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Some monotonicity properties and inequalities for Γ and ζ - functions

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Abstract. In this paper several monotonicity properties and inequalities are given for Γ and Γ_q functions as well as for their logarithmic derivatives ψ and ψ_q. A p analogue of Riemann Zeta function ζ_p is introduced. Using the generalization of Schwarz inequality and Holder's inequality some inequalities relating ζ, ζ_p, Γ and Γ_p are obtained. By the use of Laplace Convolution Theorem, some monotonicity results related to digamma function ψ and its derivatives of order n are obtained. For the Γ_p-function, defined by Euler, some properties related to monotonicity are given. Also, some properties of a p analogue of the ψ function have been established.

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1. Introduction and preliminaries

A function f is said to be completely monotonic on an interval I, if f has derivatives of all orders on I and satisfies

$$(-1)^n f^{(n)}(x) \geq 0, (x \in I, n = 0, 1, 2, \ldots). \tag{1}$$

If inequality (1) is strict, then f is said to be strictly completely monotonic on I. Theorem of Bernstein (for example, see [19]) states that f is completely monotonic if and only if $f(x) = \int_0^\infty e^{-xt} d\mu(t)$, where μ is a nonnegative measure on $[0, \infty)$ such that for all $x > 0$ the integral converges.

A positive function f is said to be logarithmically completely monotonic on an interval I, if f satisfies

$$(-1)^n [\ln f(x)]^{(n)} \geq 0, (x \in I, n = 1, 2, \ldots). \tag{2}$$

If inequality (2) is strict, then f is said to be strictly logarithmically completely monotonic.

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The Euler gamma function $\Gamma(x)$ is defined for $x > 0$ by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$. The digamma (or psi) function is defined for positive real numbers x as the logarithmic derivative of Euler's gamma function, that is $\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$. The following integral and series representations are valid (see [1]):

$$\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}}dt = -\gamma - \frac{1}{x} + \sum_{n\geq 1}\frac{x}{n(n+x)}, \tag{3}$$

where $\gamma = 0.57721\cdots$ denotes Euler's constant. Euler gave another equivalent definition for $\Gamma(x)$ (see [15]),

$$\Gamma_p(x) = \frac{p!p^x}{x(x+1)\cdots(x+p)} = \frac{p^x}{x(1+\frac{x}{1})\cdots(1+\frac{x}{p})}, \quad x > 0, \tag{4}$$

where

$$\Gamma(x) = \lim_{p\to\infty}\Gamma_p(x). \tag{5}$$

The p-analogue of the psi function is defined as the logarithmic derivative of the Γ_p function (see [12]), that is

$$\psi_p(x) = \frac{d}{dx}\ln\Gamma_p(x) = \frac{\Gamma_p'(x)}{\Gamma_p(x)}. \tag{6}$$

The function ψ_p defined in (6) satisfies the following properties (see [12]). It has the following series representation

$$\psi_p(x) = \ln p - \sum_{k=0}^{p}\frac{1}{x+k}. \tag{7}$$

It is increasing on $(0,\infty)$ and it is strictly completely monotonic on $(0,\infty)$. Its derivatives are given by

$$\psi_p^{(n)}(x) = \sum_{k=0}^{p}\frac{(-1)^{n-1}\cdot n!}{(x+k)^{n+1}}. \tag{8}$$

Jackson (see [9, 10, 11, 16]) defined the q-analogue of the gamma function as

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x},\ 0 < q < 1, \tag{9}$$

and

$$\Gamma_q(x) = \frac{(q^{-1};q^{-1})_\infty}{(q^{-x};q^{-1})_\infty}(q-1)^{1-x}q^{\binom{x}{2}},\ q > 1, \tag{10}$$

where $(a;q)_\infty = \prod_{j\geq 0}(1-aq^j)$.

A standard reference for the q-Gamma function is [4].

The q-analogue of the psi function is defined for $0 < q < 1$ as the logarithmic derivative of the q-gamma function, that is, $\psi_q(x) = \frac{d}{dx}\log\Gamma_q(x)$. Many properties of the q-gamma function were derived by Askey [5]. It is well known that $\Gamma_q(x) \to \Gamma(x)$ and $\psi_q(x) \to \psi(x)$ as $q \to 1^-$. From (9), for $0 < q < 1$ and $x > 0$ we get

$$\psi_q(x) = -\log(1-q) + \log q \sum_{n\geq 0} \frac{q^{n+x}}{1-q^{n+x}} = -\log(1-q) + \log q \sum_{n\geq 1} \frac{q^{nx}}{1-q^n} \tag{11}$$

and from (10) for $q > 1$ and $x > 0$ we obtain

$$\begin{aligned} \psi_q(x) &= -\log(q-1) + \log q \left(x - \tfrac{1}{2} - \textstyle\sum_{n\geq 0} \frac{q^{-n-x}}{1-q^{-n-x}} \right) \\ &= -\log(q-1) + \log q \left(x - \tfrac{1}{2} - \textstyle\sum_{n\geq 1} \frac{q^{-nx}}{1-q^{-n}} \right). \end{aligned} \tag{12}$$

A Stieltjes integral representation for $\psi_q(x)$ with $0 < q < 1$ is given in [7]. It is well-known that ψ' is strictly completely monotonic on $(0, \infty)$, that is,

$$(-1)^n (\psi'(x))^{(n)} > 0 \quad \text{for } x > 0 \text{ and } n \geq 0,$$

see [1, Page 260]. From (11) and (12) we conclude that ψ_q' has the same property for any $q > 0$,

$$(-1)^n (\psi_q'(x))^{(n)} > 0 \quad \text{for } x > 0 \text{ and } n \geq 0.$$

If $q \in (0,1)$, using the second representation of $\psi_q(x)$ given in (11) it can be shown that

$$\psi_q^{(k)}(x) = \log^{k+1} q \sum_{n\geq 1} \frac{n^k \cdot q^{nx}}{1-q^n} \tag{13}$$

and hence $(-1)^{k-1}\psi_q^{(k)}(x) > 0$ with $x > 1$, for all $k \geq 1$. If $q > 1$, from the second representation of $\psi_q(x)$ given in (12) we obtain

$$\psi_q'(x) = \log q \Big(1 + \sum_{n\geq 1} \frac{nq^{-nx}}{1-q^{-nx}} \Big) \tag{14}$$

and for $k \geq 2$,

$$\psi_q^{(k)}(x) = (-1)^{k-1} \log^{k+1} q \sum_{n\geq 1} \frac{n^k q^{-nx}}{1-q^{-nx}} \tag{15}$$

and hence $(-1)^{k-1}\psi_q^{(k)}(x) > 0$ with $x > 0$, for all $q > 1$.

In the next sections we derive several properties related to monotonicity and some inequalities for the functions $\Gamma_p, \Gamma_q, \Gamma, \zeta$ as well as for polygamma functions $\psi^{(m)}(x)$.

2. Inequalities for Γ, ψ, Γ_q and ψ_q functions

2.1. Inequalities for Γ and Γ_q functions

Lemma 1. *Let a, x be positive real numbers such that $x + a > 1$, and $q \in (0,1)$. Then $\gamma - \log(1-q) + \psi(x+a) - \psi_q(x+a) > 0$.*

Proof. Using series representation of function ψ and ψ_q we obtain

$$\gamma - \log(1-q) + \psi(x) - \psi_q(x) = (x-1)\sum_{k=0}^{\infty}\frac{1}{(k+1)(x+k)} - \log q \sum_{k=0}^{\infty}\frac{q^{x+k}}{1-q^{x+k}} > 0,$$

which implies that $\gamma - \log(1-q) + \psi(x) - \psi_q(x) > 0$ for all $x > 1$ and $q \in (0,1)$. The result follows by replacing x by $x + a$. □

Theorem 1. *Let*

$$f(x) = \frac{e^{\gamma x}}{(1-q)^x}\cdot\frac{\Gamma(x+a)}{\Gamma_q(x+a)}$$

with $x \in (0,1)$, *where* a *is a real number such that* $a+x > 1$ *and* $q \in (0,1)$. *Then the function* $f(x)$ *is increasing for* $x \in (0,1)$ *and the following double inequality holds*

$$\frac{(1-q)^x\Gamma(a)}{e^{\gamma x}\Gamma_q(a)} < \frac{\Gamma(x+a)}{\Gamma_q(x+a)} < \frac{e^{\gamma(1-x)}(1-q)^{x-1}\Gamma(1+a)}{\Gamma_q(1+a)}.$$

Proof. Let $g(x) = \log f(x)$ for all $x \in (0,1)$. Then

$$g'(x) = \gamma - \log(1-q) + \psi(x+a) - \psi_q(x+a).$$

By Lemma 1 we have $g'(x) > 0$. So $g(x)$ is increasing on $(0,1)$, which implies that $f(x)$ is increasing on $(0,1)$ so we have $f(0) < f(x) < f(1)$ and the result follows. □

For example, Theorem 1 for $a = 1$ together with $\Gamma_q(2) = 1$ give

$$\frac{(1-q)^x}{e^{\gamma x}} < \frac{\Gamma(x+1)}{\Gamma_q(x+1)} < \frac{2(1-q)^{1-x}}{e^{\gamma(x-1)}}.$$

2.2. Completely monotonic

Theorem 2. *The function* $G_q(x; a_1, b_1, \dots, a_n, b_n)$ *given by*

$$G_q(x) = G_q(x; a_1, b_1, \dots, a_n, b_n) = \prod_{i=1}^{n}\frac{\Gamma_q(x+a_i)}{\Gamma_q(x+b_i)}, q \in (0,1) \tag{16}$$

is a completely monotonic function on $(0,\infty)$, *for any* a_i *and* b_i, $i = 1, 2, \dots, n$, *real numbers such that* $0 < a_1 \le \cdots \le a_n$, $0 < b_1 \le b_2 \le \cdots \le b_n$ *and* $\sum_{i=1}^{k} a_i \le \sum_{i=1}^{k} b_i$ *for* $k = 1, 2, \dots, n$.

Proof. Let $h(x) = \sum_{i=1}^{n}(\log\Gamma_q(x+b_i) - \log\Gamma_q(x+a_i))$. Then for $k \geq 0$ we have

$$\begin{aligned}(-1)^k(h_q'(x))^{(k)} &= (-1)^k\sum_{i=1}^{n}(\psi_q^{(k)}(x+b_i) - \psi_q^{(k)}(x+a_i))\\ &= (-1)^k\sum_{i=1}^{n}\log^{k+1} q\sum_{n\geq 1}\frac{n^k q^{nx}}{1-q^n}(q^{bi} - q^{ai})\\ &= (-1)^k\log^{k+1} q\sum_{n\geq 1}\frac{n^k q^{nx}}{1-q^n}\sum_{i=1}^{n}(q^{bi} - q^{ai}).\end{aligned}$$

Alzer [2] showed that if f is a decreasing and convex function on $\mathbb{R}$, then there holds

$$\sum_{i=1}^{n} f(b_i) \leq \sum_{i=1}^{n} f(a_i). \tag{17}$$

Thus, since the function $z \mapsto q^z, z > 0$ is decreasing and convex on $\mathbb{R}$, we have that $\sum_{i=1}^{n}(q^{ai} - q^{bi}) \geq 0$, so $(-1)^k(G_q'(x))^{(k)} \geq 0$ for $k \geq 0$. Hence h_q' is completely monotonic on $(0,\infty)$. Using the fact that if f' is a completely monotonic function on $(0,\infty)$, then $\exp(-h)$ is also a completely monotonic function on $(0,\infty)$ (see [6]), we get the desired result. □

In a similar way one can show that the Theorem 2 also remains true for $q > 1$.

For the proof of the following lemma, see [6].

Lemma 2. *If h' is completely monotonic on $(0,\infty)$, then $\exp(-h)$ is also completely monotonic on $(0,\infty)$.*

In order to present our next theorem we need the following lemma.

Lemma 3. *For $q > 1$, the function $e^{\psi_q(x)} - x$ is convex on $(0,+\infty)$.*

Proof. Alzer and Grinshpan [3] showed that $f''(x) = (\psi_q'(x))^2 + \psi_q''(x) > 0$ for all $q > 1$ and $x > 0$. Hence $f(x) = e^{\psi_q(x)} - x$ is a convex function on $(0,+\infty)$ for $q > 1$. □

Theorem 3. *The function*

$$\theta(x) = \psi_q(x) + \log\left(e^{\frac{q^x\log q}{q^x-1}} - 1\right)$$

is strictly increasing on $(0,\infty)$ for $q > 1$ and $x > 0$.

Proof. It is well known that for $x > 0$ and $q > 1$

$$\Gamma_q(x+1) = \frac{(1-q^x)\cdot q^{\binom{x-1}{2}}}{1-q}\Gamma_q(x).$$

Taking the logarithm on both sides and differentiating yields

$$\psi_q(x+1) = \frac{q^x\log q}{q^x-1} + \psi_q(x).$$

Therefore, the exponential function of θ satisfies

$$\begin{aligned} e^{\theta(x)} &= e^{\psi_q(x)} \cdot e^{\log\left(e^{\frac{q^x \log q}{q^x-1}}-1\right)} = e^{\psi_q(x)} \cdot \left(e^{\frac{q^x \log q}{q^x-1}} - 1\right) \\ &= e^{\psi_q(x)+\frac{q^x \log q}{q^x-1}} - e^{\psi_q(x)} = e^{\psi_q(x+1)} - e^{\psi_q(x)}. \end{aligned}$$

Let $s(x) = e^{\psi_q(x+1)} - e^{\psi_q(x)}$. Then

$$s'(x) = e^{\psi_q(x+1)}\psi_q'(x+1) - e^{\psi_q(x)}\psi_q'(x) = h(x+1) - h(x),$$

where $h(x) = e^{\psi_q(x)}\psi_q'(x)$. Then $h'(x) = e^{\psi_q(x)}((\psi_q'(x))^2 + \psi_q''(x))$, so by Lemma 3 we conclude that $h'(x) > 0$ so the function h is strictly increasing. It means $s'(x) > 0$ for $x \in (0, \infty)$ and this yields that s and θ are strictly increasing functions on $(0, \infty)$. □

In the following, we denote $\psi_n(x) = \psi^{(n)}(x)$ for $n \geq 1$.

Theorem 4. *Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $x, y > 0$.*

a) If $n = 1, 3, 5, \ldots$ the following inequality holds

$$(\psi_n(x))^{\frac{1}{p}} \cdot (\psi_n(x))^{\frac{1}{q}} \geq \psi_n\Big(\frac{x}{p} + \frac{y}{q}\Big).$$

b) If $n = 2, 4, 6 \ldots$ the following inequality holds

$$(\psi_n(x))^{\frac{1}{p}} \cdot (\psi_n(x))^{\frac{1}{q}} \leq \psi_n\Big(\frac{x}{p} + \frac{y}{q}\Big).$$

Proof. a) The polygamma function has the following integral representation (see [1])

$$\psi_n(x) = (-1)^{n+1} \int_0^\infty \frac{t^n}{1 - e^{-t}} e^{-xt} dt, x > 0.$$

Thus, $\psi_n(x) = \int_0^\infty \frac{t^n}{1-e^{-t}} e^{-xt} dt$, for all $x > 0$ and n any odd positive integer number. By Holder's inequality

$$\Big| \int_0^\infty f(t)g(t)dt \Big| \leq \Big(\int_0^\infty |f(t)|^p \Big)^{\frac{1}{p}} dt \cdot \Big(\int_0^\infty |g(t)|^q \Big)^{\frac{1}{q}} dt$$

applied to functions $f(t) = \frac{t^{\frac{n}{p}} \cdot e^{-\frac{xt}{p}}}{(1-e^{-t})^{\frac{1}{p}}}$ and $g(t) = \frac{t^{\frac{n}{q}} \cdot e^{-\frac{xt}{q}}}{(1-e^{-t})^{\frac{1}{q}}}$ we obtain

$$\begin{aligned} \psi_n\Big(\frac{x}{p} + \frac{y}{q}\Big) &= \int_0^\infty \frac{t^n}{1 - e^{-t}} e^{\left(\frac{x}{p}+\frac{y}{q}\right)t} dt \\ &= \int_0^\infty \Big(\frac{t^{\frac{n}{p}}}{(1 - e^{-t})^{\frac{1}{p}}} \cdot e^{-\frac{xt}{p}} \Big) \cdot \Big(\frac{t^{\frac{n}{q}}}{(1 - e^{-t})^{\frac{1}{q}}} \cdot e^{-\frac{yt}{q}} \Big) dt \\ &\leq \Big(\int_0^\infty \frac{t^n}{1 - e^{-t}} \cdot e^{-xt} dt \Big)^{\frac{1}{p}} \cdot \Big(\int_0^\infty \frac{t^n}{1 - e^{-t}} \cdot e^{-yt} dt \Big)^{\frac{1}{q}} \\ &= (\psi_n(x))^{\frac{1}{p}} \cdot (\psi_n(y))^{\frac{1}{q}}, \end{aligned}$$

as claimed.

b) Similarly to case a). □

For example, Theorem 4 for $q = 2, p = 2$ gives

$$\psi_n\Big(\frac{x+y}{2}\Big) \leq \sqrt{\psi_n(x)\cdot\psi_n(y)}.$$

In [17] it has been shown that the function

$$f(x) = \frac{\Big(\frac{x}{e}\Big)^x}{\Gamma\Big(x+\frac{1}{2}\Big)}$$

is logarithmically completely monotonic on $(0, \infty)$. We extend this result as follows.

Theorem 5. *The function $f(x) = \frac{(x+1)^x}{e^x\Gamma(x+1)}$ is logarithmically completely monotonic on $(-1, \infty)$.*

Proof. Using the integral representation

$$\ln\Gamma(x+1) = x\ln(x+1) - x + \int_0^\infty \frac{e^{-t}}{t}\Big(\frac{1}{t} - \frac{1}{e^t-1}\Big)(1-e^{-xt})dt$$

we obtain $\ln f(x) = \int_0^\infty \frac{e^{-t}}{t}\Big(\frac{1}{t} - \frac{1}{e^t-1}\Big)(e^{-xt}-1)dt$. The function $h(y) = e^{-y} - 1$ is completely monotonic on $\mathbb{R}$. Since $\dfrac{1}{t} - \dfrac{1}{e^t-1} > 0$ for all $t > 0$, we conclude that f is logarithmically completely monotonic on $(-1, \infty)$. □

We mention that some results related to completely monotonicity have been given in [8]

2.3. Riemann zeta function and gamma function

In this section we will introduce the function ζ_p and we will prove some relations.

Definition 1. *We define the function ζ_p as*

$$\zeta_p(s) = \frac{1}{\Gamma_p(s)}\int_0^p \frac{t^{s-1}}{\Big(1+\frac{t}{p}\Big)^p - 1}dt. \tag{18}$$

Note that when $p \longrightarrow \infty$ we obtain a ζ function.

For the proof of the following Lemma see for example equation (1.4) in [13].

Lemma 4 (A generalization of Schwarz inequality)**.** *Let f, g be two nonnegative functions of a real variable and m, n real numbers such that integrals in* (19) *exist. Then*

$$\int_a^b g(t)(f(t))^m dt \cdot \int_a^b g(t)(f(t))^n dt \geq \Big(\int_a^b g(t)(f(t))^{\frac{m+n}{2}}\Big)^2 dt. \tag{19}$$

Theorem 6. *The following inequality is valid*

$$\frac{s+p+1}{s+p+2}\cdot\frac{\zeta_p(s)}{\zeta_p(s+1)} \geq \frac{s}{s+1}\cdot\frac{\zeta_p(s+1)}{\zeta_p(s+2)}. \tag{20}$$

Proof. Applying Lemma 4 with $g(t)=\dfrac{1}{\left(1+\frac{p}{t}\right)^p-1}$, $f(t)=t$.

$$\int_0^p \frac{t^{s-1}}{\left(1+\frac{p}{t}\right)^p-1}dt\cdot\int_0^p \frac{t^{s+1}}{\left(1+\frac{p}{t}\right)^p-1}dt \geq \Big(\int_0^p \frac{t^{s}}{\left(1+\frac{p}{t}\right)^p-1}\Big)^2 dt.$$

Further, using (18) we have

$$\zeta_p(s)\Gamma_p(s)\zeta_p(s+2)\Gamma_p(s+2) \geq (\zeta_p(s+1))^2(\Gamma_p(s+1))^2.$$

By using $\Gamma_p(s+1)=\dfrac{ps}{s+p+1}\Gamma_p(s)$ the result follows. □

Now when p tends to infinty, then we receive the results of [5].

Theorem 7. *We denote by $\zeta(u)$ the Riemann zeta function. Then*

$$\frac{\Gamma\Big(\frac{u}{p}+\frac{v}{q}\Big)}{\Gamma^{\frac{1}{p}}(u)\cdot\Gamma^{\frac{1}{q}}(v)} \leq \frac{\zeta^{\frac{1}{p}}(u)\cdot\zeta^{\frac{1}{q}}(u)}{\zeta\Big(\frac{u}{p}+\frac{v}{q}\Big)},$$

where $\dfrac{1}{p}+\dfrac{1}{q}=1$ *and* $\dfrac{u}{p}+\dfrac{v}{q}>1$.

Proof. For $u>1$ the Riemann zeta function satisfies the integral relation

$$\zeta(u)=\frac{1}{\Gamma(u)}\int_0^\infty \frac{t^{u-1}}{e^t-1}dt. \tag{21}$$

Using Holder's inequality for $p>1$ we have

$$\Big|\int_0^\infty f(t)\cdot g(t)dt\Big| \leq \Big(\int_0^\infty |f(t)|^p dt\Big)^{\frac{1}{p}}\Big(\int_0^\infty |g(t)|^q dt\Big)^{\frac{1}{q}}. \tag{22}$$

Using equations (21), (22) with $f(t)=\dfrac{t^{\frac{u-1}{p}}}{(e^t-1)^{\frac{1}{p}}}$ and $g(t)=\dfrac{t^{\frac{u-1}{q}}}{(e^t-1)^{\frac{1}{q}}}$ we obtain

$$\Gamma\Big(\frac{u}{p}+\frac{v}{q}\Big)\cdot\zeta\Big(\frac{u}{p}+\frac{v}{q}\Big) \leq \Gamma^{\frac{1}{p}}(u)\cdot\Gamma^{\frac{1}{q}}(v)\cdot\zeta^{\frac{1}{p}}(u)\cdot\zeta^{\frac{1}{q}}(v),$$

which completes the proof. □

For example, Theorem 7 for $p=q=2$ gives

$$\frac{\Gamma\Big(\frac{u+v}{2}\Big)}{\sqrt{\Gamma(u)\cdot\Gamma(v)}} \leq \frac{\sqrt{\zeta(u)\cdot\zeta(v)}}{\zeta\Big(\frac{u+v}{2}\Big)}.$$

2.4. Laplace transform and ψ functions

In this section, by using the convolution theorem for the Laplace transform (see [19]) we will show some monotonicity results related to ψ function. First we need the following Definition and Theorem from [18].

Definition 2. *A function g is called strongly completely monotonic on $(0,\infty)$ if*

$$x \mapsto (-1)^n x^{n+1} g^{(n)}(x)$$

is nonnegative and decreasing on $(0,\infty)$ for $n = 0, 1, 2, \ldots$.

Theorem 8. *The function g is strongly completely monotonic if and only if*

$$g(x) = \int_0^\infty p(t) e^{-xt} dt,$$

where $p(t)$ is nonnegative and increasing and the integral converges for all $x > 0$.

Now we give our results.

Theorem 9. *The function $\theta_n(x) = |\psi^{(n+1)}(x)| - \frac{n}{x}|\psi^{(n)}(x)|$ is strongly completely monotonic on $(0,\infty)$.*

Proof. Using integral representation for $\psi^{(n)}(x)$ one has

$$\theta_n(x) = \int_0^\infty \frac{e^{-xt}t^{n+1}}{1-e^{-t}} dt - n \int_0^\infty e^{-xt} dt \int_0^\infty \frac{t^n e^{-xt}}{1-e^{-t}} dt.$$

By the convolution Theorem for the Laplace transforms we have

$$\theta_n(x) = \int_0^\infty \frac{e^{-xt}t^{n+1}}{1-e^{-t}} dt - n \int_0^\infty e^{-xt} \Big[\int_0^t \frac{s^n}{1-e^{-s}} ds \Big] dt.$$

Hence $\theta_n(x) = \int_0^\infty e^{-xt} q(t) dt$, where $q(t) = \frac{t^{n+1}}{1-e^{-t}} - n \int_0^t \frac{s^n}{1-e^{-s}} ds$. Then $q'(t) = \frac{t^n e^{-t}(e^t - 1 - t)}{(1-e^{-t})^2} > 0$, so $q(t) > \lim\limits_{t\to 0} q(t) = 0$. □

Theorem 10. *Let $f(x) = \psi'(x+1) + x\psi''(x+1)$ or equivalently $f(x) = \sum_{n=1}^\infty \frac{n-x}{n+x}$. The function $\frac{f(x)}{x}$ is completely monotonic on $(-1,\infty)$.*

Proof. Clearly $\frac{f(x)}{x} = \frac{1}{x}\psi'(x+1) + \psi''(x+1)$. Using integral representation of ψ' and ψ'' one obtains

$$\frac{f(x)}{x} = \int_0^\infty e^{-xt} dt \int_0^\infty \frac{te^{-t}e^{-xt}}{1-e^{-t}} dt + \int_0^\infty \frac{e^{-xt}e^{-t}t^2}{1-e^{-t}} dt.$$

By the convolution Theorem for the Laplace Transforms we have

$$\frac{f(x)}{x} = \int_0^\infty e^{-xt} \Big[\int_0^t \frac{se^{-s}}{1-e^{-s}} ds \Big] dt + \int_0^\infty \frac{e^{-xt}e^{-t}t^2}{1-e^{-t}} dt = \int_0^\infty e^{-xt} q(t) dt,$$

where $q(t) = \int_0^t \frac{se^{-s}}{1-e^{-s}} ds + \frac{e^{-t}t^2}{1-e^{-t}} > 0$. □

3. Inequalities for the Γ_p and ψ_p

In this section we treat several inequalities for Γ_p and ψ_p.

Theorem 11. *Let n be a positive integer.*

(1) If n is even, then $\psi_p^{(n)}(x+y) \geq \psi_p^{(n)}(x) + \psi_p^{(n)}(y)$.

(2) If n is odd, then $\psi_p^{(n)}(x+y) \leq \psi_p^{(n)}(x) + \psi_p^{(n)}(y)$.

Proof. From [14] we have

$$\begin{aligned}&\psi_p^{(n)}(x+y) - \psi_p^{(n)}(x) - \psi_p^{(n)}(y)\\ &= (-1)^n \sum_{k=0}^{p} \Big(\frac{1}{(x+y+k)^{n+1}} - \frac{1}{(x+k)^{n+1}} - \frac{1}{(y+k)^{n+1}} \Big).\end{aligned}$$

Since the function $f(x) = \frac{1}{(x+k)^{n+1}}$ is convex from $f\Big(\frac{x+y}{2}\Big) \leq \frac{1}{2}\Big(f(x) + f(y)\Big)$, we obtain that

$$\frac{2 \cdot 2^{n+1}}{(x+y+k)^{n+1}} \leq \frac{1}{(x+k)^{n+1}} + \frac{1}{(y+k)^{n+1}}. \tag{23}$$

On the other hand, it is clear that

$$\frac{2 \cdot 2^{n+1}}{(x+y+k)^{n+1}} > \frac{1}{(x+y+k)^{n+1}}. \tag{24}$$

From (23) and (24) we have that $\frac{1}{(x+y+k)^{n+1}} - \frac{1}{(x+k)^{n+1}} - \frac{1}{(y+k)^{n+1}} < 0$, which implies the result. □

Lemma 5 (Integral representation for Γ_p, ψ_p and $\psi_p^{(m)}$)**.** *The following representations are valid:*

$$\Gamma_p(x) = \frac{p^x}{x(1+\frac{x}{1}) \cdot \ldots \cdot (1+\frac{x}{p})} = \int_0^p \Big(1 - \frac{t}{p}\Big)^p t^{x-1} dt \tag{25}$$

$$\psi_p(x) = \ln p - \int_0^\infty \frac{e^{-xt}(1-e^{-pt})}{1-e^{-t}} dt \tag{26}$$

and

$$\psi_p^{(m)}(x) = (-1)^{m+1} \cdot \int_0^\infty \frac{t^m \cdot e^{-xt}}{1-e^{-t}} (1-e^{-pt}) dt. \tag{27}$$

Proof. For the proof of relation (25) see [15]. Next we prove (26). From (7) one has

$$\begin{aligned}\psi_p(x) &= \ln p - \frac{1}{x} - \frac{1}{x+1} - \ldots - \frac{1}{x+p}\\ &= \ln p - \int_0^\infty e^{-xt} dt - \int_0^\infty e^{-(x+1)t} dt - \ldots - \int_0^\infty e^{-(x+p)t} dt\\ &= \ln p - \int_0^\infty \frac{e^{-xt}(1-e^{-pt})}{1-e^{-t}} dt.\end{aligned}$$

By deriving m times expression (26) one obtains (27). □

Theorem 12. *For positive integers m, n and $x > 0$ we have*

$$\psi_p^{(m)}(x) \cdot \psi_p^{(n)}(x) \geq \left(\psi_p^{(\frac{m+n}{2})}(x)\right)^2,$$

where $\frac{m+n}{2}$ is an integer.

Proof. We choose integers m, n both even or odd, so $\frac{m+n}{2}$ is an integer. By (19) with $g(t) = \frac{e^{-xt}}{1-e^{-t}} \cdot (1 - e^{-pt})$, $f(t) = t$ and $a = 0, b = \infty$ we obtain

$$\int_0^\infty g(t)t^m dt \int_0^\infty g(t)t^n dt \geq \Big(\int_0^\infty g(t) \cdot t^{\frac{m+n}{2}}\Big)^2 dt,$$

that is,

$$\psi_p^{(m)}(x) \cdot \psi_p^{(n)}(x) \geq \left(\psi_p^{(\frac{m+n}{2})}(x)\right)^2,$$

which completes the proof. □

Note that when $m = n + 2$ we have

$$\frac{\psi_p^{(n)}(x)}{\psi_p^{(n+1)}(x)} \geq \frac{\psi_p^{(n+1)}(x)}{\psi_p^{(n+2)}(x)},$$

$n = 1, 2, \ldots$ and $x > 0$. Also when $p \longrightarrow \infty$ we obtain all results of [13].

Theorem 13. *Let a_i and b_i ($i = 1, 2, \ldots, n$) be real numbers such that $0 < a_1 \leq \cdots \leq a_n$, $0 < b_1 \leq \cdots \leq b_n$ and $\sum_{i=1}^k a_i \leq \sum_{i=1}^k b_i$ for $k = 1, 2, \ldots, n$. Then the function*

$$x \mapsto \prod_{i=1}^n \frac{\Gamma_p(x + a_i)}{\Gamma_p(x + b_i)}$$

is completely monotonic on $(0, \infty)$.

Proof. Let $h(x) = \sum_{i=1}^n (\log \Gamma_p(x + b_i) - \log \Gamma_p(x + a_i))$. Then for $k \geq 0$ we have

$$\begin{aligned}
(-1)^k (h'(x))^{(k)} &= \sum_{i=1}^n (\psi_p^{(k)}(x + b_i) - \psi_p^{(k)}(x + a_i)) \\
&= (-1)^k \sum_{i=1}^n (-1)^{k+1} \sum_{n=0}^p \frac{k!}{(x + b_i + n)^{k+1}} \\
&\quad -(-1)^{k+1} \sum_{n=0}^p \frac{k!}{(x + a_i + n)^{k+1}} \\
&= (-1)^{2k+1} k! \sum_{i=1}^n \sum_{n=0}^p \Big(\frac{1}{(x + b_i + n)^{k+1}} - \frac{1}{(x + a_i + n)^{k+1}}\Big).
\end{aligned}$$

Since the function $x \mapsto \frac{1}{(x+n)^k}$ is decreasing and convex on $\mathbb{R}$, from (17) we conclude that

$$\sum_{i=1}^n \left(\frac{1}{(x + b_i + n)^{k+1}} - \frac{1}{(x + a_i + n)^{k+1}}\right) \leq 0$$

and that implies that $(-1)^k(h'(x))^{(k)} \geq 0$ for $k \geq 0$. Hence h' is completely monotonic on $(0,\infty)$. By Lemma 2 we have that

$$\exp(-h(x)) = \prod_{i=1}^{n} \frac{\Gamma_p(x+a_i)}{\Gamma_p(x+b_i)}$$

is also completely monotonic on $(0,\infty)$. □

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