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Strong convergence theorem for accretive mapping in Banach spaces*

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Abstract. Suppose K is a closed convex subset of a real reflexive Banach space E which has a uniformly Gâteaux differentiable norm and every nonempty closed convex bounded subset of E has the fixed point property for nonexpansive mappings. We prove a strong convergence theorem for an $m-$accretive mapping from K to E. The results in this paper are different from the corresponding results in [8] and they improve the corresponding results in [6, 14].

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1. Introduction and preliminaries

Let E be a real Banach space and E^* its dual space. Let J denote the normalized duality mapping from E into 2^{E^*} defined by $J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}$, where $\langle \cdot, \cdot\rangle$ denotes a generalized duality pairing between E and E^*. It is well-known that if E^* is strictly convex, then J is sing-valued. In the sequel, we shall denote the single-valued normalized duality mapping by j.

Definition 1. *$T : K \to K$ is said to be a nonexpansive mapping, if $\forall\ x, y \in K$, $\|Tx - Ty\| \le \|x - y\|$. The set of fixed points for T is denoted by $F(T) = \{x \in K : Tx = x\}$.*

Definition 2. *An operator A (possibly multivalued) with domain $D(A)$ and range $R(A)$ in E is called accretive mapping, if $\forall\ x_i \in D(A)$ and $y_i \in Ax_i$(i=1,2), there exists $j(x_2 - x_1) \in J(x_2 - x_1)$ such that $\langle y_2 - y_1, j(x_2 - x_1)\rangle \ge 0$. Especially, an accretive operator A is called m-accretive if $R(I + rA) = E$ for all $r > 0$.*

For each $r > 0$, if A is $m-$*accretive*, then $J_r := (I + rA)^{-1}$ is a nonexpansive single-valued mapping from $R(I + rA)$ to $D(A)$ and $F(J_r) = N(A)$, where $N(A) = \{x \in D(A) : Ax = 0\}$.

Iterative techniques for approximating zeros of accretive mappings have been studied by various authors (see, e.g., [3, 4, 6, 12, 14, 16], etc.), using a famous Mann iteration method, Ishikawa iteration method, and many other iteration methods such

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as viscosity approximation method [3] and steepest descent approximation method [11].

Recently, T.H. Kim and H.K. Xu [6] and H.K. Xu [14] studied the sequence generated by

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) J_{r_n} x_n, \tag{1}$$

where $x_0 \in E, J_{r_n} = (I + r_n A)^{-1}$, $\alpha_n \in [0,1]$ and obtained the following Theorem 1 and Theorem 2, respectively:

Theorem 1 (see [6], Theorem 2)**.** *Assume that E is a uniformly smooth Banach space and A is an $m-$accretive operator in E such that $N(A) \neq \emptyset$. Let$\{x_n\}$ be defined by (1). Suppose $\{\alpha_n\}$ and $\{r_n\}$ satisfy the conditions:*

(i) $\alpha_n \to 0$, $\Sigma_{n=0}^{\infty} \alpha_n = \infty$, $\Sigma_{n=0}^{\infty} |\alpha_{n+1} - a_n| < \infty$,

(ii) $r_n \geq \varepsilon > 0$, $\Sigma_{n=0}^{\infty} \left| 1 - \frac{r_{n-1}}{r_n} \right| < \infty$.

Then $\{x_n\}$ converges strongly to a zero of A.

Theorem 2 (see, e.g. [14])**.** *Suppose that E is a uniformly smooth Banach space. Suppose that A is an m-accretive operator in E such that $C = \overline{D(A)}$ is convex. Assume*

(i) $\alpha_n \to 0$, $\Sigma_{n=1}^{\infty} \alpha_n = \infty$, $\Sigma_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,

(ii) $r_n \geq \varepsilon > 0$, $\Sigma_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then $\{x_n\}$ converges strongly to a point in $N(A)$.

Inspired and motivated by the iterative sequences (1), Qin and Su [8] gave the following iterative sequences:

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) J_{r_n} x_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) y_n, \end{cases} \tag{2}$$

where $u \in K$ is an arbitrary (but fixed) element in K and sequences $\{\alpha_n\}$ in (0,1), $\{\beta_n\}$ in [0,1]. Then they obtained a strong convergence theorem as following:

Theorem 3 (see, e.g. [8])**.** *Assume that E is a uniformly smooth Banach space and A is an $m-$accretive operator in E such that $N(A) \neq \emptyset$. Given a point $u \in K$ and given sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ in [0,1], suppose that $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{r_n\}_{n=0}^{\infty}$ satisy the conditions:*

(i) $\Sigma_{n=0}^{\infty} \alpha_n = \infty$, $\alpha_n \to 0$;

(ii) $r_n \geq \varepsilon$ *for all* n *and* $\beta_n \in [0, a)$, *for some* $a \in (0, 1)$;

(iii) $\Sigma_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\Sigma_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\Sigma_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$.

Let $\{x_n\}_{n=0}^{\infty}$ be the composite process defined by (2). Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to a zero of A.

Let

$$\alpha_n = \begin{cases} 0, \text{ if } n = 2k \\ \frac{1}{n}, \text{ if } n = 2k-1, \end{cases} \quad \beta_n = \begin{cases} 0, & \text{if } n = 2k \\ \frac{1}{2} + \frac{1}{n+1}, & \text{if } n = 2k-1, \end{cases} \quad r_n = \begin{cases} \frac{1}{2}, \text{ if } n = 2k \\ \frac{1}{4}, \text{ if } n = 2k-1, \end{cases}$$

where k is some positive integer. Obviously, the coefficient α_n, β_n and r_n do not satisfy condition (iii) of Theorem 3 and conditions (i-ii) of Theorem 1. Hence, if we can remove condition (iii) of Theorem 3, then the coefficient α_n, β_n and r_n have a more extensively applicable scope.

Using the technique in [15, 5], algorithm (2) is analyzed from a new perspective in this paper, then a strong convergence theorem is obtained in the framework of real reflexive Banach spaces E with uniformly Gâteaux differentiable norms and condition (iii) of Theorem 3 is substituted by a new condition which is $0 < a \leq \beta_n \leq b < 1$ and $r_n \geq \varepsilon > 0$ for all n, $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$. At the same time, our proof is more simpler than that of Theorem 3 and our theorem also improves and extends Theorem 1 and Theorem 2 to more general real Banach spaces with uniformly Gâteaux differentiable norms.

In what follows, we shall make use of the following Lemmas.

Lemma 1 (see [2])**.** *Let E be a real normed linear space and J the normalized duality mapping on E; then for each $x, y \in E$ and $j(x+y) \in J(x+y)$, we have $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle$.*

Lemma 2 (Suzuki, see [9])**.** *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in [0,1] with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = \beta_n y_n + (1-\beta_n) x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$, then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 3 (see [13])**.** *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n \sigma_n + \gamma_n, n \geq 0,$$

if (i) $\alpha_n \in [0,1]$, $\sum \alpha_n = \infty$; (ii) $\limsup \sigma_n \leq 0$; (iii) $\gamma_n \geq 0$, $\sum \gamma_n < \infty$, then $a_n \to 0$, as $n \to \infty$.

Lemma 4 (see [7])**.** *Let K be a nonempty closed convex subset of a reflexive Banach space E which has uniformly Gâteaux differentiable norms and $T : K \to K$ a nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that every nonempty closed convex bounded subset of E has the fixed point property for nonexpansive mappings. Then there exists a continuous path $t \to z_t$, $0 < t < 1$, satisfying $z_t = tu + (1-t)Tz_t$, for arbitrary but fixed $u \in K$, which converges to a fixed point of T.*

Lemma 5 (see [1, 8])**.** *For $\lambda > 0$ and $\mu > 0$ and $x \in E$,*

$$J_\lambda x = J_\mu \left(\frac{\mu}{\lambda} x + \left(1 - \frac{\mu}{\lambda}\right) J_\lambda x\right).$$

2. Main results

Throughout this paper, suppose that

(a) E is a real reflexive Banach space E which has uniformly Gâteaux differentiable norms;

(b) K is a nonempty closed convex subset of E;

(c) every nonempty closed bounded convex subset of E has the fixed point property for nonexpansive mappings.

Theorem 4. *Let $A : K \to E$ be an $m-$accretive mapping with $N(A) \neq \emptyset$. For given $u, x_0 \in K$, let $\{x_n\}$ be generated by the algorithm (2). If $\alpha_n \in [0,1]$, $\{\beta_n\}$, $\{r_n\}$ satisfy the following conditions:*

(i) $\alpha_n \to 0$, $\Sigma_{n=0}^{\infty}\alpha_n = \infty$,

(ii) $0 < a \leq \beta_n \leq b < 1$; *(iii)* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$, $r_n \geq \varepsilon > 0$,

then $\{x_n\}$ converges strongly to a zero of A.

Proof. We know that $F(J_{r_n}) = N(A) \neq \emptyset$ and J_{r_n} is nonexpansive. Let $p \in F(J_{r_n})$, it follows from (2)

$$\|y_n - p\| \leq \|x_n - p\|, \ \|x_{n+1} - p\| \leq \alpha_n\|u - p\| + (1 - \alpha_n)\|x_n - p\|,$$

which yields that $\|x_n - p\| \leq \max\{\|x_0 - p\|, \|u - p\|\}$. Hence, $\{x_n\}$ is bounded and so is $\{y_n\}$.

Now, we shall show $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. For the purpose, let $\gamma_n = 1 - (1-\alpha_n)\beta_n$, $\overline{y}_n = \frac{x_{n+1}-x_n+\gamma_n x_n}{\gamma_n}$, i.e. $\overline{y}_n = \frac{\alpha_n u+(1-\alpha_n)(1-\beta_n)J_{r_n}x_n}{\gamma_n}$, then

$$\begin{aligned}
\overline{y}_{n+1} - \overline{y}_n &= \left(\frac{\alpha_{n+1}}{\gamma_{n+1}} - \frac{\alpha_n}{\gamma_n}\right)u + \frac{(1-\alpha_{n+1})(1-\beta_{n+1})J_{r_{n+1}}x_{n+1}}{\gamma_{n+1}} \\
&\quad - \frac{(1-\alpha_n)(1-\beta_n)J_{r_n}x_n}{\gamma_n} \\
&= \left(\frac{\alpha_{n+1}}{\gamma_{n+1}} - \frac{\alpha_n}{\gamma_n}\right)u + \frac{(1-\alpha_n)(1-\beta_n)(J_{r_{n+1}}x_{n+1} - J_{r_n}x_n)}{\gamma_n} \\
&\quad + \left(\frac{\alpha_n(1-(1-\alpha_{n+1})\beta_{n+1}) - \alpha_{n+1}(1-\beta_n(1-\alpha_n))}{\gamma_{n+1}\gamma_n}\right)J_{r_{n+1}}x_{n+1}.
\end{aligned} \tag{3}$$

It follows from (3) and Lemma 5 that

$$\begin{aligned}\|\overline{y}_{n+1}-\overline{y}_n\| &\le \left|\frac{\alpha_{n+1}}{\gamma_{n+1}}-\frac{\alpha_n}{\gamma_n}\right|\|u\|+\frac{(1-\beta_n)\|J_{r_{n+1}}x_{n+1}-J_{r_n}x_n\|}{\gamma_n}\\ &\quad+\frac{\alpha_n+\alpha_{n+1}}{\gamma_{n+1}\gamma_n}\|J_{r_{n+1}}x_{n+1}\|\\ &\le \frac{\|J_{r_{n+1}}x_{n+1}-J_{r_{n+1}}x_n+J_{r_n}(\frac{r_n}{r_{n+1}}x_n+(1-\frac{r_n}{r_{n+1}})J_{r_{n+1}}x_n)-J_{r_n}x_n\|}{\gamma_n}\\ &\quad\times(1-\beta_n)+\frac{\alpha_n+\alpha_{n+1}}{\gamma_{n+1}\gamma_n}(\|J_{r_{n+1}}x_{n+1}\|+\|u\|)\\ &\le \frac{(1-\beta_n)(\|x_{n+1}-x_n\|+|1-\frac{r_n}{r_{n+1}}|M_0)}{\gamma_n}\\ &\quad+\frac{\alpha_n+\alpha_{n+1}}{\gamma_{n+1}\gamma_n}(\|J_{r_{n+1}}x_{n+1}\|+\|u\|). \qquad (4)\end{aligned}$$

where $\|J_{r_{n+1}}x_n-x_n\|\le M_0$. By (i-iii) and boundedness of $\{x_n\}$, from (4) we get that

$$\limsup_{n\to\infty}\{\|\overline{y}_{n+1}-\overline{y}_n\|-\|x_{n+1}-x_n\|\}\le 0. \qquad (5)$$

Based on Lemma 2 and (5), we have $\lim_{n\to\infty}\|\overline{y}_n-x_n\|=0$, which implies $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. Since $\|x_{n+1}-y_n\|=\alpha_n\|u-y_n\|\to 0$ as $n\to\infty$, then $\|x_n-y_n\|\to 0$ and

$$\|x_n-J_{r_n}x_n\|=\frac{1}{1-\beta_n}\|x_n-y_n\|\to 0 \text{ as } n\to\infty. \qquad (6)$$

Take a fixed number r such that $0<r<\varepsilon$, from Lemma 5 we obtain

$$\|J_{r_n}x_n-J_rx_n\|=\left\|J_r\left(\frac{r}{r_n}x_n+\left(1-\frac{r}{r_n}\right)J_{r_n}x_n\right)-J_rx_n\right\|\le\|x_n-J_{r_n}x_n\|,$$

which implies that

$$\|x_n-J_rx_n\|\le\|x_n-J_{r_n}x_n\|+\|J_{r_n}x_n-J_rx_n\|\le 2\|x_n-J_{r_n}x_n\|\to 0 \text{ as n}\to\infty.$$

Let z_t denote the fixed point of contraction mapping H_t given by

$$H_tx=tu+(1-t)J_rx,\ x\in E, \quad \forall\ t\in(0,1).$$

Then, using Lemma 1, we have

$$\begin{aligned}\|z_t-x_n\|^2&=\|t(u-x_n)+(1-t)(J_rz_t-x_n)\|^2\\ &\le(1-t)^2\|J_rz_t-x_n\|^2+2t\langle u-x_n,j(z_t-x_n)\rangle\\ &\le(1-t)^2(\|J_rz_t-J_rx_n\|+\|J_rx_n-x_n\|)^2\\ &\quad+2t\langle u-z_t+z_t-x_n,j(z_t-x_n)\rangle\\ &\le(1+t^2)\|z_t-x_n\|^2+\|J_rx_n-x_n\|(2\|z_t-x_n\|+\|J_rx_n-x_n\|)\\ &\quad+2t\langle u-z_t,j(z_t-x_n)\rangle,\end{aligned}$$

hence,

$$\langle u - z_t, j(x_n - z_t)\rangle \le \frac{t}{2}\|z_t - x_n\|^2 + \frac{\|J_r x_n - x_n\|}{2t}(2\|z_t - x_n\| + \|J_r x_n - x_n\|),$$

let $n \to \infty$ in the last inequality, then we obtain

$$\limsup_{n\to\infty}\langle u - z_t, j(x_n - z_t)\rangle \le \frac{t}{2}M,$$

where $M > 0$ is a constant such that$\|z_t - x_n\|^2 \le M$ for all $t \in (0,1)$ and $n \ge 0$. Now letting $t \to 0^+$, then we have that

$$\limsup_{t\to 0^+}\limsup_{n\to\infty}\langle u - z_t, j(x_n - z_t)\rangle \le 0.$$

Thus , for $\forall\,\varepsilon > 0$, there exists a positive number δ' such that for any $t \in (0, \delta')$,

$$\limsup_{n\to\infty}\langle u - z_t, j(x_n - z_t)\rangle \le \frac{\varepsilon}{2}.$$

On the other hand, by Lemma 4 we have $z_t \to p \in F(J_{r_n}) = N(A)$ as $t \to 0^+$. In addition, j is norm-to-weak* uniformly continuous on bounded subsets of E, so there exists $\delta'' > 0$ such that, for any $t \in (0, \delta'')$, we have

$$\begin{aligned}|\langle u-p, j(x_n - p)\rangle - \langle u-z_t, j(x_n - z_t)\rangle| &\le |\langle u-p, j(x_n - p)\rangle - \langle u - p, j(x_n - z_t)\rangle| \\ &\quad + |\langle u - p, j(x_n - z_t)\rangle - \langle u - z_t, j(x_n - z_t)\rangle| \\ &\le \|u - p\|\|j(x_n - p) - j(x_n - z_t)\| \\ &\quad + \|\, z_t - p\|\|x_n - z_t\| \\ &< \frac{\varepsilon}{2}.\end{aligned}$$

Taking $\delta = \min\{\delta', \delta''\}$, for $t \in (0, \delta)$, we have that

$$\langle u - p, j(x_n - p)\rangle \le \langle u - z_t, j(x_n - z_t)\rangle + \frac{\varepsilon}{2}.$$

Hence,

$$\limsup_{n\to\infty}\langle u - p, j(x_n - p)\rangle \le \varepsilon, \quad \text{where } \varepsilon > 0 \text{ is arbitrary},$$

which yields that

$$\limsup_{n\to\infty}\langle u - p, j(x_n - p)\rangle \le 0. \tag{7}$$

Now we prove that $\{x_n\}$ converges strongly to p. It follows from Lemma 1 and 2 that

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\alpha_n(u - p) + (1 - \alpha_n)(y_n - p)\|^2 \\ &\le (1 - \alpha_n)\|y_n - p\|^2 + 2\alpha_n\langle u - p, j(x_{n+1} - p)\rangle \\ &\le (1 - \alpha_n)\|x_n - p\|^2 + 2\alpha_n\langle u - p, j(x_{n+1} - p)\rangle \end{aligned} \tag{8}$$

By condition (i) and Lemma 3, $\{x_n\}$ converges strongly to p. The proof is complete. □

Remark 1. *If E is uniformly smooth, then E is reflexive and has a uniformly Gâteaux differentiable norm with the property that every nonempty closed and bounded subset of E has the fixed point property for nonexpansive mappings (see Remark 3.5 of [16]). Thus, if E in Theorem 4 is a real uniformly smooth Banach space, then Theorem 4 is true, too.*

Using the proof method of Theorem 4, we may improve Theorem 1 and Theorem 2 as follows:

Theorem 5. *Let $A : K \to E$ be an $m-$accretive mapping with $N(A) \neq \emptyset$. Let$\{x_n\}$ be defined by (1). Suppose $\{\alpha_n\}$ and $\{r_n\}$ satisfy the conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\Sigma_{n=0}^{\infty}\alpha_n = \infty$,

(ii) $r_n \geq \varepsilon > 0$, $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$, $r_n < r_{n+1}$.

Then $\{x_n\}$ converges strongly to a zero of A.

Proof. By using the proof method of Theorem 4, we can also obtain that $\{x_n\}$ is bounded. Now, we shall show $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. For the purpose, let $\gamma_n = 1 - \delta$, $0 < \delta < \frac{1}{2}\left(1 - \frac{r_n}{r_{n+1}}\right)$, $\overline{y}_n = \frac{x_{n+1}-x_n+\gamma_n x_n}{\gamma_n}$, i.e. $\overline{y}_n = \frac{\alpha_n u+(1-\alpha_n)J_{r_n}x_n-\delta x_n}{1-\delta}$, then

$$\begin{aligned}\overline{y}_{n+1}-\overline{y}_n &= \frac{\alpha_{n+1}u - \alpha_n u}{1-\delta} + \frac{(1-\alpha_{n+1})J_{r_{n+1}}x_{n+1}}{1-\delta} - \frac{(1-\alpha_n)J_{r_n}x_n}{1-\delta} \\ &\quad + \frac{\delta}{1-\delta}(x_n - x_{n+1}) \qquad (9) \\ &= \frac{(\alpha_{n+1}-\alpha_n)(u - J_{r_n}x_n)+(1-\alpha_{n+1})(J_{r_{n+1}}x_{n+1} - J_{r_n}x_n) + \delta x_n - \delta x_{n+1}}{1-\delta}.\end{aligned}$$

It follows from (9) and Lemma 5 that

$$\begin{aligned}\|\overline{y}_{n+1}-\overline{y}_n\| &\leq \frac{\alpha_{n+1}+\alpha_n}{1-\delta}\|u - J_{r_n}x_n\| + \frac{\|J_{r_{n+1}}x_{n+1} - J_{r_n}x_n\|}{1-\delta} + \frac{\delta}{1-\delta}\|x_{n+1}-x_n\| \\ &= \frac{\alpha_{n+1}+\alpha_n}{1-\delta}\|u - J_{r_n}x_n\| + \frac{\delta}{1-\delta}\|x_{n+1}-x_n\| \\ &\quad + \frac{\left\|J_{r_n}\left(\frac{r_n}{r_{n+1}}x_{n+1} + (1-\frac{r_n}{r_{n+1}})J_{r_{n+1}}x_{n+1}\right) - J_{r_n}x_n\right\|}{1-\delta} \\ &\leq \frac{\alpha_{n+1}+\alpha_n}{1-\delta}\|u - J_{r_n}x_n\| \\ &\quad + \frac{(\delta+\frac{r_n}{r_{n+1}})\|x_{n+1} - x_n\| + (1-\frac{r_n}{r_{n+1}})\|J_{r_{n+1}}x_{n+1} - x_n\|}{1-\delta},\end{aligned}$$

which implies that

$$\begin{aligned}\|\overline{y}_{n+1} - \overline{y}_n\| - \|x_{n+1}-x_n\| &\leq \frac{\alpha_{n+1}+\alpha_n}{1-\delta}\|u - J_{r_n}x_n\| \qquad (10) \\ &\quad + \frac{(1-\frac{r_n}{r_{n+1}})\|J_{r_{n+1}}x_{n+1} - x_n\|}{1-\delta}.\end{aligned}$$

By condition (i-ii) and boundedness of $\{x_n\}$, from (10) we get that

$$\limsup_{n\to\infty}\{\|\overline{y}_{n+1}-\overline{y}_n\| - \|x_{n+1}-x_n\|\} \leq 0. \tag{11}$$

Based on Lemma 2 and (11), we have $\lim_{n\to\infty}\|\overline{y}_n - x_n\| = 0$, which implies $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Since $\|x_{n+1} - J_{r_n}x_n\| = \alpha_n\|u - J_{r_n}x_n\| \to 0$ as $n \to \infty$, then

$$\|x_n - J_{r_n}x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - J_{r_n}x_n\| \to 0 \text{ as } n \to \infty. \tag{12}$$

The rest of the argument is similar to the corresponding part of Theorem 4 and so it is omitted. This completes the proof of Theorem 5. □

Remark 2. *From Remark 1, if E is uniformly smooth, then Theorem 5 is true.*

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