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Cacti with minimum, second-minimum, and third-minimum Kirchhoff indices

Hongzhuan Wang[1], Hongbo Hua[1,*] and Dongdong Wang[1]

[1] *Faculty of Mathematics and Physics, Huaiyin Institute of Technology, Huaian, Jiangsu-223 003, P. R. China*

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Abstract. Resistance distance was introduced by Klein and Randić. The Kirchhoff index $Kf(G)$ of a graph G is the sum of resistance distances between all pairs of vertices. A graph G is called a cactus if each block of G is either an edge or a cycle. Denote by $Cat(n;t)$ the set of connected cacti possessing n vertices and t cycles. In this paper, we give the first three smallest Kirchhoff indices among graphs in $Cat(n;t)$, and characterize the corresponding extremal graphs as well.

AMS subject classifications: 05C12, 05C90

Key words: cactus, Kirchhoff index, resistance distance

1. Introduction

Let $G = (V(G), E(G))$ be a graph whose sets of vertices and edges are $V(G) = \{v_1, v_2, \cdots, v_n\}$ and $E(G)$, respectively.

In 1993, Klein and Randić [1] posed a new distance function named resistance distance on the basis of electrical network theory. The term resistance distance was used for the physical interpretation: one imagines unit resistors on each edge of a connected graph G with vertices $v_1, v_2, \cdots, v_n$ and takes the resistance distance between vertices v_i and v_j of G to be the effective resistance between vertices v_i and v_j, denoted by $r_G(v_i, v_j)$. Recall that the conventional distance between vertices v_i and v_j, denoted by $d_G(v_i, v_j)$, is the length of a shortest path between them and the famous Wiener index [2] is the sum of distances between all pairs of vertices; that is,

$$W(G) = \sum_{i<j} d_G(v_i, v_j).$$

Analogous to the Wiener index, the Kirchhoff index [3] is defined as

$$Kf(G) = \sum_{i<j} r_G(v_i, v_j).$$

Similarly to the conventional distance, the resistance distance is also intrinsic to the graph, not only with some nice purely mathematical and physical interpretations

*Corresponding author. *Email addresses:* `hongzhuanwang@gmail.com` (H. Wang), `hongbo.hua@gmail.com` (H. Hua), `dongdong.wang.hyit@gmail.com` (D. Wang)

[4,5], but with a substantial potential for chemical applications. In fact, for those two distance functions, the shortest-path might be imagined to be more relevant when there is corpuscular communication (along edges) between two vertices, whereas the resistance distance might be imagined to be more relevant when the communication is wave- or fluid-like. Then, that chemical communication in molecules is rather wavelike suggests the utility of this concept in chemistry. So in recent years, the resistance distance has become much studied in the chemical literature [6-17]. It is found that the resistance distance is closely related with many well-known graph invariants, such as the connectivity index, the Balaban index, etc. This further suggests the resistance distance is worthy of being investigated. The resistance distance is also well studied in the mathematical literature. Much work has been done to compute the Kirchhoff index of some classes of graphs, or give bounds for the Kirchhoff index of graphs and characterize extremal graphs [10,15,18]. For instance, unicyclic graphs with the extremal Kirchhoff index are characterized and sharp bounds for the Kirchhoff index of such graphs are obtained [19]. In [20], the authors further investigated the Kirchhoff index in unicyclic graphs, and determined graphs with second- and third-minimal Kirchhoff indices as well as those graphs with second- and third-maximal Kirchhoff indices.

A graph G is called a *cactus* if each block of G is either an edge or a cycle. Denote by $Cat(n,t)$ the set of connected cacti possessing n vertices and t cycles. In this paper, we give the first three smallest Kirchhoff indices among graphs in $Cat(n,t)$ and characterize the corresponding extremal graphs as well.

2. Lemmas and main results

Before we state and prove our main results, we introduce some lemmas.

Lemma 1 (see [7]). *Let x be a cut vertex of a connected graph G and a and b vertices occurring in different components which arise upon deletion of x. Then $r_G(a,b) = r_G(a,x) + r_G(x,b)$.*

Lemma 2. *Let G_1 and G_2 be connected graphs. If we identify any vertex, say x_1, of G_1 with any other vertex, say x_2, of G_2 as a new common vertex x, and we obtain a new graph G, then*

$$Kf(G) = Kf(G_1) + Kf(G_2) + n_1 Kf_{x_2}(G_2) + n_2 Kf_{x_1}(G_1),$$

where $Kf_{x_i}(G_i) = \sum_{y_i \in G_i} r_{G_i}(x_i, y_i)$ and $n_i = |V(G_i)| - 1$ for $i = 1,2$.

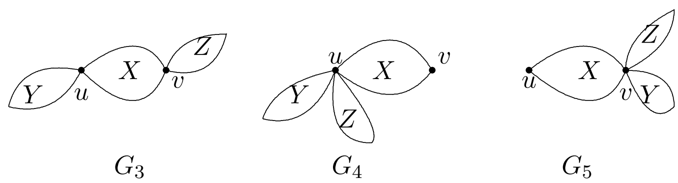


Figure 1.

Let G_3, G_4 and G_5 be connected graphs as depicted in Fig. 1. By **Operation I** we mean the graph transformation from G_3 to G_4, or from G_3 to G_5.

The following lemma is our main lemma, which shall play an important role in proving our main results.

Lemma 3. *Let G_3, G_4 and G_5 be connected graphs as depicted in Fig. 1. Then the following holds:*

(a) If $Kf_v(X) \geq Kf_u(X)$, then $Kf(G_3) > Kf(G_4)$;

(b) If $Kf_u(X) \geq Kf_v(X)$, then $Kf(G_3) > Kf(G_5)$.

Proof. By Lemmas 1 and 2, one obtains

$$\begin{aligned}
Kf(G_4) &= Kf(X) + Kf(Y \cup Z) + (|X| - 1)Kf_u(Y \cup Z) + (|Y| + |Z| - 2)Kf_u(X)\\
&= Kf(X) + Kf(Y) + Kf(Z) + (|Y| - 1)Kf_u(Z) + (|Z| - 1)Kf_u(Y)\\
&\quad +(|X| - 1)(Kf_u(Y) + Kf_u(Z)) + (|Y| + |Z| - 2)Kf_u(X)\\
&= Kf(X) + Kf(Y) + Kf(Z) + (|Y| - 1)Kf_v(Z) + (|Z| - 1)Kf_u(Y)\\
&\quad +(|X| - 1)(Kf_u(Y) + Kf_v(Z)) + (|Y| + |Z| - 2)Kf_u(X),
\end{aligned}$$

$$\begin{aligned}
Kf(G_3) &= Kf(Y) + Kf(X \cup Z) + (|Y| - 1)Kf_u(X \cup Z) + (n - |Y|)Kf_u(Y)\\
&= Kf(X) + Kf(Y) + Kf(Z) + (|X| - 1)Kf_v(Z) + (|Z| - 1)Kf_v(X)\\
&\quad +(|Y| - 1)(Kf_u(X) + Kf_v(Z) + (|Z| - 1)r_X(u, v)) + (n - |Y|)Kf_u(Y)\\
&= Kf(X) + Kf(Y) + Kf(Z) + (|Y| - 1)Kf_v(Z) + (|Z| - 1)Kf_u(Y)\\
&\quad +(|X| - 1)(Kf_u(Y) + Kf_v(Z)) + (|Y| + |Z| - 2)Kf_u(X).
\end{aligned}$$

Therefore,

$$\begin{aligned}
Kf(G_4) - Kf(G_3) &= |Y| - 1)Kf_v(Z) + (|Z| - 1)Kf_u(Y) + (|X| - 1)Kf_u(Y)\\
&\quad +(|X| - 1)Kf_v(Z) + (|Y| + |Z| - 2)Kf_u(X)\\
&\quad -(|X| - 1)Kf_v(Z) - (|Z| - 1)Kf_v(X) - (|Y| - 1)Kf_u(X)\\
&\quad -(|Y| - 1)Kf_v(Z) - (|Y| - 1)(|Z| - 1)r_X(u, v))\\
&\quad -(n - |Y|)Kf_u(Y)\\
&= (|Z| - 1)(Kf_u(X) - Kf_v(X)) - (|Y| - 1)(|Z| - 1)r_X(u, v).
\end{aligned}$$

If $Kf_v(X) \geq Kf_u(X)$, it then follows from the above equation that

$$Kf(G_4) - Kf(G_3) \leq -(|Y| - 1)(|Z| - 1)r_X(u, v) < 0,$$

which proves (a).

The same reasoning can be used to prove (b). □

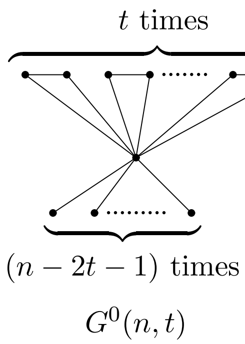


Figure 2.

Now, we are in a position to state and prove our main results of this paper.

Theorem 1. *Among all graphs in $Cat(n,t)$ with $n \geq 13$ and $t \geq 1$, $G^0(n,t)$ is the unique graph having the minimum Kirchhoff index.*

Proof. Let G_{min} be the graph in $Cat(n,t)$ with the minimal Kirchhoff index. Our goal is to prove that $G_{min} \cong G^0(n,t)$.

By contradiction. Suppose to the contrary that $G_{min} \ncong G^0(n,t)$.

We first assume that G_{min} has no cut-edges. From Lemma 3, all cycles in G_{min} must share exactly one common vertex, say v.

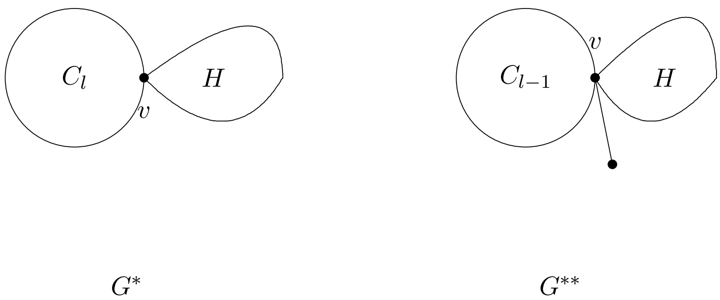


Figure 3.

By **Operation II** we mean the graph transformation from G^* to G^{**}.

We first prove the following assertion.

Assertion 1. *Let G^* and G^{**} be two graphs as depicted in Fig.3., then $Kf(G^*) > Kf(G^{**})$.*

Proof. From Lemma 2 we obtain

$$Kf(G^*) = Kf(C_l) + Kf(H) + (l-1)Kf_v(H) + (|H|-1)Kf_v(C_l),$$
$$Kf(G^{**}) = Kf(S_l^{l-1}) + Kf(H) + (l-1)Kf_v(H) + (|H|-1)Kf_v(S_l^{l-1}),$$

where S_l^{l-1} is the graph obtained from C_{l-1} by attaching a pendant edge to one of its vertices, C_{l-1} is an $l-1$-vertex cycle. Recall that $Kf_v(C_l) = (l^2-1)/6$ and $Kf(C_l) = (l^3-l)/(12)$, where v is any vertex of C_l. So we have

$$\begin{aligned} Kf(G^*) - Kf(G^{**}) &= \frac{(l^3-l)}{12} - \frac{(l-1)^3-(l-1)}{12} - 1 - \frac{(l-1)^2-1}{6} - (l-2) \\ &\quad +(|H|-1)(\frac{l^2-1}{6} - \frac{(l-1)^2-1}{6} - 1) \\ &= \frac{l^2-11l+12}{12} + (|H|-1)(\frac{2l-7}{6}) \\ &= \frac{(l-\frac{11}{2})^2}{12} - \frac{73}{48} + (n-l)(\frac{2l-7}{6}). \end{aligned}$$

• If $n-l=0$, then $l \geq 13$, so $Kf(G^*) - Kf(G^{**}) \geq \frac{(13-\frac{11}{2})^2}{12} - \frac{73}{48} > 0$;
• If $n-l=1$, then $l \geq 12$, so $Kf(G^*) - Kf(G^{**}) \geq -\frac{73}{48} + (\frac{2\times 12-7}{6}) > 0$;
• If $n-l=2$, then $l \geq 11$, so $Kf(G^*) - Kf(G^{**}) \geq -\frac{73}{48} + 2 \times (\frac{2\times 11-7}{6}) > 0$;
• If $n-l=3$, then $l \geq 10$, so $Kf(G^*) - Kf(G^{**}) \geq -\frac{73}{48} + 3 \times (\frac{2\times 10-7}{6}) > 0$;
• If $n-l=4$, then $l \geq 9$, so $Kf(G^*) - Kf(G^{**}) \geq -\frac{73}{48} + 4 \times (\frac{2\times 9-7}{6}) > 0$;
• If $n-l=5$, then $l \geq 8$, so $Kf(G^*) - Kf(G^{**}) \geq -\frac{73}{48} + 5 \times (\frac{2\times 8-7}{6}) > 0$;
• If $n-l=6$, then $l \geq 7$, so $Kf(G^*) - Kf(G^{**}) \geq -\frac{73}{48} + 6 \times (\frac{2\times 7-7}{6}) > 0$;
• If $n-l=7$, then $l \geq 6$, so $Kf(G^*) - Kf(G^{**}) \geq -\frac{73}{48} + 7 \times (\frac{2\times 6-7}{6}) > 0$;
• If $n-l=8$, then $l \geq 5$. So $Kf(G^*) - Kf(G^{**}) \geq -\frac{73}{48} + 8 \times (\frac{2\times 5-7}{6}) > 0$;
• If $n-l=9$, then $l \geq 4$. If $l=4$, then $Kf(G^*) - Kf(G^{**}) = \frac{9}{48} - \frac{73}{48} + 9 \times(\frac{2\times 4-7}{6}) > 0$; If $l \geq 5$, then $Kf(G^*) - Kf(G^{**}) \geq -\frac{73}{48} + 9 \times (\frac{2\times 5-7}{6}) > 0$;
• If $n-l \geq 10$, then $Kf(G^*) - Kf(G^{**}) \geq -\frac{73}{48} + 10 \times (\frac{2\times 4-7}{6}) > 0$, because $l \geq 4$.
This proves the assertion. □

By the above Assertion, G_{min} has no cycles of length greater than 3. As G_{min} has no cut-edges and all cycles in G_{min} must share exactly one common vertex, we have $G_{min} \cong G^0(n,t)$ with $n = 2t+1$, a contradiction to our assumption that $G_{min} \ncong G^0(n,t)$. So we may suppose now that G_{min} contains at least one cut-edge. If G_{min} contains non-pendant cut-edges, then we can use Operation I in Fig. 1, and we shall obtain a new graph G', which is still in $Cat(n,t)$. But by Lemma 3, $Kf(G') < Kf(G_{min})$, a contradiction to the minimality of G_{min}. So all cut-edges in G_{min} are pendant edges. Moreover, all cycles in G_{min} share exactly one common vertex, say v, and there exists no cycle in G_{min} of length greater than 3. Furthermore, all pendant edges in G_{min} are incident to v. Thus, $G_{min} \cong G^0(n,t)$, a contradiction once again. These contradictions force us to draw a conclusion that $G_{min} \cong G^0(n,t)$ if G has the minimal $Kf(G)-$value, completing the proof. □

Denote by $\underbrace{C_3vC_3\cdots vC_3}_{t\,times}$ the graph obtained from t copies of C_3 by fusing one vertex of each C_3 into a new common vertex v.

Corollary 1. *For any graph G in $Cat(n,t)$ with $n \geq 13$ and $t \geq 1$, $Kf(G) \geq n^2 - 2t^2 - 3n + t + 2 + \frac{nt}{3}$ with equality if and only if $G \cong G^0(n,t)$.*

Proof. According to Lemma 2, we obtain

$$\begin{aligned}
Kf(G^0(n,t) &= Kf(\underbrace{C_3vC_3\cdots vC_3}_{t\,times}) + Kf(S_{n-2t}) + 2tKf_v(S_{n-2t}) \\
&\quad +(n-2t-1)Kf_v(\underbrace{C_3vC_3\cdots vC_3}_{t\,times}) \\
&= Kf(\underbrace{C_3vC_3\cdots vC_3}_{t\,times}) + (n-2t-1) + 2t(n-2t-1) \\
&\quad +\frac{4t}{3}(n-2t-1) + 2\binom{n-2t-1}{2}, \\
Kf(\underbrace{C_3vC_3\cdots vC_3}_{t\,times}) &= Kf(C_3) + Kf(\underbrace{C_3vC_3\cdots vC_3}_{(t-1)\,times}) \\
&\quad +2Kf_v(\underbrace{C_3vC_3\cdots vC_3}_{(t-1)\,times}) + 2Kf_v(C_3) \\
&= Kf(\underbrace{C_3vC_3\cdots vC_3}_{(t-1)\,times}) + \frac{16t-10}{3}.
\end{aligned}$$

Note that $Kf(C_3) = 2$. Hence, we obtain $Kf(\underbrace{C_3vC_3\cdots vC_3}_{t\,times}) = \frac{8t^2-2t}{3}$ by an elementary calculation, and then $Kf(G^0(n,t)) = n^2-2t^2-3n+t+2+\frac{nt}{3}$. Combining this fact and Theorem 1, we get the desired result. □

In the following we shall consider the cacti with the second and the third smallest Kirchhoff indices.

Suppose first that G has the second smallest Kirchhoff index among all elements of $Cat(n,t)$. Evidently, G can be changed into $G^0(n,t)$ by using exactly one step of Operation I or II, for otherwise, one can employ one step of Operation I or II on G, and obtain a new graph G', which is still in $Cat(n,t)$ but not isomorphic to $G^0(n,t)$, which gives

$$Kf(G) > Kf(G') > Kf(G^0(n,t)),$$

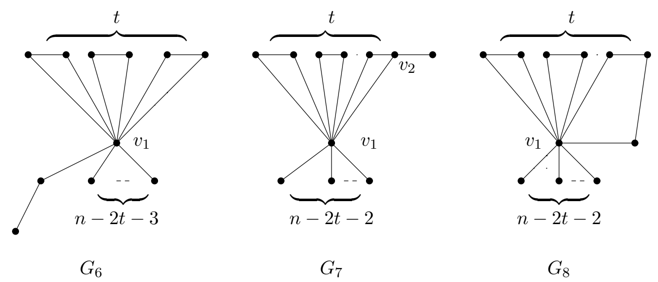


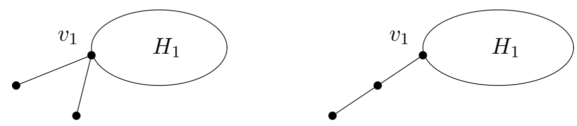
Figure 4.

contradicting our choice of G.

By the above arguments, one can conclude that G must be one of the graphs G_6, G_7, and G_8 as shown in Fig. 4.

Theorem 2. *Among all graphs in $Cat(n,t)$ with $n \geq 13$ and $t \geq 1$, the cactus with the second-minimum Kirchhoff index is G_8 (see Fig. 4).*

Proof. (i): Let H_1 denote the common subgraph of G_6 and $G^0(n,t)$. Thus, we can view graphs G_6 and $G^0(n,t)$ as the graphs depicted in Fig. 5.



$G^0(n,t)$ $\qquad$ G_6

Figure 5.

Using Lemma 2, we have

$$\begin{aligned} Kf(G^0(n,t)) &= Kf(S_3) + Kf(H_1) + 2Kf_{v_1}(H_1) + (n-3)Kf_{v_1}(S_3) \\ &= 4 + Kf(H_1) + 2Kf_{v_1}(H_1) + 2(n-3), \end{aligned}$$

$$\begin{aligned} Kf(G_6) &= Kf(P_3) + Kf(H_1) + 2Kf_{v_1}(H_1) + (n-3)Kf_{v_1}(P_3) \\ &= 4 + Kf(H_1) + 2Kf_{v_1}(H_1) + 3(n-3). \end{aligned}$$

Therefore,

$$Kf(G_6) = Kf(G^0(n,t)) + (n-3).$$

(ii): Let H_2 be the common subgraph of G_7 and $G^0(n,t)$(see Fig. 6). Here we also let v_1 denote the unique maximum-degree vertex in $G^0(n,t)$.

$G^0(n,t)$ $\qquad$ G_7

Figure 6.

In view of Lemma 2,

$$\begin{aligned} Kf(G^0(n,t)) &= Kf(P_2) + Kf(H_2) + Kf_{v_1}(H_2) + (n-2)Kf_{v_1}(P_2) \\ &= Kf(H_2) + Kf_{v_1}(H_2) + (n-1), \end{aligned}$$

$$\begin{aligned} Kf(G_7) &= Kf(H_2) + Kf_{v_2}(H_2) + (n-1) \\ &= Kf(H_2) + Kf_{v_1}(H_2) + \frac{2}{3}(n-4) + (n-1). \end{aligned}$$

Therefore,

$$Kf(G_7) = Kf(G^0(n,t)) + \frac{2}{3}(n-4). \tag{1}$$

(iii): Let H_3 be the common subgraph of G_8 and $G^0(n,t)$(see Fig. 7).

Figure 7.

Denote by S_4^3 the graph obtained by attaching one pendant edge to any vertex of C_3. So,

$$Kf(G_8) = kf(C_4) + Kf(H_3) + 3Kf_{v_1}(H_3) + (n-4)Kf_{v_1}(C_4),$$
$$Kf(G^0(n,t)) = Kf(S_4^3) + Kf(H_3) + 3Kf_{v_1}(H_3) + (n-4)Kf_{v_1}(S_4^3).$$

Recall that $Kf_{v_1}(C_4) = \frac{17}{6}$, and that $Kf(C_4) = \frac{4^3-4}{12} = 5$, we thus have

$$\begin{aligned} Kf(G_8) - Kf(G^0(n,t)) &= \frac{4^2 - 11 \times 4 + 12}{12} + (n-4)(\frac{2 \times 4 - 7}{6}) \\ &= \frac{32}{12} + \frac{n-4}{6} \\ &= \frac{n}{6} - 2. \end{aligned}$$

Therefore, $Kf(G_8) = Kf(G^0(n,t)) + \frac{n}{6} - 2$.

By the above expressions obtained for the Kirchhoff indices of G_6, G_7 and G_8, we immediately have the desired result. □

From Theorem 2 we immediately have the following result.

Corollary 2. *For a graph G, not isomorphic to $G^0(n,t)$, in $Cat(n,t)$ with $n \geq 13$ and $t \geq 1$, it holds that $Kf(G) \geq n^2 - 2t^2 - \frac{17n}{6} + t + \frac{nt}{3}$, with equality if and only if $G \cong G_8$ (see Fig.4).*

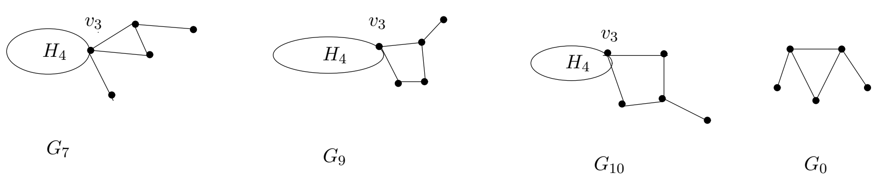


Figure 8.

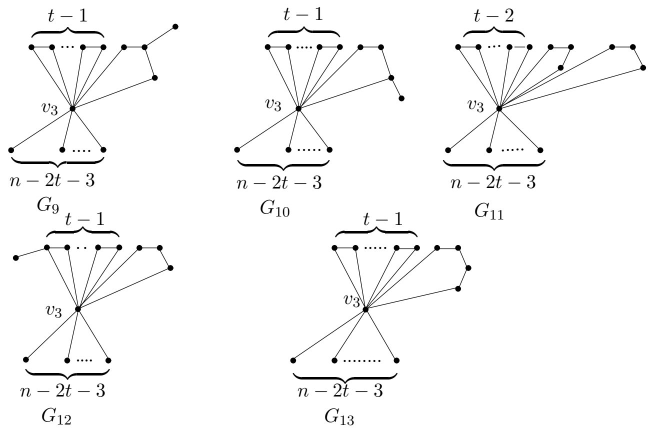


Figure 9.

By the same reasonings as those used in Theorem 2, we conclude that the possible candidates having the third smallest Kirchhoff index must come from one graph of G_7, $G_9 - G_{13}$(see Fig.s 4 and 9).

Theorem 3. *Among all graphs in $Cat(n,t)$ with $n \geq 13$ and $t \geq 1$, the cactus with the third-minimum Kirchhoff index is G_{11} (see Fig. 9).*

Proof. By above discussions, we need only to determine the minimum cardinality among $Kf(G_7)$, $Kf(G_9)$, $Kf(G_{10})$, $Kf(G_{11})$, $Kf(G_{12})$ and $Kf(G_{13})$.

Let H_4 be the common subgraph of G_7, G_9 and G_{10}(see Fig.s 4, 8 and 9). Also, we let G_0 be a subgraph of G_7(see Fig. 8). It is now reduced to

$$\begin{aligned} Kf(G_9) &= Kf(S_5^4) + Kf(H_4) + 4Kf_{v_3}(H_4) + (n-5)Kf_{v_3}(S_5^4) \\ &= \frac{23}{2} + Kf(H_4) + 4Kf_{v_3}(H_4) + \frac{17}{4}(n-5), \\ Kf(G_7) &= Kf(G_0) + Kf(H_4) + 4Kf_{v_3}(H_4) + (n-5)Kf_{v_3}(G_0) \\ &= \frac{40}{3} + Kf(H_4) + 4Kf_{v_3}(H_4) + 4(n-5), \end{aligned}$$

where S_5^4 denotes the graph obtained by attaching one pendant edge to any vertex of C_4.

Therefore,

$$Kf(G_9) - Kf(G_7) = \frac{3n-37}{12} > 0,$$

since $n \geq 13$.

Furthermore,

$$\begin{aligned} Kf(G_{10}) &= Kf((S_5^4)) + Kf(H_4) + 4Kf_{v_3}(H_4) + (n-5)Kf_{v_3}((S_5^4)) \\ &= \frac{23}{2} + Kf(H_4) + 4Kf_{v_3}(H_4) + \frac{9}{2}(n-5). \end{aligned}$$

Therefore

$$Kf(G_{10}) - Kf(G_9) = \frac{n-5}{4} > 0 \quad (n \geq 13).$$

Then

$$Kf(G_{10}) > Kf(G_9) > Kf(G_7) \quad (n \geq 13).$$

In the following, we need only to compare G_7, G_{11}, G_{12} and G_{13}.

Evidently, G_{11} can be changed into G_8 by using exactly one step of Operation II. Hence, by the Assertion in Theorem 1, we have

$$Kf(G_{11}) = Kf(G_8) + \frac{l^2 - 11l + 12}{12} + (\mid H \mid -1)(\frac{2l-7}{6}).$$

Here, $l = 4$ and $\mid H \mid -1 = n - 4$. Therefore,

$$\begin{aligned} Kf(G_{11}) &= Kf(G_8) + \frac{4^2 - 11 \times 4 + 12}{12} + (n-4)(\frac{2 \times 4 - 7}{6}) \\ &= Kf(G_8) + \frac{n}{6} - 2 \\ &= Kf(G^0(n,t)) + \frac{n}{3} - 4. \end{aligned} \tag{2}$$

From Eqs.(1) and (2) it follows that

$$Kf(G_{11}) - Kf(G_7) = -\frac{1}{3}n - \frac{4}{3} < 0.$$

Evidently, G_{12} can be changed into G_7 by using exactly one step of Operation II. So we have

$$Kf(G_{12}) = Kf(G_7) + \frac{n}{6} - 2 = Kf(G^0(n,t)) + \frac{5n-28}{6}.$$

Thus

$$Kf(G_{12}) > Kf(G_7) > Kf(G_{11}) \quad (n \geq 13).$$

It is obvious that G_{13} can be changed into G_8 by using exactly one step of Operation II, thus by the same reasoning as above, we get

$$Kf(G_{13}) = Kf(G_8) + \frac{l^2 - 11l + 12}{12} + (\mid H \mid -1)(\frac{2l-7}{6}).$$

Here, $l = 5$ and $\mid H \mid -1 = n - 5$. Therefore,

$$\begin{aligned} Kf(G_{13}) &= Kf(G_8) + \frac{5^2 - 11 \times 5 + 12}{12} + (n-5)(\frac{2 \times 5 - 7}{6}) \\ &= Kf(G_8) + \frac{n}{2} - 4. \end{aligned}$$

Combining this fact and the expression that $Kf(G_8) = Kf(G^0(n,t)) + \frac{n}{6} - 2$, we arrive at

$$Kf(G_{13}) - Kf(G_{11}) = \frac{1}{3}n - 2 = \frac{n-6}{3}.$$

So $Kf(G_{11}) < Kf(G_{13})$ for $n \geq 13$.

The proof of Theorem 3 is thus completed. □

Corollary 3. *For a graph G in $Cat(n,t)$ with $n \geq 13$ and $t \geq 1$, not isomorphic to $\{G^0(n,t), G_8\}$,*

$$Kf(G) \geq n^2 - 2t^2 - \frac{8n}{3} + t - 2 + \frac{nt}{3}$$

with equality if and only if $G \cong G_{11}$ (see Fig. 9).

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References

[1] D. BABIĆ, D. J. KLEIN, I. LUKOVITS, S. NIKOLIĆ, N. TRINAJSTIĆ, *Resistance-distance matrix: A computational algorithm and its application*, Int. J. Quantum Chem. **90**(2002), 166–176.

[2] D. BONCHEV, A. T. BALABAN, X. LIU, D. J. KLEIN, *Molecular cyclicity and centricity of polycyclic graphs. I. Cyclicity based on resistance distances or reciprocal distances*, Int. J. Quantum Chem. **50**(1994), 1–20.

[3] A. A. DOBRYNIN, R. ENTRINGER, I. GUTMAN, *Wiener index of trees: theory and applications*, Acta Appl. Math. **66**(2001), 211–249.

[4] P. W. FOWLER, *Resistance distances in fullerene graphs*, Croat. Chem. Acta **75**(2002), 401–408.

[5] I. GUTMAN, B. MOHAR, *The quasi-Wiener and the Kirchhoff indices coincide*, J. Chem. Inf. Comput. Sci. **36**(1996), 982–985.

[6] D. J. KLEIN, *Graph geometry, graph metrics and Wiener*, MATCH Commun. Math. Comput. Chem. **35**(1997), 7–27.

[7] D. J. KLEIN, M. RANDIĆ, *Resistance distance*, J. Math. Chem. **12**(1993), 81–95.

[8] D. J. KLEIN, H. Y. ZHU, *Distances and volumina for graphs*, J. Math. Chem. **23**(1998), 179–195.

[9] I. LUKOVITS, S. NIKOLIĆ, N. TRINAJSTIĆ, *Resistance distance in regular graphs*, Int. J. Quantum Chem. **71**(1999), 217–225.

[10] J. L. PALACIOS, *Resistance distance in graphs and random walks*, Int. J. Quantum Chem. **81**(2001), 29–33.

[11] J. L. PALACIOS, *Closed-form formulas for Kirchhoff index*, Int. J. Quantum Chem. **81**(2001), 135–140.

[12] H. WIENER, *Structural determination of paraffin boiling points*, J. Amer. Chem. Soc. **69**(1947), 17–20.

[13] W. XIAO, I. GUTMAN, *On resistance matrices*, MATCH Commun. Math. Comput. Chem. **49**(2003), 67–81.

[14] W. XIAO, I. GUTMAN, *Relation between resistance and Laplacian matrices and their application*, Theor. Chem. Acta. **110**(2003), 284–289.

[15] W. XIAO, I. GUTMAN, *Relation between resistance and Laplacian matrices and their application*, MATCH Commun. Math. Comput. Chem. **51**(2004), 119-127.

[16] Y. J. YANG, X. Y. JIANG, *Unicyclic graphs with extremal Kirchhoff index*, MATCH Commun. Math Comput. Chem. **60**(2008), 107–120.

[17] W.ZHANG H. DENG, *The second maximal and minimal Kirchhoff indices of unicyclic graphs*, MATCH Commun. Math. Comput. Chem. **61**(2009), 683–695.

[18] H. P. ZHANG, Y. J. YANG, *Resistance distance and Kirchhoff index in circulant graphs*, Int. J. Quantum Chem. **107**(2007), 330–339.
[19] H. Y. ZHU, D. J. KLEIN, I. LUKOVITS, *Extensions of the Wiener number*, J. Chem. Inf. Comput. Sci. **36**(1996), 420–428.