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The number of $D(-1)$-quadruples

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Abstract. In this paper, we first show that for any fixed $D(-1)$-triple $\{1, b, c\}$ with $b < c$, there exist at most two d's such that $\{1, b, c, d\}$ is a $D(-1)$-quadruple with $c < d$. Using this result, we further show that there exist at most 10^{356} $D(-1)$-quadruples.

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1. Introduction

Let n be a nonzero integer. A set of m distinct positive integers $\{a_1, \ldots, a_m\}$ is called a Diophantine m-tuple with the property $D(n)$, or simply a $D(n)$-m-tuple, if $a_i a_j + n$ is a perfect square for each i, j with $1 \le i < j \le m$. The cases of $n = \pm 1$ and $n = 4$ are topics of active research.

A folklore conjecture says that there does not exist a $D(1)$-quintuple. The first result supporting this conjecture is due to Baker and Davenport ([2]), which asserts that if $\{1, 3, 8, d\}$ is a $D(1)$-quadruple, then $d = 120$. This result has been generalized to those concerning several kinds of parametrized $D(1)$-quadruples (cf. [6, 16, 9, 26, 4, 28, 29]). Dujella ([11]) in general showed that there does not exist a $D(1)$-sextuple and that there exist only finitely many $D(1)$-quintuples. In [12], he further bounded the number of $D(1)$-quintuples explicitly. The second author improved the bound using Okazaki's gap principle (cf. [3, Lemma 2.2]) and the uniqueness of d in a $D(1)$-quintuple $\{a, b, c, d, e\}$ with $a < b < c < d < e$ for a fixed a, b, c (cf. [27]). Similar results on $D(4)$-tuples have been obtained mainly by the first author (cf. [17, 24, 20, 21, 22, 23]).

In the case of $n = -1$, it is conjectured that there does not exist a $D(-1)$-quadruple (cf. [7]). We are only a step away from the solution to this conjecture. More precisely, if $\{a, b, c, d\}$ is a $D(-1)$-quadruple with $a < b < c < d$, then $a = 1$ (cf. [14]), and there exist only finitely many $D(-1)$-quadruples (cf. [13]). Moreover, if $\{1, b, c, d\}$ is a $D(-1)$-quadruple with $b < c < d$, then $b \ge 101$ (cf. [8, 1, 18, 25, 30]). Note that the validity of the conjecture on $D(-1)$-quadruples implies that there does not exist a $D(-4)$-quadruple ([5, Remark 3]).

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For general n, Dujella ([10]) gave upper bounds for the size of sets with the property $D(n)$, which are logarithmic in $|n|$. Moreover, if $|n|$ is prime, Dujella and Luca ([15]) showed that the size of sets with the property $D(n)$ is bounded by the absolute constant $3 \cdot 2^{168}$.

Our theorems in this paper are the following.

Theorem 1. *Let $\{1, b, c\}$ be a $D(-1)$-triple with $b < c$. Then, there exist at most two d's such that $\{1, b, c, d\}$ is a $D(-1)$-quadruple with $c < d$.*

Theorem 2. *There exist at most 10^{356} $D(-1)$-quadruples.*

Theorem 1 is proved by transforming the problem into a system of Pell equations with right-hand sides equal to 1, to which one can apply the theorem in [3]. The proof of Theorem 2 goes along the same lines as [12, Theorem 4], and it is completed by using Theorem 1.

It is to be noted that Theorem 2 improves a result of the first author in [19], which asserts that there exist at most 10^{902} $D(-1)$-quadruples. The improvement comes from the use of Theorem 1.

2. Proof of Theorem 1

Let $\{1, b, c\}$ be a $D(-1)$-triple with $b < c$ and let $b - 1 = r^2$, $c - 1 = s^2$, $bc - 1 = t^2$ with positive integers r, s, t. Suppose that $\{1, b, c, d\}$ is a $D(-1)$-quadruple with $c < d$. Then, there exist positive integers x, y, z such that

$$d - 1 = x^2,\ bd - 1 = y^2,\ cd - 1 = z^2.$$

Eliminating d, we obtain the system of Diophantine equations

$$y^2 - bx^2 = b - 1, \tag{1}$$
$$z^2 - cx^2 = c - 1, \tag{2}$$
$$bz^2 - cy^2 = c - b. \tag{3}$$

The positive solutions of each equation above can be expressed as follows:

$$y + x\sqrt{b} = (y_0 + x_0\sqrt{b})(r + \sqrt{b})^{2l}, \tag{4}$$
$$z + x\sqrt{c} = (z_1 + x_1\sqrt{c})(s + \sqrt{c})^{2m}, \tag{5}$$
$$z\sqrt{b} + y\sqrt{c} = (z_2\sqrt{b} + y_2\sqrt{c})(t + \sqrt{bc})^{2n} \tag{6}$$

with some integers $l, m, n \geq 0$, where

$$0 < y_0 < b,\ \ |x_0| < r,$$
$$0 < z_1 < c,\ \ |x_1| < s,$$
$$0 < z_2 < c,\ \ |y_2| < t$$

(cf. [14, Lemma 1]). By [13, Theorem 1], we may assume that $c \leq b^9$. Then, [13, Lemma 5] implies that

$$z_1 = z_2 = s,\ x_1 = 0,\ y_2 = \pm r.$$

By (4) and (6), we may write $y = \alpha_l = \beta_n$, where

$$\alpha_0 = y_0,\ \alpha_1 = (2b-1)y_0 + 2rbx_0,\ \alpha_{l+2} = 2(2b-1)\alpha_{l+1} - \alpha_l$$

and

$$\beta_0 = \pm r,\ \beta_1 = \pm r(2bc-1) + 2stb,\ \beta_{n+2} = 2(2bc-1)\beta_{n+1} - \beta_n.$$

Hence, $\alpha_l \equiv (-1)^l y_0 \pmod{2b}$ and $\beta_n \equiv \pm(-1)^n r \pmod{2b}$, which yield $y_0 \equiv \pm(-1)^{l+n} r \pmod{2b}$. Since $0 < y_0 < b$, we obtain $y_0 = r$ and $x_0 = 0$. By (4) and (5), we may write $x = u_l = v_m$, where

$$u_0 = 0,\ u_1 = 2r^2,\ u_{l+2} = 2(2r^2+1)u_{l+1} - u_l$$

and

$$v_0 = 0,\ v_1 = 2s^2,\ v_{m+2} = 2(2s^2+1)v_{m+1} - v_m.$$

It follows that $x \equiv 0 \pmod{2M^2}$, where $M = \operatorname{lcm}(r,s)$. By (1) and (2), we may write $y = ry'$ and $z = sz'$ for some integers y', z'. Putting $M_1 = M/r$, $M_2 = M/s$ and $x' = x/M$, we obtain the system of Pell equations

$$(y')^2 - bM_1^2(x')^2 = 1, \tag{7}$$
$$(z')^2 - cM_2^2(x')^2 = 1. \tag{8}$$

The theorem of [3] says that the system of Pell equations (7) and (8) has at most two positive solutions. This completes the proof of Theorem 1.

3. Proof of Theorem 2

As seen in Introduction, it suffices to bound the number of $D(-1)$-quadruples $\{1,b,c,d\}$ with $b < c < d$ and $b \geq 101$. By [13, Theorem 1] we have $b < c^{1/1.1} < (10^{491})^{1/1.1} < 3 \cdot 10^{446}$. Since $\sqrt{b-1}$ is an integer bounded by $\sqrt{3 \cdot 10^{446}} < 2 \cdot 10^{223}$, the number of $D(-1)$ pairs $\{1,b\}$ is bounded by $2 \cdot 10^{223}$. For a fixed pair $\{1,b\}$ the integer c such that $\{1,b,c\}$ $(b < c)$ is a $D(-1)$-triple belongs to the union of finitely many binary recurrent sequences, and the number of the sequences is less than or equal to the number of solutions of the congruence $t_0^2 \equiv -1 \pmod b$ with $0 < t_0 < b$ (cf. [14, Lemma 1]). The number of solutions of the congruence is less than or equal to $2^{\omega(b)}$, where $\omega(b)$ denotes the number of distinct prime factors of b (cf. [31, g, §4, ch. V]). If $b \leq 10^{124}$, then the number of sequences is bounded by 10^{124}. Assume that $b > 10^{124}$. We know by (8) in [12] that

$$\log b > \frac{1}{2}\omega(b)\log\omega(b). \tag{9}$$

If $2^{\omega(b)} \geq b^{0.29}$, then (9) implies that $2^{\omega(b)} > \omega(b)^{0.145\omega(b)}$, which yields $\omega(b) \leq 119$. We then see from $2^{\omega(b)} \geq b^{0.29}$ that $b < 10^{124}$, a contradiction. Thus we have $2^{\omega(b)} < b^{0.29} < 3 \cdot 10^{129}$. Hence, the number of the sequences is bounded by $3 \cdot 10^{129}$. Moreover, our sequences grow exponentially; in fact, we have

$$(b-1)(4b-3)^{m-1} < t_m = \sqrt{bc-1} < 5 \cdot 10^{468}.$$

Since $b \geq 101$, we obtain $m \leq 180$. Hence, the number of c's that extend our $D(-1)$-pair $\{1, b\}$ is bounded by $3 \cdot 10^{129} \cdot 180 < 6 \cdot 10^{131}$. It follows from Theorem 1 that the number of $D(-1)$-quadruples is bounded by

$$2 \cdot 10^{223} \cdot 6 \cdot 10^{131} \cdot 2 < 10^{356}.$$

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