\mu(y) \tag{14}$$

is in $L^p(\mathbb{S}^{n-1})$ *or* $C(\mathbb{S}^{n-1})$,

$$\|f * d\mu\|_p \;\le\; \|f\|_{p,\lambda} \|\mu\|_M \tag{15}$$

and

$$Y_j(f * d\mu; x) \;=\; \frac{\lambda}{j+\lambda} \hat{f}(j) Y_j(d\mu; x). \tag{16}$$

Moreover, if μ *is a zonal function, so is* $f * \mu$.

To get the saturation class, we need to discuss spherical convolution and a singular integral together (see [1]). Let $f \in X, \mathcal{B}_\rho \in L^1_\lambda$, and also

$$c(0,\rho) \int_0^\pi \mathcal{B}_\rho(\cos\theta) \sin^{2\lambda}\theta d\theta \;=\; 1, \tag{17}$$

$$c(0,\rho) \int_0^\pi |\mathcal{B}_\rho(\cos\theta)| \sin^{2\lambda}\theta d\theta \;\le\; M, \tag{18}$$

(for $\rho > 0$, it holds uniformly and also $M \ge 1$)

$$\lim_{\rho\to\infty} \sup_{\delta \le \theta \le \pi} |\mathcal{B}_\rho(\cos\theta)| \;=\; 0. \tag{19}$$

Then

$$H_\rho(f; x) \;=\; \frac{1}{\Omega_{n-1}} \int_{\mathbb{S}^{n-1}} f(y) \mathcal{B}_\rho(x \cdot y) dy \tag{20}$$

is a singular integral with kernel $\mathcal{B}_\rho (\rho > 0)$, where $c(0, \lambda) = \Omega_{n-2}/\Omega_{n-1}$. It is easy to know that the operator $(Q^d_{k,s} f)(\mu)$ is a singular integral with classical Jackson kernel $\mathcal{D}_{k,s}$.

Now we give the fifth lemma (see [1]).

Lemma 5. *Let* $f \in X$. *Suppose that the kernel* $\mathcal{B}_\rho$ *of singular integral (20) satisfies (17),(18) and*

$$\lim_{\rho\to\infty} \frac{\frac{\lambda}{j+\lambda} \hat{\mathcal{B}}_\rho(j) - 1}{\varphi_(\rho)} \;=\; \psi(j), \tag{21}$$

where the meaning of $\varphi(\rho)$ *is the same as that of Lemma 3. Then the following statements hold:*

a) If there exists $g \in X$, *such that*

$$\lim_{\rho\to\infty} \left\| \frac{1}{\varphi(\rho)} \{H_\rho f - f\} - g \right\|_X = 0, \tag{22}$$

then

$$\psi_j Y_j(f;x) \;=\; Y_j(g;x), \quad j=0,1,2,\dots.$$

Especially, if $g=0$, then f is constant.

b) If

$$\|H_\rho f-f\|_X \;=\; O(\varphi(\rho)), \quad \rho\to\infty,$$

then $f\in\mathcal{H}(X,\psi(j))$.

To represent the functions in the saturation class (see Theorem 3), we define the following kernel:

$$k_d(1,t) \;:=\; \begin{cases} \dfrac{1}{2\lambda}\displaystyle\int_0^1 \frac{1-r^{2\lambda}}{r}\left[\frac{1-r^2}{(1-2rt+r^2)^{\lambda+1}}-1\right]dr, & d=1;\\ \dfrac{1}{2\lambda}\displaystyle\int_0^1 \frac{1-r^{2\lambda}}{r^{2\lambda+1}}k_{d-1}(r,t)dr, & d=2,3,\dots \end{cases}$$

where $|r|<1$,$|t|\le 1$,

$$k_1(r,t)= \;\frac{1}{2\lambda}\int_0^r\left[\frac{1-r_1^2}{(1-2r_1t+r_1^2)^{\lambda+1}}-1\right]\frac{r^{2\lambda}-r_1^{2\lambda}}{r_1}dr_1$$

and

$$k_d(r,t)= \;\frac{1}{2\lambda}\int_0^r \frac{r^{2\lambda}-r_1^{2\lambda}}{r_1^{2\lambda+1}}k_{d-1}(r_1,t)dr_1,\; d=2,3\dots.$$

Lemma 6 gives some important properties of k_d.

Lemma 6. *Let $t=\cos\theta$. For $\lambda>\frac{1}{2}$, $1\le q<(2\lambda+1)/(2\lambda-1)$, we have*

$$\|k_d(1,\cos(\cdot))\|_{q,\lambda}<\infty. \tag{23}$$

Furthermore, (23) also holds for $\lambda=\frac{1}{2}$, $1\le q<\infty$.

Proof. From (3) it follows that

$$\frac{1-r^2}{(1-2rt+r^2)^{\lambda+1}}=\sum_{j=0}^{\infty}\frac{j+\lambda}{\lambda}r^jG_j^\lambda(t), \tag{24}$$

where $-1\le t\le 1,\;\; 0\le r<1$ and $\lambda>-\frac{1}{2}$, $\lambda\ne 0$. By (24) we have

$$\sum_{j=1}^{\infty}r_1^{j+\beta}\frac{j+\lambda}{\lambda}G_j^\lambda(t) \;=\; \left[\frac{1-r_1^2}{(1-2r_1t+r_1^2)^{\lambda+1}}-1\right]r_1^\beta, \quad 0\le r_1<1,\; \beta\ge -1.$$

Since the left series converges uniformly, by termwise integration with respect to r_1 on $[0,r]$ we have

$$\sum_{j=1}^{\infty}\frac{r^{j+\beta+1}}{j+\beta+1}\frac{j+\lambda}{\lambda}G_j^\lambda(t) \;=\; \int_0^r\left[\frac{1-r_1^2}{(1-2r_1t+r_1^2)^{\lambda+1}}-1\right]r_1^\beta dr_1.$$

We first take $\beta=-1$ and multiply $r^{2\lambda}$, and take $\beta=2\lambda-1$ in the above formula, respectively. Then their difference is given by

$$\sum_{j=1}^{\infty}\left(\frac{1}{j}-\frac{1}{j+2\lambda}\right)r^{j+2\lambda}\frac{j+\lambda}{\lambda}G_j^{\lambda}(t) \;=\; \int_0^r\left[\frac{1-r_1^2}{(1-2r_1t+r_1^2)^{\lambda+1}}-1\right]\frac{r^{2\lambda}-r_1^{2\lambda}}{r_1}dr_1.$$

For $0\le r<1$ and $-1\le t\le 1$, we have

$$\begin{aligned} k_1(r,t) &= \frac{1}{2\lambda}\int_0^r\left[\frac{1-r_1^2}{(1-2r_1t+r_1^2)^{\lambda+1}}-1\right]\frac{r^{2\lambda}-r_1^{2\lambda}}{r_1}dr_1 \\ &= \sum_{j=1}^{\infty}\frac{1}{j(j+2\lambda)}r^{j+2\lambda}\frac{j+\lambda}{\lambda}G_j^{\lambda}(t). \end{aligned}$$

Similarly,

$$\begin{aligned} k_2(r,t) &= \frac{1}{2\lambda}\int_0^r\frac{r^{2\lambda}-r_1^{2\lambda}}{r_1^{2\lambda+1}}k_1(r_1,t)dr_1 \\ &= \sum_{j=1}^{\infty}\frac{1}{(j(j+2\lambda))^2}r^{j+2\lambda}\frac{j+\lambda}{\lambda}G_j^{\lambda}(t). \end{aligned}$$

It follows from induction that

$$k_d(r,t) \;=\; \frac{1}{2\lambda}\int_0^r\frac{r^{2\lambda}-r_1^{2\lambda}}{r_1^{2\lambda+1}}k_{d-1}(r_1,t)dr_1 \tag{25}$$

$$\;=\; \sum_{j=1}^{\infty}\frac{1}{(j(j+2\lambda))^d}r^{j+2\lambda}\frac{j+\lambda}{\lambda}G_j^{\lambda}(t). \tag{26}$$

For $0\le r\le 1$, $-1\le t<1$, $1-2rt+r^2>0$ holds. So from the integral expression of $k_1(r,t)$ we can attain that for any $t\in[-1,1)$, $\lim_{r\to 1-}k_1(r,t)$ exists and $k_1(1,t)=\lim_{r\to 1-}k_1(r,t)$ is continuous for t. Furthermore, $k_d(1,t)=\lim_{r\to 1-}k_d(r,t)$ also holds for $t\in[-1,1)$. Therefore,

$$k_d(1,t)=\frac{1}{2\lambda}\int_0^1\frac{1-r_1^{2\lambda}}{r_1^{2\lambda+1}}k_{d-1}(r_1,t)dr_1.$$

Next we prove the boundary of $k_d(1,t)$. From (25), $k_d(1,t)$ can be written as

$$\begin{aligned} k_d(1,t) &= \frac{1}{2\lambda}\int_0^{\frac12}\frac{1-r_1^{2\lambda}}{r_1^{2\lambda+1}}k_{d-1}(r_1,t)dr_1+\frac{1}{2\lambda}\int_{\frac12}^1\frac{1-r_1^{2\lambda}}{r_1^{2\lambda+1}}k_{d-1}(r_1,t)dr_1 \\ &:= I_1+I_2. \end{aligned} \tag{27}$$

Applying (7) and (26), we have

$$\begin{aligned}
|I_1| &= \frac{1}{2\lambda}\sum_{j=1}^{\infty}\frac{1}{(j(j+2\lambda))^{d-1}}\frac{j+\lambda}{\lambda}\left|G_j^\lambda(t)\right|\int_0^{\frac12}\frac{1-r_1^{2\lambda}}{r_1^{2\lambda+1}}\ r_1^{j+2\lambda}dr_1\\
&\le \frac{1}{2\lambda}\sum_{j=1}^{\infty}\frac{1}{(j(j+2\lambda))^{d-1}}\frac{j+\lambda}{\lambda}\binom{j+2\lambda-1}{j}\left(\frac{1}{2^j j}-\frac{1}{(j+2\lambda)2^{j+2\lambda}}\right)\\
&\le C_1(\lambda)\left(\sum_{j=1}^{\infty}\frac{j^{2\lambda}}{(j(j+2\lambda))^{d-1}}\frac{1}{2^j j}\ -\ \sum_{j=1}^{\infty}\frac{j^{2\lambda}}{(j(j+2\lambda))^{d-1}}\frac{1}{(j+2\lambda)\ 2^{j+2\lambda}}\right)\\
&= C_2(\lambda)<\infty, \qquad (28)
\end{aligned}$$

where $C_i(\lambda)(i=1,2,\dots)$ are constants depending on λ. Thus I_1 is uniformly bounded for t.

Next, we discuss I_2. For $r_1\in\left[\frac12,1\right)$,

$$\begin{aligned}
k_1(r_1,t) &= \frac{1}{2\lambda}\int_0^{r_1}\left[\frac{1-r_2^2}{(1-2r_2t+r_2^2)^{\lambda+1}}-1\right]\frac{r_1^{2\lambda}-r_2^{2\lambda}}{r_2}dr_2\\
&= \frac{1}{2\lambda}\int_0^{\frac12}\left[\frac{1-r_2^2}{(1-2r_2t+r_2^2)^{\lambda+1}}-1\right]\frac{r_1^{2\lambda}-r_2^{2\lambda}}{r_2}dr_2\\
&\quad+\frac{1}{2\lambda}\int_{\frac12}^{r_1}\left[\frac{1-r_2^2}{(1-2r_2t+r_2^2)^{\lambda+1}}-1\right]\frac{r_1^{2\lambda}-r_2^{2\lambda}}{r_2}dr_2\\
&:= B_1(t)+B_2(t).
\end{aligned}$$

From the above discussion, B_1 is uniformly bounded for t. Also

$$\begin{aligned}
|B_2(t)| &= \left|\frac{1}{2\lambda}\int_{\frac12}^{r_1}\left[\frac{1-r_2^2}{(1-2r_2t+r_2^2)^{\lambda+1}}-1\right]\frac{r_1^{2\lambda}-r_2^{2\lambda}}{r_2}dr_2\right|\\
&\le \frac{1}{2\lambda}\int_{\frac12}^{r_1}\frac{r_1^{2\lambda}-r_2^{2\lambda}}{r_2}dr_2\ +\ \frac{1}{2\lambda}\int_{\frac12}^{r_1}\frac{1-r_2^2}{(1-2r_2t+r_2^2)^{\lambda+1}}\frac{r_1^{2\lambda}-r_2^{2\lambda}}{r_2}dr_2.
\end{aligned}$$

It is easy to verify that the first integral of the right–hand side is uniformly bounded. Hence we consider the second integral on the right–hand side. Let

$$I(t)=\frac{1}{2\lambda}\int_{\frac12}^{r_1}\frac{1-r_2^2}{(1-2r_2t+r_2^2)^{\lambda+1}}\frac{r_1^{2\lambda}-r_2^{2\lambda}}{r_2}dr_2.$$

For $-1\le t<\frac34$, $I(t)$ is bounded uniformly. For $\frac34\le t\le 1$, we divide $I(t)$ into two parts, $(1-r_2)^2\ge 1-t$ and $(1-r_2)^2\le 1-t$, and estimate each part respectively. Since $1-2r_2t+r_2^2=(1-r_2)^2+2r_2(1-t)$, we have the following inequalities (see [14]):

$$\begin{aligned}
(1-2r_2t+r_2^2)^{\lambda+1} &\ge (1-r_2)^{2\lambda+2},\\
(1-2r_2t+r_2^2)^{\lambda+1} &\ge (1-t)^{\lambda+1},\quad r\ge 1/2.
\end{aligned}$$

It follows from $1-r_2^{2\lambda}=(1-r_2)(1+r_2+r_2^2+\cdots+r_2^{2\lambda-1})\le 2\lambda(1-r_2)$ that

$$\begin{aligned}
|I(t)| &= \frac{1}{2\lambda}\left(\int_{\frac{1}{2}}^{1-(1-t)^{\frac{1}{2}}} + \int_{1-(1-t)^{\frac{1}{2}}}^{r_1}\right)\frac{1-r_2^2}{(1-2r_2t+r_2^2)^{\lambda+1}}\frac{r_1^{2\lambda}-r_2^{2\lambda}}{r_2}dr_2\\
&\le 4\int_{\frac{1}{2}}^{1-(1-t)^{\frac{1}{2}}}(1-r_2)^{-2\lambda}dr_2 + 4(1-t)^{-\lambda-1}\int_{1-(1-t)^{\frac{1}{2}}}^{1}(1-r_2)^2dr_2\\
&\le \begin{cases} C_2+C_3\left|\ln(1-t)\right|, & \lambda=\frac{1}{2},\\ C_2(\lambda)+C_3(\lambda)(1-t)^{-\lambda+\frac{1}{2}}, & \lambda>\frac{1}{2}.\end{cases}
\end{aligned}$$

Thus as $\theta\to 0+$,

$$|k_1(r_1,\cos\theta)| = \begin{cases} O(\ln(\sin\frac{\theta}{2})), & \lambda=\frac{1}{2},\\ O((\sin\frac{\theta}{2})^{-2\lambda+1}), & \lambda>\frac{1}{2}.\end{cases}$$

From the above estimations we can see that $k_1(r_1,\cos\theta)$ is uniformly bounded for r_1. Analogously, by (25), (26) and (28), we have

$$|k_2(r_2,\cos\theta)| \le C_4(\lambda)+\int_{\frac{1}{2}}^{r_2}\frac{r_2^{2\lambda}-r_1^{2\lambda}}{r_1^{2\lambda+1}}|k_1(r_1,\cos\theta)|dr_1<\infty.$$

This implies that for $1\le q<\frac{2\lambda+1}{2\lambda-1}$, and $\lambda>1/2$, or $1\le q<\infty$ and $\lambda=1/2$, there holds

$$\|k_1(1,\cos(\cdot))\|_{q,\lambda}^q=\frac{\Omega_{n-2}}{\Omega_{n-1}}\int_0^\pi |k_1(1,\cos\theta)|^q\sin^{2\lambda}\theta d\theta<\infty.$$

Using (25), (26) and (28), by induction we can prove $|k_d(1,\cos\theta)|<\infty$. Then by (27), we can know that for $1\le q<\frac{2\lambda+1}{2\lambda-1}$, and $\lambda>1/2$, or $1\le q<\infty$ and $\lambda=1/2$, there holds

$$\|k_d(1,\cos(\cdot))\|_{q,\lambda}^q=\frac{\Omega_{n-2}}{\Omega_{n-1}}\int_0^\pi |k_d(1,\cos\theta)|^q\sin^{2\lambda}\theta d\theta<\infty.$$

□

4. Main results and their proofs

In this section, we first give an estimation of the rate of approximation by the Jackson-type operator $(Q_{k,s}^d f)(\mu)$. Then, we prove the saturation theorem of the Jackson-type operator on $L^p(\mathbb{S}^{n-1})$.

Theorem 1. *Let d, s, k be positive integers, $2s>2d+n-1$, and $f\in L^p(\mathbb{S}^{n-1})$, $1\le p\le\infty$. For the Jackson-type operator $(Q_{k,s}^d f)(\mu)$ defined above, we have*

$$\|Q_{k,s}^d f-f\|_p\le C_1\omega^{2d}\left(f,\frac{1}{k}\right)_p \tag{29}$$

where the constant C_1 only depends on d and s.

Proof. By Definition 1 we have

$$(Q^d_{k,s}f)(\mu) - f(\mu) \;=\; \int_0^\pi (-1)^{d-1}(S_\theta - I)^d f(\mu)\mathcal{D}_{k,s}(\theta)\sin^{n-2}\theta d\theta.$$

Therefore

$$\begin{aligned}
\|Q^d_{k,s}f \;-\; f\|_p \;&\leq\; \int_0^\pi \|(S_\theta - I)^d f\|_p \mathcal{D}_{k,s}(\theta)\sin^{n-2}\theta d\theta \\
&\leq\; \int_0^\pi \omega^{2d}(f,\theta)_p \mathcal{D}_{k,s}(\theta)\sin^{n-2}\theta d\theta \\
&\leq\; \int_0^\pi (k\theta+1)^{2d}\omega^{2d}\left(f,\frac{1}{k}\right)_p \mathcal{D}_{k,s}(\theta)\sin^{n-2}\theta d\theta \\
&\leq \omega^{2d}\left(f,\frac{1}{k}\right)_p \sum_{j=0}^{2d}\binom{2d}{j}C_{j,s} \\
&= C_1\omega^{2d}\left(f,\frac{1}{k}\right)_p.
\end{aligned}$$

The last inequality follows from Lemma 1, since $2s > 2d+n-1 > j+n-1$, $j = 0,1,2,\ldots,2d$. This completes the proof of Theorem 1. □

Theorem 2. *Let d be a positive integer, $2s > 2d+n-1$. Then $(Q^d_{k,s}f)(\mu)$ is saturated on $L^p(\mathbb{S}^{n-1})$ with order k^{-2d} and the collection of constants is their invariant class.*

Proof. For $j = 0,1,2,\ldots,$ we first prove that

$$\lim_{k\to\infty}\frac{1-\xi^d_{k,s}(j)}{1-\xi^d_{k,s}(1)} = \left(\frac{j(j+2\lambda)}{2\lambda+1}\right)^d. \tag{30}$$

In fact, for any $0 < \delta < \pi$, there exists β such that $2s > \beta+n-1 > 2d+n-1$, then it follows from (11) that

$$\begin{aligned}
\int_\delta^\pi \mathcal{D}_{k,s}(\theta)\sin^{2\lambda}\theta d\theta \;&\leq\; \int_\delta^\pi (\frac{\theta}{\delta})^\beta \mathcal{D}_{k,s}(\theta)\sin^{2\lambda}\theta d\theta \\
&\leq\; \delta^{-\beta}\int_0^\pi \theta^\beta \mathcal{D}_{k,s}(\theta)\sin^{2\lambda}\theta d\theta \\
&\leq\; C_{\beta,s}(\delta k)^{-\beta}.
\end{aligned}$$

Since $\beta > 2d$, then

$$\int_\delta^\pi \mathcal{D}_{k,s}(\theta)\sin^{2\lambda}\theta d\theta \;=\; o(k^{-2d}), \qquad k\to\infty. \tag{31}$$

And using (11) again, we have

$$\begin{aligned}
1-\xi_{k,s}^{d}(1) &= \int_0^\pi \mathcal{D}_{k,s}(\theta)\left(1-\frac{G_1^\lambda(\cos\theta)}{G_1^\lambda(1)}\right)^d \sin^{2\lambda}\theta d\theta \\
&= 2^d\int_0^\pi \mathcal{D}_{k,s}(\theta)\sin^{2d}\frac{\theta}{2}\sin^{2\lambda}\theta d\theta \\
&\le \int_0^\pi \mathcal{D}_{k,s}(\theta)\theta^{2d}\sin^{2\lambda}\theta d\theta \\
&\le Ck^{-2d}.
\end{aligned}$$

So we obtain

$$1-\xi_{k,s}^{d}(1) = O(k^{-2d}). \tag{32}$$

Let

$$T_j^\lambda(\cos\theta) = 1-\left(1-\frac{G_j^\lambda(\cos\theta)}{G_j^\lambda(1)}\right)^d.$$

We deduce from (8) that

$$\lim_{\theta\to 0}\frac{1-T_j^\lambda(\cos\theta)}{1-T_1^\lambda(\cos\theta)} = \left(\frac{j(j+2\lambda)}{2\lambda+1}\right)^d.$$

Hence for any $\varepsilon>0$, there exists $\delta>0$ such that for $0<\theta<\delta$

$$\left|(1-T_j^\lambda(\cos\theta))-\left(\frac{j(j+2\lambda)}{2\lambda+1}\right)^d(1-T_1^\lambda(\cos\theta))\right| \le \varepsilon\,(1-T_1^\lambda(\cos\theta)).$$

Then

$$\begin{aligned}
&\left|1-\xi_{k,s}^{d}(j)-\left(\frac{j(j+2\lambda)}{2\lambda+1}\right)^d(1-\xi_{k,s}^{d}(1))\right| \\
&= \left|\int_0^\pi \mathcal{D}_{k,s}(\theta)\left[\left(1-\frac{G_j^\lambda(\cos\theta)}{G_j^\lambda(1)}\right)^d-\left(\frac{j(j+2\lambda)}{2\lambda+1}\right)^d\left(1-\frac{G_1^\lambda(\cos\theta)}{G_1^\lambda(1)}\right)^d\right]\right. \\
&\qquad \left.\times\sin^{2\lambda}\theta d\theta\right| \\
&\le \int_0^\delta \mathcal{D}_{k,s}(\theta)\,\varepsilon\,\left(1-\frac{G_1^\lambda(\cos(\theta))}{G_1^\lambda(1)}\right)^d\sin^{2\lambda}\theta d\theta \\
&\quad +\int_\delta^\pi \mathcal{D}_{k,s}(\theta)\left(1+\left(\frac{j(j+2\lambda)}{2\lambda+1}\right)^d\right)\sin^{2\lambda}\theta d\theta \\
&\le \varepsilon\cdot O(k^{-2d})+o(k^{-2d}) \\
&= o(k^{-2d}), \qquad k\to\infty.
\end{aligned}$$

Therefore (30) holds and is not equal to zero.

According to Lemma 3, we finally finish the proof of Theorem 2. $\square$

From Lemma 5, it is easy to see that the saturation class of $(Q^d_{k,s}f)(\mu)$ is

$$\mathcal{H}(X, -(j(j+2\lambda))^d).$$

Next we give an equivalent theorem about the saturation class.

Theorem 3. *Let $f \in X$. The following statements are equivalent.*
(i) $f \in \mathcal{H}(X, -(j(j+2\lambda))^d)$.
(ii) For $f \in L^p(\mathbb{S}^{n-1})$ and $g \in L^p(\mathbb{S}^{n-1})$ $(1 < p < \infty)$

$$f(x) = Y_0(f;x) - \frac{1}{\Omega_{n-1}} \int_{\mathbb{S}^{n-1}} k_d(1, (x\cdot y))g(y)dy \tag{33}$$

holds almost everywhere, and for $f \in C(\mathbb{S}^{n-1})$ and $g \in L^\infty(\mathbb{S}^{n-1})$, (33) follows too. For $f \in L^1(\mathbb{S}^{n-1})$ and the measure $\mu \in M(\mathbb{S}^{n-1})$

$$f(x) = Y_0(f;x) - \frac{1}{\Omega_{n-1}} \int_{\mathbb{S}^{n-1}} k_d(1, (x\cdot y))d\mu(y) \tag{34}$$

holds almost everywhere, where $k_d(1,t)(-1 \le t \le 1)$ is given in Section 3.

Proof. We first prove that (i) implies (ii). From (26), we obtain the Gegenbauer coefficient of $k_d(1,t)$:

$$\hat{k}_d(1,j) = \frac{j+\lambda}{\lambda}\frac{1}{(j(j+2\lambda))^d}, \quad j = 1, 2\ldots, \quad \hat{k}_d(1,0) = 0.$$

If $f \in \mathcal{H}(X, -(j(j+2\lambda))^d)$, and $X = L^p(\mathbb{S}^{n-1}), 1 < p < \infty$ or $X = C(\mathbb{S}^{n-1})$, then for any $j = 1, 2\ldots$, we have

$$Y_j(f;x) = -\frac{1}{(j(j+2\lambda))^d}\, Y_j(g;x) = -\frac{\lambda}{j+\lambda}\hat{k}_d(1,j)Y_j(g;x).$$

If $f \in \mathcal{H}(X, -(j(j+2\lambda))^d)$ and $X = L^1(\mathbb{S}^{n-1})$, then we also have for $j = 1, 2\ldots$

$$Y_j(f;x) = -\frac{\lambda}{j+\lambda}\hat{k}_d(1,j)Y_j(d\mu;x).$$

Therefore, by (10), Lemma 4 and the uniqueness theorem of Laplace series expansion, there hold (33) and (34).

Using (9), (10) and Lemma 4, we can deduce (i) from (ii). This completes the proof. □

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