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Dotson′s convexity, Banach operator pair and best simultaneous approximations

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Abstract. The existence of common fixed points is established for three mappings where T is either generalized (f, g)-nonexpansive or asymptotically (f, g)-nonexpansive on a set of fixed points which is not necessarily starshaped. As applications, the invariant best simultaneous approximation results are proved.
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1. Introduction and Preliminaries

We first review needed definitions. Let M be a subset of a normed space $(X, \|.\|)$. The set $P_M(u) = \{x \in M : \|x - u\| = dist(u, M)\}$ is called the set of best approximants to $u \in X$ out of M, where $dist(u, M) = inf\{\|y - u\| : y \in M\}$. Suppose that A and G are bounded subsets of X. Then we write

$$r_G(A) = inf_{g\in G} sup_{a\in A} \parallel a - g \parallel$$
$$cent_G(A) = \{g_0 \in G : sup_{a\in A} \parallel a - g_0 \parallel = r_G(A)\}.$$

The number $r_G(A)$ is called the Chebyshev radius of A w.r.t G and an element $y_0 \in cent_G(A)$ is called a best simultaneous approximation of A w.r.t. G. If $A = \{u\}$, then $r_G(A) = dist(u, G)$ and $cent_G(A)$ is the set of all best approximations, $P_G(u)$, of u from G. We also refer the reader to Milman [24] and Vijayaraju [26] for further details. We denote by $\mathbb{N}$ and $cl(M)$ ($wcl(M)$) the set of positive integers and the closure (weak closure) of a set M in X, respectively. Let $f,\ g\ \ T : M \to M$ be mappings. Then T is called an (f, g)-Fisher contraction [11] if there exists $0 \le k < 1$ such that $\|Tx - Ty\| \le k\|fx - gy\|$ for any $x, y \in M$. If $k = 1$, then T is called (f, g)-nonexpansive. The map T is called asymptotically (f, g)-nonexpansive if there exists a sequence $\{k_n\}$ of real numbers with $k_n \ge 1$ and $\lim_n k_n = 1$ such that $\|T^n x - T^n y\| \le k_n\|fx - gy\|$ for all $x, y \in M$ and $n = 1, 2, 3, ...$; if $g = f$, then

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T is called asymptotically f-nonexpansive [26]. The map T is called uniformly asymptotically regular [4, 26] on M, if for each $\eta > 0$, there exists $N(\eta) = N$ such that $\|T^n x - T^{n+1}x\| < \eta$ for all $n \geq N$ and all $x \in M$. The set of fixed points of T is denoted by $F(T)$. A point $x \in M$ is a coincidence point (common fixed point) of f and T if $fx = Tx$ ($x = fx = Tx$). The pair $\{f, T\}$ is called
(1) commuting, if $Tfx = fTx$ for all $x \in M$,
(2) compatible, if $\lim_n \|Tfx_n - fTx_n\| = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_n Tx_n = \lim_n fx_n = t$ for some t in M,
(3) weakly compatible, if they commute at their coincidence points, i.e., if $fTx = Tfx$ whenever $fx = Tx$,
(4) Banach operator pair, if the set $F(f)$ is T-invariant, namely $T(F(f)) \subseteq F(f)$.
Obviously, commuting pair (T, f) is a Banach operator pair but converse is not true in general, see [8, 12]. If (T, f) is a Banach operator pair, then (f, T) need not be a Banach operator pair (cf. Example 1 [8, 10, 25]).
The set M is called q-starshaped with $q \in M$, if the segment $[q, x] = \{(1-k)q + kx : 0 \leq k \leq 1\}$ joining q to x is contained in M for all $x \in M$. The map f defined on a q-starshaped set M is called affine if

$$f((1-k)q + kx) = (1-k)fq + kfx, \quad \textit{for all} \quad x \in M.$$

Suppose that M is q-starshaped with $q \in F(f)$ and is both T- and f-invariant. Then T and f are called
(5) pointwise R-subweakly commuting, if for given $x \in M$, there exists a real number $R > 0$ such that $\|fTx - Tfx\| \leq Rdist(fx, [q, Tx])$
(6) R-subweakly commuting on M, if for all $x \in M$, there exists a real number $R > 0$ such that $\|fTx - Tfx\| \leq Rdist(fx, [q, Tx])$
(7) uniformly R-subweakly commuting on $M \setminus \{q\}$ (see [4]), if there exists a real number $R > 0$ such that $\|fT^n x - T^n fx\| \leq Rdist(fx, [q, T^n x])$, for all $x \in M \setminus \{q\}$ and $n \in \mathbb{N}$.

The following important extension of the concept of starshapedness was defined by Dotson [7] and has been studied by many authors.

Definition 1 (Dotson′s convexity). *Let M be a subset of a normed space X and $\mathbb{F} = \{h_x\}_{x \in M}$ a family of functions from $[0, 1]$ into M such that $h_x(1) = x$ for each $x \in M$. The family $\mathbb{F}$ is said to be contractive [3, 6, 7, 13, 22, 23] if there exists a function $\varphi : (0, 1) \to (0, 1)$ such that for all $x, y \in M$ and all $t \in (0, 1)$, we have $\|h_x(t) - h_y(t)\| \leq \varphi(t)\|x - y\|$. The family $\mathbb{F}$ is said to be jointly (weakly) continuous if $t \to t_0$ in [0,1] and $x \to x_o$ ($x \rightharpoonup x_0$) in M, then $h_x(t) \to h_{x_0}(t_0)$ ($h_x(t) \rightharpoonup h_{x_0}(t_o)$) in M (here $\rightharpoonup$ denotes weak convergence). We observe that if $M \subset X$ is q-starshaped and $h_x(t) = (1-t)q + tx$, ($x \in M; t \in [0, 1]$), then $\mathbb{F} = \{h_x\}_{x \in M}$ is a contractive jointly continuous and jointly weakly continuous family with $\varphi(t) = t$. Thus the class of subsets of X with the property of contractiveness and joint continuity contains the class of starshaped sets which in turn contains the class of convex sets (see [7, 14, 22, 23]).*

Definition 2 (see [6]). *The family $\mathbb{F} = \{h_x\}_{x \in M}$ is said to be φ-contractive if for all $x, y \in M$ and all $t \in (0, 1)$, there exists a comparison function φ_t such that*

$$\|h_x(t) - h_y(t)\| \leq \varphi_t(\|x - y\|).$$

Definition 3 (see [14, 22]). *Let T be a selfmap of the set M having a family of functions $\mathbb{F} = \{h_x\}_{x\in M}$ as defined above. Then T is said to satisfy the property (A), if $T(h_x(t)) = h_{Tx}(t)$ for all $x \in M$ and $t \in [0,1]$.*

Example 1. *An affine map T defined on a q-starshaped set with $Tq = q$ satisfies the property (A). For this note that each q-starshaped set M has a contractive jointly continuous family of functions $\mathbb{F} = \{h_x\}_{x\in M}$ defined by $h_x(t) = tx+(1-t)q$, for each $x \in M$ and $t \in [0,1]$. Thus $h_x(1) = x$ for all $x \in M$. Also, if the selfmap T of M is affine and $Tq = q$, we have $T(h_x(t)) = T(tx + (1-t)q) = tTx + (1-t)q = h_{Tx}(t)$ for all $x \in M$ and all $t \in [0,1]$. Thus T satisfies the property (A).*

Recently, Chen and Li [8] introduced the class of Banach operator pairs, as a new class of noncommuting maps and it has been further studied by Ćirić et al. [10], Hussain [12], Khan and Akbar [20, 21] and Pathak and Hussain [25]. In this paper, we improve and extend the recent common fixed point and invariant approximation results of Beg et al. [4], Berinde [6], Chen and Li [8], Ćirić et al. [10], Hussain [12], Khan and Akbar [20, 21], Pathak and Hussain [25], and Vijayaraju [26] to the classes of generalized (f,g)-nonexpansive and asymptotically (f,g)-nonexpansive maps where $F(f)\cap F(g)$ need not be starshaped. As applications, invariant best simultaneous approximation results are obtained.

2. Main results

We shall need the following recent result.

Lemma 1 (see [20]). *Let M be a nonempty subset of a metric space (X,d), and let T, f and g be self-maps of M. If $F(f)\cap F(g)$ is nonempty, $clT(F(f)\cap F(g)) \subseteq F(f)\cap F(g)$, $cl(T(M))$ is complete, and T, f and g satisfy for all $x,y \in M$ and $0 \le k < 1$,*

$$d(Tx,Ty) \le k\max\{d(fx,gy), d(Tx,fx), d(Ty,gy), d(Tx,gy), d(Ty,fx)\} \quad (1)$$

Then $M\cap F(T)\cap F(f)\cap F(g)$ is singleton.

Inequality (1) is known as (f,g)-Ciric contraction [9]. We shall denote by $Y_q^{Tx} = \{h_{Tx}(k) : 0 \le k \le 1\}$ where $q = h_{Tx}(0)$.

The following result properly contains Theorems 3.2-3.3 of [8], Theorem 2.11 in [12] and Theorem 2.2 in [25] and improves Theorem 2.2 of [2] and Theorem 6 of [19].

Theorem 1. *Let M be a nonempty subset of a normed [resp. Banach] space X and let T, f and g be self-maps of M. Suppose that $F(f)\cap F(g)$ is nonempty and has a contractive, jointly continuous [resp. jointly weakly continuous] family of functions $\mathbb{F} = \{h_x\}_{x\in F(f)\cap F(g)}$, $clT(F(f)\cap F(g)) \subseteq F(f)\cap F(g)$[resp. $wclT(F(f)\cap F(g)) \subseteq F(f)\cap F(g)$], $cl(T(M))$ is compact [resp. $wcl(T(M))$ is weakly compact], T is continuous [resp. weakly continuous] on M and*

$$\begin{aligned}\|Tx - Ty\| \le \max\{&\|fx - gy\|, dist(fx, Y_q^{Tx}), dist(gy, Y_q^{Ty}),\\ &dist(gy, Y_q^{Tx}), dist(fx, Y_q^{Ty})\},\end{aligned} \quad (2)$$

for all $x, y \in M$. Then $M\cap F(T)\cap F(f)\cap F(g) \ne \emptyset$.

Proof. For $n \in N$, let $k_n = \frac{n}{n+1}$. Define $T_n : F(f) \cap F(g) \to F(f) \cap F(g)$ by $T_n x = h_{Tx}(k_n)$ for all $x \in F(f) \cap F(g)$. Since $F(f) \cap F(g)$ has a contractive family and $clT(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$ [resp. $wclT(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$], so $clT_n(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$] [resp. $wclT_n(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$] for each $n \geq 1$. By (2), and contractiveness of the family $\mathbb{F} = \{h_x\}_{x \in F(f) \cap F(g)}$, we have

$$\begin{aligned}
\|T_n x - T_n y\| &= \|h_{Tx}(k_n) - h_{Ty}(k_n)\| \\
&= \phi(k_n)\|Tx - Ty\| \\
&\leq \phi(k_n) \max\{\|fx - gy\|, dist(fx, Y_q^{Tx}), \\
&\quad dist(gy, Y_q^{Ty}), dist(gy, Y_q^{Tx}), dist(fx, Y_q^{Ty})\} \\
&\leq \phi(k_n)\ max\{\|fx - gy\|, \|fx - T_n x\|, \|gy - T_n y\|, \\
&\quad \|gy - T_n x\|, \|fx - T_n y\|\},
\end{aligned}$$

for each $x, y \in F(f) \cap F(g)$.

If $cl(T(M))$ is compact, for each $n \in \mathbb{N}$, $cl(T_n(M))$ is compact and hence complete. By Lemma 1, for each $n \geq 1$, there exists $x_n \in F(f) \cap F(g)$ such that $x_n = fx_n = gx_n = T_n x_n$. Compactness of $cl(T(M))$ implies that there exists a subsequence $\{Tx_m\}$ of $\{Tx_n\}$ such that $Tx_m \to z \in cl(T(M))$ as $m \to \infty$. Since $\{Tx_m\}$ is a sequence in $T(F(f) \cap F(g))$ and $clT(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$, therefore $z \in F(f) \cap F(g)$. Further, the joint continuity of $\mathbb{F}$ implies that

$$x_m = T_m x_m = h_{Tx_m}(k_m) \to h_z(1) = z$$

as $m \to \infty$. By the continuity of T, we obtain $Tz = z$. Thus, $M \cap F(T) \cap F(f) \cap F(g) \neq \emptyset$ proves the first case.

Weak compactness of $wcl(T(M))$ implies that $wcl(T_n(M))$ is weakly compact and hence complete due to completeness of X. From Lemma 1, for each $n \geq 1$, there exists $x_n \in F(f) \cap F(g)$ such that $x_n = fx_n = gx_n = T_n x_n$. Weak compactness of $wcl(T(M))$ implies that there is a subsequence $\{Tx_m\}$ of $\{Tx_n\}$ converging weakly to $y \in wcl(T(M))$ as $m \to \infty$. Since $\{Tx_m\}$ is a sequence in $T(F(f) \cap F(g))$, therefore $y \in wcl(T(F(f) \cap F(g))) \subseteq F(f) \cap F(g)$. By the joint weak continuity of $\mathbb{F}$ we obtain,

$$x_m = T_m x_m = h_{Tx_m}(k_m) \rightharpoonup h_y(1) = y$$

as $m \to \infty$. By the weak continuity of T, we get $Ty = y$. Thus $M \cap F(T) \cap F(f) \cap F(g) \neq \emptyset$. □

Corollary 1. *Let M be a nonempty subset of a normed [resp. Banach] space X and let T, f and g be self-maps of M. Suppose that $F(f) \cap F(g)$ is q-starshaped, $clT(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$[resp. $wclT(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$], $cl(T(M))$ is compact [resp. $wcl(T(M))$ is weakly compact], T is continuous [resp. weakly continuous] on M and*

$$\begin{aligned}
\|Tx - Ty\| \leq \max\{&\|fx - gy\|, dist(fx, [q, Tx]), dist(gy, [q, Ty]), \\
&dist(gy, [q, Tx]), dist(fx, [q, Ty])\}, \qquad (3)
\end{aligned}$$

for all $x, y \in M$. Then $M \cap F(T) \cap F(f) \cap F(g) \neq \emptyset$.

Remark 1.

(1) *By comparing Theorem 2.2(i) of Akbar and Sultana [1] with the first case of Theorem 1, their assumptions "M is closed, and has a contractive jointly continuous family $\mathbb{F}$, $f = g$ satisfies property (A) and is continuous on M, $f(M) = M$, $h_{Tx}(0) = q \in F(f)$ and (T, f) is pointwise R-subweakly commuting on M" are replaced with "M is a nonempty subset, $F(f)\cap F(g)$ is nonempty and has a contractive jointly continuous family $\mathbb{F}$ and $clT(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$".*

(2) *By comparing Theorem 2.2(v) of Akbar and Sultana [1] with the second case of Theorem 1, their assumptions "M is weakly compact, and has a contractive jointly weakly continuous family $\mathbb{F}$, $f = g$ satisfies property (A) and $f(M) = M$, $h_{Tx}(0) = q \in F(f)$, f, T are weakly continuous and (T, f) is pointwise R-subweakly commuting on M" are replaced with "$wcl(T(M))$ is weakly compact, $F(f)\cap F(g)$ is nonempty and has a contractive jointly weakly continuous family $\mathbb{F}$, $wclT(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$ and T is weakly continuous".*

(3) *By comparing the results in [1, 13, 14, 22, 23] with Theorem 1 we notice that property (A) is a key assumption in the results of [1, 13, 14, 22, 23].*

Corollary 2. *Let M be a nonempty subset of a normed [resp. Banach] space X and let T, f and g be self-maps of M. Suppose that $F(f) \cap F(g)$ is nonempty, closed [resp. weakly closed], has a contractive, jointly continuous [resp. jointly weakly continuous] family of functions $\mathbb{F} = \{h_x\}_{x\in F(f)\cap F(g)}$, $cl(T(M))$ is compact [resp. $wcl(T(M))$ is weakly compact], T is continuous [resp. weakly continuous] on M. If (T, f) and (T, g) are Banach operator pairs and satisfy* (2) *for all $x, y \in M$, then $M \cap F(T) \cap F(f) \cap F(g) \neq \emptyset$.*

Let $C = P_M(u) \cap C_M^{f,g}(u)$, where $C_M^{f,g}(u) = C_M^f(u) \cap C_M^g(u)$ and $C_M^f(u) = \{x \in M : \ fx \in P_M(u)\}$.

Corollary 3. *Let X be a normed [resp. Banach] space and let T, f and g be self-maps of X. If $u \in X$, $D \subseteq C$, $D_0 := D \cap F(f) \cap F(g)$ is nonempty, has a contractive, jointly continuous [resp. jointly weakly continuous] family of functions $\mathbb{F} = \{h_x\}_{x\in D_0}$, $cl(T(D_0)) \subseteq D_0$ [resp. $wcl(T(D_0)) \subseteq D_0$], $cl(T(D))$ is compact [resp. $wcl(T(D))$ is weakly compact], T is continuous [resp. weakly continuous] on D and* (2) *holds for all $x, y \in D$, then $P_M(u) \cap F(T) \cap F(f) \cap F(g) \neq \emptyset$.*

Corollary 4. *Let X be a normed [resp. Banach] space and let T, f and g be self-maps of X. If $u \in X$, $D \subseteq P_M(u)$, $D_0 := D \cap F(f) \cap F(g)$ is nonempty, has a contractive, jointly continuous [resp. jointly weakly continuous] family of functions $\mathbb{F} = \{h_x\}_{x\in D_0}$, $cl(T(D_0)) \subseteq D_0$ [resp. $wcl(T(D_0)) \subseteq D_0$], $cl(T(D))$ is compact [resp. $wcl(T(D))$ is weakly compact], T is continuous [resp. weakly continuous] on D and* (2) *holds for all $x, y \in D$, then $P_M(u) \cap F(T) \cap F(f) \cap F(g) \neq \emptyset$.*

Remark 2. *Corollaries 2.7 and 2.8 in [20] and Theorems 4.1 and 4.2 of Chen and Li in [8] are particular cases of Corollaries 3 and 4.*

Definition 4. *A subset M of a linear space X is said to have the property (N) with respect to T [17, 18] if* (*i*) $T : M \to M$,

(*ii*) $(1 - k_n)q + k_nTx \in M$, *for some* $q \in M$ *and a fixed sequence of real numbers* $k_n (0 < k_n < 1)$ *converging to* 1 *and for each* $x \in M$.

Hussain et al. [17] noted that each T*-invariant* q*-starshaped set* M *has the property* (N) *but converse does not hold in general.*

The following result constitutes an extension of Theorem 2.10 in [20] for non-starshaped domain.

Theorem 2. *Let* f, g, T *be self-maps of a subset* M *of a normed [resp. Banach] space* X. *Assume that* $F(f) \cap F(g)$ *is nonempty and has property* (N) *w.r.t.* T, T *is uniformly asymptotically regular and asymptotically* (f, g)*-nonexpansive on* M. *If* $clT(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$ *[resp.* $wclT(F(f) \cap F(g)) \subseteq F(f) \cap F(g)$*],* $cl(T(M))$ *is compact [resp.* $wcl(T(M))$ *is weakly compact and* $id - T$ *is demiclosed at* 0*], then* $F(T) \cap F(f) \cap F(g) \neq \emptyset$.

Proof. For each $n \geq 1$, define a self-map T_n on $F(f) \cap F(g)$ by

$$T_nx = h_{T^nx}(\mu_n),$$

where $\mu_n = \frac{\lambda_n}{k_n}$ and $\{\lambda_n\}$ is a sequence of numbers in $(0, 1)$ such that $\lim_{n\to\infty} \lambda_n = 1$. For each $x, y \in F(f) \cap F(g)$, we have

$$\begin{aligned} \|T_nx - T_ny\| &= \phi(\mu_n)\|T^nx - T^ny\| \\ &\leq \phi(\lambda_n)\|fx - gy\|. \end{aligned}$$

Since $T^n(F(f) \cap F(g)) \subset F(f) \cap F(g)$ and $F(f) \cap F(g)$ has a contractive family of function $\mathbb{F} = \{h_x\}_{x\in F(f)\cap F(g)}$, it follows that T_n maps $F(f) \cap F(g)$ into $F(f) \cap F(g)$. The rest of the proof is similar to that of Theorem 2.10 in [20] and so it is omitted. □

Corollary 5. *Let* f, g, T *be self-maps of a subset* M *of a normed [resp. Banach] space* X. *Assume that* $F(f) \cap F(g)$ *is nonempty closed [resp. weakly closed] and has property* (N) *w.r.t.* T, T *is uniformly asymptotically regular and asymptotically* (f, g)*-nonexpansive on* M. *If* $cl(T(M))$ *is compact [resp.* $wcl(T(M))$ *is weakly compact and* $id - T$ *is demiclosed at* 0*] and* (T, f) *and* (T, g) *are Banach operator pairs, then* $F(T) \cap F(f) \cap F(g) \neq \emptyset$.

Remark 3. *By comparing Theorem 3.4 of Beg et al. [4] with the first case of Theorem 2 with* $g = f$, *their assumptions "*M *is closed and* q*-starshaped,* $fM = M$, $T(M \setminus \{q\}) \subset f(M) \setminus \{q\}$, f, T *are continuous,* f *is linear,* $q \in F(f)$, $clT(M \setminus \{q\})$ *is compact and* T *and* f *are uniformly R-subweakly commuting on* M*" are replaced with "*M *is a nonempty set,* $F(f)$ *has property* (N) *w.r.t.* T, $clT(F(f)) \subseteq F(f)$ *and* $clT(M)$ *is compact".*

Corollary 6. *Let* X *be a normed [resp. Banach] space and let* T, f *and* g *be self-maps of* X. *If* $u \in X$, $D \subseteq P_M(u)$, $D_0 := D \cap F(f) \cap F(g)$ *has property* (N) *w.r.t.* T, $cl(T(D_0)) \subseteq D_0$ *[resp.* $wcl(T(D_0)) \subseteq D_0$*],* $cl(T(D))$ *is compact [resp.* $wcl(T(D))$ *is weakly compact* $id - T$ *is demiclosed at* 0*],* T *is uniformly asymptotically regular and asymptotically* (f, g)*-nonexpansive on* D, *then* $P_M(u) \cap F(T) \cap F(f) \cap F(g) \neq \emptyset$.

Remark 4. *Corollary 2.12 in [20] and Theorems 4.1 and 4.2 of Chen and Li in [8] are particular cases of Corollary 6.*

Corollary 7. *Let X be a normed [resp. Banach] space and let T, f and g be self-maps of X. If $y_1,\ y_2 \in X$, $D \subseteq cent_K(\{y_1, y_2\})$, where $cent_K(A)$ is the set of best simultaneous approximations of A w.r.t. K. Assume that $D_0 := D \cap F(f) \cap F(g)$ has property (N) w.r.t. T, $cl(T(D_0)) \subseteq D_0$ [resp. $wcl(T(D_0)) \subseteq D_0$], $cl(T(D))$ is compact [resp. $wcl(T(D))$ is weakly compact and $id - T$ is demiclosed at 0], T is uniformly asymptotically regular and asymptotically (f, g)-nonexpansive on D, then $cent_K(\{y_1, y_2\} \cap F(T) \cap F(f) \cap F(g) \neq \emptyset$.*

Remark 5. *Theorems 2.2 and 2.7 of Khan and Akbar in [21] and Theorem 2.3 of Vijayraju in [26] are particular cases of Corollary 7.*

Corollary 8 (see [26], Theorem 2.3)**.** *Let K be a nonempty subset of a normed space X and $y_1, y_2 \in X$. Suppose that T and f are selfmaps of K such that T is asymptotically $f-$nonexpansive. Suppose that the set $F(f)$ is nonempty. Let the set D of best simultaneous K-approximants to y_1 and y_2 be nonempty, compact and starshaped with respect to an element q in $F(f)$ and D is invariant under T and f. Assume further that T and f are commuting, T is uniformly asymptotically regular on D and f is an affine continuous mapping on D with $f(D) = D$. Then D contains a $T-$ and $f-$invariant point.*

Proof. As f is continuous and D is closed, $F(f)$ is therefore closed. Using the commutativity of T with f, we obtain $T(F(f)) \subseteq F(f)$. Thus $clT(F(f)) \subseteq cl(F(f)) = F(f)$. Since f is affine and $q \in F(f)$, so $F(f)$ is q-starshaped and hence $F(f)$ has property (N) w.r.t. T. The desired conclusion follows now from Theorem 2. □

The following result is a consequence of Theorem 2.1 in [16].

Theorem 3. *Let K be a subset of a metric space (X, d), and let T be a self mapping of K. Assume that $cl(T(K)) \subset K$, $cl(T(K))$ is complete, and there exists a continuous non-decreasing function $\varphi : [0,\infty) \to [0,\infty)$ satisfying $\phi(t) < t$ for $t > 0$ such that*

$$d(Tx, Ty) \le \varphi(\max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(y, Tx) + d(x, Ty)]\})$$

Then $F(T)$ is a singleton.

The following result generalizes Theorems 3.2 and 3.3 of [8].

Theorem 4. *Let M be a nonempty subset of a normed [resp. Banach] space X and let T and f be self-maps of M. Suppose that $F(f)$ is nonempty and has a φ-contractive, jointly continuous [resp. jointly weakly continuous] family of functions $\mathbb{F} = \{h_x\}_{x\in F(f)}$, $clT(F(f)) \subseteq F(f)$[resp. $wclT(F(f)) \subseteq F(f)$], $cl(T(M))$ is compact [resp. $wcl(T(M))$ is weakly compact], T is continuous [resp. weakly continuous] on M and*

$$\|Tx - Ty\| \le \varphi(\max\{\|fx - fy\|, dist(fx, Y_q^{Tx}), dist(fy, Y_q^{Ty}), \\ \frac{1}{2}[dist(fy, Y_q^{Tx}) + dist(fx, Y_q^{Ty})]\}), \tag{4}$$

for all $x, y \in M$, where $\varphi : [0, +\infty) \to [0, +\infty)$ is continuous and has the following properties:

(1) for each $k \in (0,1)$, $\varphi(t) < \frac{t}{k}$ for $t > 0$,

(2) $\varphi(t)$ is non-decreasing.

Then $M \cap F(T) \cap F(f) \neq \emptyset$.

Proof. For $n \in N$, let $k_n = \frac{n}{n+1}$. Define $T_n : F(f) \to F(f)$ as in the proof of Theorem 1. By (4) and the φ-contractiveness of the family $\mathbb{F} = \{h_x\}_{x \in F(f)}$, we have

$$\begin{aligned} \|T_n x - T_n y\| &= \|h_{Tx}(k_n) - h_{Ty}(k_n)\| \\ &= \varphi_{k_n}(\|Tx - Ty\|) \\ &\leq \varphi_{k_n}(\max\{\|fx - gy\|, dist(fx, Y_q^{Tx}), \\ &\quad dist(fy, Y_q^{Ty}), \frac{1}{2}[dist(fy, Y_q^{Tx}) + dist(fx, Y_q^{Ty})]\}) \\ &\leq \varphi_{k_n}(\ max\{\|fx - fy\|, \|fx - T_n x\|, \|fy - T_n y\|, \\ &\quad \frac{1}{2}[\|fy - T_n x\| + \|fx - T_n y\|]\}), \end{aligned}$$

for each $x, y \in F(f)$, since φ is increasing and T satisfies (4). By Theorem 3, for each $n \geq 1$, there exists $x_n \in F(f)$ such that $x_n = fx_n = T_n x_n$. A desired conclusion now follows as in the proof of Theorem 1. □

For $\varphi(t) = t$, $t \in [0, \infty)$, from Theorem 4 we obtain:

Corollary 9. *Let M be a nonempty subset of a normed [resp. Banach] space X and let T and f be self-maps of M. Suppose that $F(f)$ is nonempty and has a φ-contractive, jointly continuous [resp. jointly weakly continuous] family of functions $\mathbb{F} = \{h_x\}_{x \in F(f)}$, $clT(F(f)) \subseteq F(f)$[resp. $wclT(F(f)) \subseteq F(f)$], $cl(T(M))$ is compact [resp. $wcl(T(M))$ is weakly compact], T is continuous [resp. weakly continuous] on M and T is f-nonexpansive. Then $M \cap F(T) \cap F(f) \neq \emptyset$.*

Corollary 10 (see Theorems 1-2 in [6])**.** *Let M be a nonempty subset of a normed [resp. Banach] space X and let T be a self-map of M. Suppose that M has a φ-contractive, jointly continuous [resp. jointly weakly continuous] family of functions $\mathbb{F} = \{h_x\}_{x \in M}$, M is compact [resp. M is weakly compact and T is weakly continuous] and T is nonexpansive on M. Then $M \cap F(T) \neq \emptyset$.*

Remark 6.

(1) Locally bounded topological vector spaces provide an active area of research and have the following nice characterization.
A topological vector space X is Hausdorff locally bounded if and only its topology is defined by some p-norm, $0 < p \leq 1$.
All of the results proved above for normed(Banach) spaces hold for the p-normed(complete p-normed) spaces.

(2) *Our results represent very strong variants of the results of [2, 19, 22] in the sense that the commutativity of the maps T and f is replaced by the general hypothesis that (T, f) is a Banach operator pair. The comparison of Theorems 1-3 with the corresponding results in [1, 2, 13, 14, 15, 19, 22, 23] indicates that the concept of a Banach operator pair is more useful for the study of common fixed points in best approximation theory in the sense that here we are able to prove the results without the linearity or affinity or Property (A) of f or g.*

(3) *Banach operator pairs are different from those of weakly compatible, C_q-commuting and R-subweakly commuting maps, so our results are different from those in [1, 13, 14, 22, 23]. Consider $M = \mathbb{R}^2$ with the usual norm. Define T and f on M as follows:*

$$T(x,y) = \Big(x^3 + x - 1, \frac{\sqrt[3]{x^2+y^3-1}}{3}\Big),$$
$$f(x,y) = \Big(x^3 + x - 1, \sqrt[3]{x^2+y^3-1}\Big).$$

Then

$$F(T) = \{(1,0)\}, \quad F(f) = \{(1,y) : y \in R^1\},$$
$$C(T,f) = \{(x,y) : y = \sqrt[3]{1-x^2}, \quad x \in R^1\},$$
$$T(F(f)) = \{T(1,y) : y \in R^1\} = \{(1, \frac{y}{3}) : y \in R^1\}$$
$$\subseteq \{(1,y) : y \in R^1\} = F(f).$$

Thus, (T, f) is a Banach operator pair. It is easy to see that T and f do not commute on the set $C(T, f)$. Thus T and f are not weakly compatible and hence not pointwise R-subweakly commuting maps.

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