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On the stirling expansion into negative powers of a triangular number

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Abstract. The aim of this paper is to answer an open problem posed by M. B. Villarino [arXiv:0707.3950v2]. We also introduce a new accurate approximation formula for big factorials.

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1. Introduction

Maybe one of the most known and most used formula for approximation of big factorials is the Stirling's formula

$$n! \sim \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n} = \sigma_n.$$

It was first discovered by the French mathematician Abraham de Moivre (1667-1754) (with a missing constant), then the English mathematician James Stirling (1692-1770) found the constant $\sqrt{2\pi}$.

The Stirling's formula has important applications in many branches of science, being satisfactory in probability theory, statistical physiscs, or mechanics, while in pure mathematics more accurate approximations are required. In fact, the Stirling's formula is the first approximation of the following series

$$n! \sim \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n} \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \frac{571}{2488320n^4} + ...\right).$$

For details see [1, p. 257].

The gamma function Γ is defined for $x > 0$, by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

and it is a natural extension of the factorial function, since $\Gamma(n+1) = n!$, for all $n = 1, 2, 3, ...$. The factorial and gamma function are related with the harmonic sum

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$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$$

in the sense we describe next. The digamma function ψ defined as the logarithmic derivative of the gamma function

$$\psi(x) = \frac{d}{dx}(\ln \Gamma(x)) = \frac{\Gamma'(x)}{\Gamma(x)},$$

satisfies the recurrence relation

$$\psi(x+1) = \psi(x) + \frac{1}{x};$$

thus, implying the formula $\psi(n) = H_{n-1} - \gamma$, where $\gamma = 0.577215664901532...$ is the Euler-Mascheroni constant.

In 1755, the Swiss mathematician Leonhard Euler (1707-1783) found the asymptotic expansion for H_n,

$$H_n \sim \ln n + \gamma - \sum_{k=1}^{\infty} \frac{B_k}{n^k},$$

where B_k denotes the k^{th} Bernoulli number [1, p. 804]. Ramanujan [3, p. 521] found the asymptotic expansion of H_n into powers of the reciprocal of the n^{th} triangular number $m = \frac{n(n+1)}{2}$. Precisely, as n approaches infinity,

$$H_n \sim \frac{1}{2}\ln(2m) + \gamma + \frac{1}{12m} - \frac{1}{120m^2} + \frac{1}{630m^3} - \frac{1}{1680m^4} + \frac{1}{2310m^5} - ... .$$

M. B. Villarino finishes his work [22] with the remark that it would be interesting to develop an expansion for $n!$ into powers of m, that is, a new Stirling expansion of the form

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \sum_{k=1}^{\infty} \frac{a_k}{m^k}\right). \tag{1}$$

Motivated by this fact, in this paper we try to introduce a new method for constructing such a series. Until now, we were able to give the first term of that expansion. More precisely, we propose the approximation

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12}\left(\frac{\pi^2}{6} - \sum_{k=1}^{n-1}\frac{1}{k^2}\right)} \left(1 - \frac{1}{48m}\right) = \mu_n, \tag{2}$$

which gives better results than Stirling's formula and other known results. As it follows from our study, a performant series (1) can be constructed only if an additional factor of the form

$$\exp\left(\frac{1}{12}\left(\frac{\pi^2}{6} - \sum_{k=1}^{n-1}\frac{1}{k^2}\right)\right)$$

is considered.

2. The results

In what follows, we need the following result, which is a measure of the rate of convergence.

Lemma 1. *If* $(\lambda_n)_{n\geq 1}$ *is convergent to zero and there exists the limit*

$$\lim_{n\to\infty} n^k(\lambda_n - \lambda_{n+1}) = l \in \overline{\mathbb{R}}, \tag{3}$$

with $k > 1$, *then there exists the limit:*

$$\lim_{n\to\infty} n^{k-1}\lambda_n = \frac{l}{k-1}. \tag{4}$$

Limit (4) offers the rate of convergence of the sequence $(\lambda_n)_{n\geq 1}$ and we can see that the sequence $(\lambda_n)_{n\geq 1}$ converges faster to zero, as the value k satisfying (3) is greater. This Lemma was first used by Mortici [6–20] for constructing asymptotic expansions, or to accelerate some convergences. For proof and other details, see, e.g., [7], or [8].

First, let us define the sequence $(\lambda_n)_{n\geq 1}$ by the Stirling's approximation

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \exp\lambda_n. \tag{5}$$

We have

$$\lambda_n - \lambda_{n+1} = \left(n + \frac{1}{2}\right)\ln\left(1 + \frac{1}{n}\right) - 1$$

and from routine calculations we obtain $\lim_{n\to\infty} n^2(\lambda_n - \lambda_{n+1}) = \frac{1}{12}$. By applying Lemma 1, the sequence $(\lambda_n)_{n\geq 1}$ converges to zero as n^{-1}.

Now, if we are interested in constructing an approximation of the form (1), then we expect that already the first approximation

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1 + \frac{a}{m}\right) \tag{6}$$

is much better than the Stirling's formula. In our language introduced here, we ask that the sequence $(\omega_n)_{n\geq 1}$ defined by

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1 + \frac{a}{m}\right)\exp\omega_n \tag{7}$$

should be faster convergent to zero than the sequence $(\lambda_n)_{n\geq 1}$, that is, it should converge to zero faster than n^{-1}.

From (7), we deduce that

$$\omega_n - \omega_{n+1} = \left(n + \frac{1}{2}\right)\ln\left(1 + \frac{1}{n}\right) - 1 + \ln\frac{1 + \frac{2a}{(n+1)(n+2)}}{1 + \frac{2a}{n(n+1)}}$$

or, using computer software,

$$\omega_n - \omega_{n+1} = \frac{1}{12n^2} - \left(4a + \frac{1}{12}\right)\frac{1}{n^3} + \left(12a + \frac{3}{40}\right)\frac{1}{n^4} + O\left(\frac{1}{n^5}\right). \tag{8}$$

We have $\lim_{n\to\infty} n^2(\omega_n-\omega_{n+1}) = \frac{1}{12}$. From Lemma 1, $\lim_{n\to\infty} n\omega_n = \frac{1}{12}$, independently of the value a, and consequently, approximation (6) cannot be made better than the Stirling's formula. In other words, a series of the form (1) cannot be constructed in the background we present later.

We can improve the rate of convergence by annihilating the term $\frac{1}{12n^2}$. After a careful analysis, we introduce a class of approximations of the form

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12}\left(\frac{\pi^2}{6}-\sum_{k=1}^{n-1}\frac{1}{k^2}\right)}\left(1+\frac{a}{m}\right), \tag{9}$$

with $m=\frac{n(n+1)}{2}$ and $a\in\mathbb{R}$. Let us define the sequence $(\tau_n)_{n\ge1}$ by

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12}\left(\frac{\pi^2}{6}-\sum_{k=1}^{n-1}\frac{1}{k^2}\right)}\left(1+\frac{a}{m}\right)\exp\tau_n, \tag{10}$$

associated with the approximation formula (9). Now we are in the position to state the following

Theorem 1. *Let*

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left[1+\frac{2a}{n(n+1)}\right]\exp\left[\frac{1}{12}\left(\frac{\pi^2}{6}-\sum_{i=1}^{n-1}\frac{1}{i^2}\right)+\tau_n\right]$$

for $n\in\mathbb{N}$. Then

$$\lim_{n\to\infty}\left(n^2\tau_n\right) = -\left(2a+\frac{1}{24}\right)\neq 0$$

for $a\neq-\frac{1}{48}$ and

$$\lim_{n\to\infty}\left(n^3\tau_n\right) = -\frac{7}{120}$$

for $a=-\frac{1}{48}$.

Proof. From (10), we have

$$\tau_n-\tau_{n+1} = \left(n+\frac{1}{2}\right)\ln\left(1+\frac{1}{n}\right)-1+\ln\frac{1+\frac{2a}{(n+1)(n+2)}}{1+\frac{2a}{n(n+1)}}-\frac{1}{12n^2},$$

or, again using a computer software,

$$\tau_n-\tau_{n+1} = -\left(4a+\frac{1}{12}\right)\frac{1}{n^3}+\left(12a+\frac{3}{40}\right)\frac{1}{n^4}+O\left(\frac{1}{n^5}\right).$$

Now (i) follows immediately from Lemma 1. In case $a=-\frac{1}{48}$, we obtain

$$\tau_n-\tau_{n+1} = -\frac{7}{40n^4}+O\left(\frac{1}{n^5}\right),$$

which justifies the statement (ii). $\square$

This Theorem 1 shows that the best approximation of the form (9) is (2), which is obtained for $a = -\frac{1}{48}$. The corresponding asymptotic formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left[1 + \frac{2a}{n\,(n+1)}\right] \exp\left[\frac{1}{12}\left(\frac{\pi^2}{6} - \sum_{i=1}^{n-1}\frac{1}{i^2}\right)\right], \quad n \to \infty$$

is better than some known results. Some works about the approximating $n!$, gamma and related functions investigate this problem from the viewpoint of inequalities and logarithmically complete monotonicity. See, for example [6–21].

3. Conclusions

Some of slightly more accurate approximation formula than Stirling's result, are the following:

$$n! \sim \sqrt{2\pi}\left(\frac{n+1/2}{e}\right)^{n+1/2} \quad \text{(W. Burnside [4]).} \tag{11}$$

$$n! \sim \sqrt{\frac{2\pi}{e}}\left(\frac{n+1}{e}\right)^{n+1/2} \quad \text{(C. Mortici [6]).} \tag{12}$$

Better results were recently established by N. Batir [2]

$$n! \sim \frac{\sqrt{2\pi} n^{n+1} e^{-n}}{\sqrt{n-1/6}} \tag{13}$$

and R. W. Gosper [5]

$$n! \sim \sqrt{2\pi\left(n+\frac{1}{6}\right)}\left(\frac{n}{e}\right)^n = \gamma_n. \tag{14}$$

The best approximations of (11)-(14) is the Gosper's formula (14). The following numerical computations show the great superiority of our new formula (2) over the Gosper's formula (14). For the sake of completeness, we also consider the Stirling's approximation $n! \sim \sigma_n$.

$n!$	$n! - \sigma_n$	$n! - \gamma_n$	$\mu_n - n!$
10	30104	239.18	**197.55**
25	5.1615×10^{22}	1.6883×10^{20}	$\mathbf{5.628 \times 10^{19}}$
50	5.0647×10^{61}	8.362×10^{58}	$\mathbf{1.399 \times 10^{58}}$
100	7.7739×10^{154}	6.4479×10^{151}	$\mathbf{5.4047 \times 10^{150}}$
200	3.2854×10^{371}	1.3657×10^{368}	$\mathbf{5.7298 \times 10^{366}}$
500	2.0334×10^{1130}	3.3858×10^{1126}	$\mathbf{5.6857 \times 10^{1124}}$
1000	3.3531×10^{2563}	2.7929×10^{2559}	$\mathbf{2.3455 \times 10^{2557}}$

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