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Dirac operators on Weil representations I*

Pavle Pandžić[1,†]

[1] *Department of Mathematics, University of Zagreb, Bijenička 30, HR-10 000 Zagreb, Croatia*

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Abstract. Let G be the metaplectic double cover of the group of four-by-four real symplectic matrices. Let $\mathfrak{g}$ be the complexified Lie algebra of G. Let W_0 and W_1 be the Harish-Chandra modules of the even and odd Weil representations of G, respectively. We find the Dirac cohomology of W_0 and W_1 with respect to the Dirac operator corresponding to a maximal compact subalgebra of $\mathfrak{g}$, and then also with respect to the Kostant's cubic Dirac operator corresponding to a compact Cartan subalgebra of $\mathfrak{g}$. The results can be considered as examples illustrating the main results of [11].
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1. Introduction

Dirac operators were introduced into representation theory of real reductive Lie groups by Parthasarathy [16]. In this paper we consider their algebraic versions due to Vogan [18] and Kostant [13]. The main goal is to illustrate the main results of [11] by very concrete examples. In the sequel [15] we show further examples which are counterexamples to certain generalizations of the results of [11] that one might attempt. We believe that the results we obtain and the calculations we need to get them are also interesting in their own right.

All our examples concern the two Weil representations of the metaplectic double cover $G = \mathrm{Mp}(4,\mathbb{R})$ of the group of symplectic four by four real matrices $\mathrm{Sp}(4,\mathbb{R})$. More precisely, we study the Harish-Chandra modules of these representations. Recall that Harish-Chandra modules are modules for the complexified Lie algebra $\mathfrak{g}$ of G, which also carry an action of the maximal compact subgroup K of G. In our case, $\mathfrak{g}$ is the symplectic Lie algebra $\mathfrak{sp}(4,\mathbb{C})$, while K is a double cover of the unitary group $U(2)$.

Weil representations and their Harish-Chandra modules are well known and much studied representations, with many important applications. They are also called Segal-Shale-Weil representations. See [19] or [2], Ch. VIII. For the approach we adopt, see [1, 3, 6] and [17].

We are concerned with actions of several Dirac operators on these representations: first the "ordinary" Dirac operator corresponding to the complexified Lie algebra $\mathfrak{k}$

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†Corresponding author. *Email address:* `pandzic@math.hr` (P. Pandžić)

of K, then the Kostant's cubic Dirac operator corresponding to the compact Cartan subalgebra, and finally, in [15], the Kostant's cubic Dirac operator corresponding to a certain noncompact Levi subalgebra.

Our results can be generalized in various directions. For example, [10] contains a generalization of the results of Section 5 below. The argument in a more general case is however less direct, and consequently the results are also not quite so explicit.

2. Definition and basic properties of Dirac cohomology

Let G be a connected reductive Lie group with Cartan involution Θ, such that $K = G^\Theta$ is a maximal compact subgroup of G. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decompositions of the Lie algebra of G and its complexification. We denote by B a fixed invariant nondegenerate symmetric bilinear form on $\mathfrak{g}_0$ and $\mathfrak{g}$. Then $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal with respect to B, and hence B restricts nondegenerately to both $\mathfrak{k}$ and $\mathfrak{p}$.

Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, and let $C(\mathfrak{p})$ be the Clifford algebra of $\mathfrak{p}$ with respect to B. We use the conventions of [12] and [11]. In particular, $C(\mathfrak{p})$ is the associative algebra with a unit generated by $\mathfrak{p}$, with relations

$$XY + YX = 2B(X,Y), \qquad X, Y \in \mathfrak{p}.$$

(Notice the opposite sign compared to [8] or [9]).

The Dirac operator corresponding to $\mathfrak{k} \subseteq \mathfrak{g}$ is

$$D = D(\mathfrak{g},\mathfrak{k}) = \sum_i b_i \otimes d_i \in U(\mathfrak{g}) \otimes C(\mathfrak{p}), \tag{1}$$

where b_i is any basis of $\mathfrak{p}$ and d_i is the dual basis with respect to B. It is easy to see that D is independent of the choice of b_i and K-invariant for the adjoint action on both factors.

Let X be a $(\mathfrak{g}, K)$-module and let S be a spin module for $C(\mathfrak{p})$. Recall that S is constructed as the exterior algebra of a maximal isotropic subspace U of $\mathfrak{p}$. The elements of U act on S by wedging and the elements of the dual isotropic subspace U^* act by contractions. Finally, if $\mathfrak{p}$ is odd-dimensional, there are two choices for the action of the orthogonal of $U \oplus U^*$. For more facts about Cliford algebras and spinors, see [4], [12] or [9]. Then D acts on $X \otimes S$ and we define Dirac cohomology of X as

$$H_D(X) = \operatorname{Ker} D / \operatorname{Im} D \cap \operatorname{Ker} D.$$

Then $H_D(X)$ is a module for the spin double cover $\tilde{K}$ of K. If X is unitary, then D is skew symmetric with respect to a natural inner product on $X \otimes S$, and therefore

$$H_D(X) = \operatorname{Ker} D = \operatorname{Ker} D^2. \tag{2}$$

By a result of Parthasarathy [16], D^2 is given by

$$D^2 = (\operatorname{Cas}_\mathfrak{g} \otimes 1 + \|\rho_\mathfrak{g}\|^2) - (\Delta(\operatorname{Cas}_\mathfrak{k}) + \|\rho_\mathfrak{k}\|^2). \tag{3}$$

Here $\mathrm{Cas}_{\mathfrak{g}}$ resp. $\mathrm{Cas}_{\mathfrak{k}}$ denote the Casimir elements of $U(\mathfrak{g})$ resp. $U(\mathfrak{k})$, $\rho_{\mathfrak{g}}$ and $\rho_{\mathfrak{k}}$ are the half sums of positive roots for $\mathfrak{g}$ resp. $\mathfrak{k}$, and Δ denotes the diagonal embedding of $\mathfrak{k}$ into $U(\mathfrak{g})\otimes C(\mathfrak{p})$. More precisely,

$$\Delta(X)=X\otimes 1+1\otimes\alpha(X),$$

where $\alpha:\mathfrak{k}\to\mathfrak{so}(\mathfrak{p})\cong\bigwedge^2\mathfrak{p}\hookrightarrow C(\mathfrak{p})$ is the action map followed by the skew symmetrization map $X\wedge Y\mapsto\frac{1}{2}(XY-YX)$.

To make use of (3), let $\mathfrak{h}=\mathfrak{t}\oplus\mathfrak{a}$ be a fundamental Cartan subalgebra of $\mathfrak{g}$. In other words, $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$ while $\mathfrak{a}$ is the centralizer of $\mathfrak{t}$ in $\mathfrak{p}$. We view $\mathfrak{t}^*$ as a subspace of $\mathfrak{h}^*$ by extending functionals on $\mathfrak{t}$ to functionals on $\mathfrak{h}$ which act by zero on $\mathfrak{a}$. Then if X is unitary and has infinitesimal character $\Lambda\in\mathfrak{h}^*$, we can use (2) and (3) to conclude that an irreducible $\tilde{K}$-submodule $E(\tau)$ of $X\otimes S$ with highest weight $\tau\in\mathfrak{t}^*\subset\mathfrak{h}^*$ is contained in $H_D(X)$ if and only if

$$\|\Lambda\|^2=\|\tau+\rho_{\mathfrak{k}}\|^2.$$

By the main result of [8], this is in fact equivalent to the seemingly stronger requirement that

$$\Lambda=w(\tau+\rho_{\mathfrak{k}})$$

for some w in the Weyl group of $\mathfrak{g}$.

Let us now assume that the pair $(\mathfrak{g},\mathfrak{k})$ is Hermitian symmetric. Then $\mathfrak{p}=\mathfrak{p}^+\oplus\mathfrak{p}^-$, where $\mathfrak{p}^\pm$ are K-invariant abelian subalgebras of $\mathfrak{g}$. It follows that we can pick the dual maximal isotropic subspaces of $\mathfrak{p}$ to be $\mathfrak{p}^+$ and $\mathfrak{p}^-$, and the spin module S to be $\bigwedge\mathfrak{p}^+$. Let v_i be a basis of $\mathfrak{p}^+$ and let v_i^* be the dual basis of $\mathfrak{p}^-$. Then the Dirac operator D can be written as the sum $C+C^-$, where

$$C=C(\mathfrak{g},\mathfrak{k})=\sum_i v_i^*\otimes v_i\qquad\text{and}\qquad C^-=C^-(\mathfrak{g},\mathfrak{k})=\sum_i v_i\otimes v_i^*\tag{4}$$

are K-invariant elements of $U(\mathfrak{g})\otimes C(\mathfrak{p})$, both squaring to zero. As shown in [11], Section 2, $X\otimes S$ can be interpreted as the space of chains for the $\mathfrak{p}^+$-homology of X, or as the space of cochains for the $\mathfrak{p}^-$-cohomology of X. Under these identifications, C corresponds to the $\mathfrak{p}^-$-cohomology differential, while C^- corresponds to a multiple of the $\mathfrak{p}^+$-homology differential.

Moreover, by the results of Section 7 of [11], if X is unitary, then the Dirac cohomology of X can be expressed as $\operatorname{Ker}C\cap\operatorname{Ker}C^-$, and it can be viewed as the space of harmonic representatives for both the $\mathfrak{p}^-$-cohomology and the $\mathfrak{p}^+$-homology of X. In particular, the Dirac cohomology, the $\mathfrak{p}^-$-cohomology and the $\mathfrak{p}^+$-homology of X all coincide as vector spaces, and also as $\mathfrak{k}$-modules if one introduces appropriate modular twists.

More generally, if $\mathfrak{q}=\mathfrak{l}\oplus\mathfrak{u}$ is a θ-stable parabolic subalgebra of $\mathfrak{g}$, such that $\mathfrak{l}\subseteq\mathfrak{k}$ and $\mathfrak{u}\supseteq\mathfrak{p}^+$, then analogous results are true for the $\bar{\mathfrak{u}}$-cohomology of X, the $\mathfrak{u}$-homology of X, and the Dirac cohomology of X with respect to Kostant's cubic Dirac operator $D(\mathfrak{g},\mathfrak{l})$. Moreover, each of these three kinds of cohomology can be calculated in stages, by first calculating cohomology with respect to the pair $(\mathfrak{g},\mathfrak{k})$, and then the cohomology with respect to the pair $(\mathfrak{k},\mathfrak{l})$. See [11], Theorem 6.1, Corollary 7.9 and Theorem 7.11.

3. A description of the Lie algebra $\mathfrak{sp}(4)$

We use the construction of symplectic Lie algebras inside Weyl algebras, parallel to the construction of orthogonal Lie algebras inside Clifford algebras. See [1, 3, 6] or [17].

Let $\mathbb{D}(2)$ be the Weyl algebra of differential operators on $\mathbb{C}^2$ with polynomial coefficients. We will identify $\mathfrak{g} = \mathfrak{sp}(4, \mathbb{C})$ with the Lie subalgebra of $\mathbb{D}(2)$ spanned by the following (total) degree two elements:

$$\begin{array}{lll} h_1 = z_1\partial_1 + \frac{1}{2}, & h_2 = z_2\partial_2 + \frac{1}{2}; & \\ u_1 = z_1\partial_2, & u_1^* = -2z_2\partial_1; & \\ v_1 = z_1^2, & v_2 = z_1 z_2, & v_3 = z_2^2; \\ v_1^* = \partial_1^2, & v_2^* = 2\partial_1\partial_2, & v_3^* = \partial_2^2. \end{array}$$

Note that the elements of $\mathbb{D}(2)$ of pure degree are obtained by symmetrization from their symbols. This explains the presence of summands $\frac{1}{2}$ in the definition of h_1 and h_2.

Using the commutation rules of $\mathbb{D}(2)$ (all generators commute, except $[\partial_i, z_i] = 1$), it is easy to get the commutator table for the above basis of $\mathfrak{g}$:

	h_2	u_1	u_1^*	v_1	v_2	v_3	v_1^*	v_2^*	v_3^*
h_1	0	u_1	$-u_1^*$	$2v_1$	v_2	0	$-2v_1^*$	$-v_2^*$	0
h_2		$-u_1$	u_1^*	0	v_2	$2v_3$	0	$-v_2^*$	$-2v_3^*$
u_1			$-2h_1 + 2h_2$	0	v_1	$2v_2$	$-v_2^*$	$-2v_3^*$	0
u_1^*				$-4v_2$	$-2v_3$	0	0	$4v_1^*$	$2v_2^*$
v_1					0	0	$-4h_1$	$-4u_1$	0
v_2						0	u_1^*	$-2h_1 - 2h_2$	$-2u_1$
v_3							0	$2u_1^*$	$-4h_2$
v_1^*								0	0
v_2^*									0

(These commutators can be easily calculated using Maple.)

Let us denote by $\mathfrak{k}$ the copy of $\mathfrak{gl}(2, \mathbb{C})$ spanned by h_1, h_2, u_1 and u_1^*, and by $\mathfrak{p}$ the subspace of $\mathfrak{g}$ spanned by v_i and v_i^* $(i = 1, 2, 3)$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition corresponding to the group $\mathrm{Sp}(4, \mathbb{R})$. Moreover, $\mathfrak{k}$ is the complexified Lie algebra of the maximal compact subgroup K of G mentioned in the introduction. The elements h_1 and h_2 span the Cartan subalgebra $\mathfrak{t}$ for both $\mathfrak{g}$ and $\mathfrak{k}$; in particular, both $\mathfrak{g}$ and $\mathfrak{k}$ are of rank 2. Furthermore, $h_1 + h_2$ spans the center of $\mathfrak{k}$, v_i span the abelian subalgebra $\mathfrak{p}^+$ of $\mathfrak{p}$ and v_i^* span the abelian subalgebra $\mathfrak{p}^-$ of $\mathfrak{p}$.

In terms of roots, we can identify $\mathfrak{t}^*$ with $\mathbb{C}^2$ using the basis $\varepsilon_1 = (1, 0)$, $\varepsilon_2 = (0, 1)$ dual to the basis h_1, h_2 of $\mathfrak{t}$. This gives exactly standard coordinates for both $\mathfrak{k} = \mathfrak{gl}(2, \mathbb{C})$ and $\mathfrak{g} = \mathfrak{sp}(4, \mathbb{C})$. Then u_1 and v_2 correspond to the short positive roots $\varepsilon_1 - \varepsilon_2$ resp. $\varepsilon_1 + \varepsilon_2$, while v_1 and v_3 correspond to the long positive roots $2\varepsilon_1$ resp. $2\varepsilon_2$. An element with a star always corresponds to the opposite root from the corresponding element without a star.

The normalizations are picked so that u_1^* resp. v_i^* are dual to u_1 resp. v_i with respect to the (suitably normalized) Killing form.

4. The Weil representation

With the above description of $\mathfrak{g}$, describing the (($\mathfrak{g}, K$)-module of the) Weil representation is natural and easy: just take the natural representation of $\mathbb{D}(2)$ on the space $W = P(\mathbb{C}^2)$ of polynomials in z_1 and z_2, and restrict the action to $\mathfrak{g}$. See [1, 3, 6] or [17].

Let us check that the restriction of the action of $\mathfrak{g}$ on W to $\mathfrak{k}$ exponentiates to the group K, the maximal compact subgroup of $\mathrm{Mp}(4, \mathbb{R})$.

It is clear that u_1 sends $z_1^i z_2^j$ to $j z_1^{i+1} z_2^{j-1}$, while u_1^* sends $z_1^i z_2^j$ to $i z_1^{i-1} z_2^{j+1}$. It follows that the space of polynomials of a fixed degree is invariant and irreducible for $\mathfrak{k}$. Moreover, the highest weight vectors are of the form z_1^n and since h_1 acts on z_1^n by the scalar $n + \frac{1}{2}$ and h_2 by the scalar $\frac{1}{2}$, the highest weights appearing are of the form $(n + \frac{1}{2}, \frac{1}{2})$. Here we are using the standard coordinates on $\mathfrak{t}^*$ described at the end of the previous section. Since K is the double cover of $U(2)$, any $\mathfrak{k}$-weight with integral or half-integral coordinates is analytically integral for K. So indeed K acts on W, and this action is locally finite.

The above calculation also shows that all the K-types lie on the line $y = \frac{1}{2}$ in $\mathfrak{t}^*$, and that the multiplicity of each K-type is one.

It is now also clear that W decomposes as the sum of two irreducible modules for $\mathfrak{g}$, W_0 resp. W_1, consisting of polynomials of even resp. odd degree. Namely, the action of each of the v_i's and v_i^*'s from $\mathfrak{p}$ changes the degree of a polynomial by two. Finally, we also see that both W_0 and W_1 are highest weight modules with respect to the parabolic subalgebra $\mathfrak{q}^- = \mathfrak{k} \oplus \mathfrak{p}^-$ of $\mathfrak{g}$.

All this can be summarized by the following two pictures representing the bases of W_0 and W_1:

$$\begin{array}{ccccc}
 & & & & \cdots \\
 & & & z_1^6 & \cdots \\
 & & z_1^4 & z_1^5 z_2 & \cdots \\
 & z_1^2 & z_1^3 z_2 & z_1^4 z_2^2 & \cdots \\
1 & z_1 z_2 & z_1^2 z_2^2 & z_1^3 z_2^3 & \cdots \\
 & z_2^2 & z_1 z_2^3 & z_1^2 z_2^4 & \cdots \\
 & & z_2^4 & z_1 z_2^5 & \cdots \\
 & & & z_2^6 & \cdots \\
 & & & & \cdots
\end{array}
\qquad\qquad
\begin{array}{cccc}
 & & & \cdots \\
 & & z_1^5 & \cdots \\
 & z_1^3 & z_1^4 z_2 & \cdots \\
z_1 & z_1^2 z_2 & z_1^3 z_2^2 & \cdots \\
z_2 & z_1 z_2^2 & z_1^2 z_2^3 & \cdots \\
 & z_2^3 & z_1 z_2^4 & \cdots \\
 & & z_2^5 & \cdots \\
 & & & \cdots
\end{array}
\tag{5}$$

The columns of each picture represent the K-types. The action of $\mathfrak{p}^+$ is to the right (raising the degree), while the action of $\mathfrak{p}^-$ is to the left (lowering the degree). We leave it to the reader to plot the weights of each of the monomials in our standard coordinates; the pictures are analogous but different from the ones above.

5. Dirac cohomology of W with respect to $D(\mathfrak{g}, \mathfrak{k})$

We choose $S = \bigwedge(\mathfrak{p}^+)$ for the space of spinors for $\mathfrak{p}$. So v_i act on S by exterior multiplication, while v_i^* are antiderivations defined on generators by

$$v_i^* \cdot v_j = 2\delta_{ij}.$$

In view of (1), we can write the Dirac operator for the pair $(\mathfrak{g},\mathfrak{k})$ as

$$\begin{aligned} D = D(\mathfrak{g},\mathfrak{k}) &= \sum v_i^* \otimes v_i + \sum v_i \otimes v_i^* \\ &= \partial_1^2 \otimes v_1 + 2\partial_1\partial_2 \otimes v_2 + \partial_2^2 \otimes v_3 + z_1^2 \otimes v_1^* + z_1 z_2 \otimes v_2^* + z_2^2 \otimes v_3^*. \end{aligned}$$

Here the first three summands give the $\mathfrak{p}^-$-cohomology operator C, while the last three summands give the $\mathfrak{p}^+$-homology operator C^-.

By the results of [11] explained in Section 2,

$$H_D(W) = \operatorname{Ker} D = \operatorname{Ker} C \cap \operatorname{Ker} C^-.$$

So we need to look for solutions X of $C(X) = C^-(X) = 0$. Clearly, we can assume that X is homogeneous with respect to the second (spinor) factor.

If a solution has degree 0 with respect to the second variable, then it is of the form $P\otimes 1$ for some polynomial P. This is automatically killed by C^-, while $C(P\otimes 1) = 0$ gives

$$\partial_1^2 P = 0; \qquad 2\partial_1\partial_2 P = 0; \qquad \partial_2^2 P = 0.$$

This is satisfied if and only if $\deg P \leq 1$, and we get the solutions

$$\begin{gathered} 1 \otimes 1; \\ z_1 \otimes 1, \qquad z_2 \otimes 1. \end{gathered}$$

If a solution has degree 1 with respect to the second variable, then it is of the form $X = P \otimes v_1 + Q \otimes v_2 + R \otimes v_3$ for some polynomials P, Q and R. Writing out $C(X) = 0$, we get

$$\partial_1^2 Q - 2\partial_1\partial_2 P = 0; \tag{6}$$

$$\partial_1^2 R - \partial_2^2 P = 0; \tag{7}$$

$$2\partial_1\partial_2 R - \partial_2^2 Q = 0, \tag{8}$$

and writing out $C^-(X) = 0$ we get

$$z_1^2 P + z_1 z_2 Q + z_2^2 R = 0. \tag{9}$$

To solve this system of equations, let us first note that all the operators applied to P, Q and R in the above equations are homogeneous with respect to each of the variables. Consequently, we can assume P, Q and R are monomials.

Note that (7) can be satisfied either trivially, when $\partial_1^2 R = \partial_2^2 P = 0$, or nontrivially, when both $\partial_1^2 R$ and $\partial_2^2 P$ are nonzero. If it is satisfied nontrivially, then we can assume

$$P = p z_1^i z_2^{j+2}, \qquad R = r z_1^{i+2} z_2^j,$$

where $p, r \in \mathbb{C}^*$, i, j are nonnegative integers, and

$$p(j+2)(j+1) = r(i+2)(i+1).$$

In particular, p/r is a positive real number.

Substituting these expressions for P and R in (9) and simplifying, we see that

$$Q = -(p+r)z_1^{i+1}z_2^{j+1}.$$

Now (6) becomes $-(p+r)(i+1)i = 2pi(j+2)$, that is

$$-ri(i+1) = pi(2j+i+5).$$

If $i \neq 0$, we get that p/r is a negative real number, a contradiction. Hence $i = 0$. Similarly, (8) implies that $j = 0$. Since (7) now gives $p = r$, we see that we are getting a unique solution (up to a scalar):

$$z_2^2 \otimes v_1 - 2z_1z_2 \otimes v_2 + z_1^2 \otimes v_3.$$

If (7) is satisfied trivially, then $P = pz_1^iz_2$ while $R = rz_1z_2^j$, where $p, r \in \mathbb{C}$ and i, j are nonnegative integers. Namely, by (9), P is divisible by z_2, while R is divisible by z_1. Now (9) becomes

$$Q = -pz_1^{i+1} - rz_2^{j+1}.$$

So if we want a nonzero solution, p and r cannot both be 0. From (6), we get

$$pi(i+3) = 0,$$

so either $p = 0$ or $i = 0$. Similarly, from (8) we see that either $r = 0$ or $j = 0$. If $p = 0$, then $r \neq 0$, so $j = 0$. Taking $r = 1$, we get $R = z_1$ and $Q = -z_2$, i.e., we obtain the solution

$$-z_2 \otimes v_2 + z_1 \otimes v_3.$$

If $r = 0$, then $p \neq 0$, so $i = 0$. Taking $p = 1$, we get $P = z_2$ and $Q = -z_1$, i.e., we obtain the solution

$$z_2 \otimes v_1 - z_1 \otimes v_2.$$

If $p \neq 0$ and $r \neq 0$, then i and j are 0, so $P = pz_2$, $R = rz_1$, $Q = -pz_1 - rz_2$; this solution is a linear combination of the above solutions.

A solution of degree 2 with respect to the second variable is of the form $X = P \otimes v_1v_2 + Q \otimes v_1v_3 + R \otimes v_2v_3$. Writing out $C(X) = 0$, we get

$$\partial_1^2 R - 2\partial_1\partial_2 Q + \partial_2^2 P = 0, \tag{10}$$

and writing out $C^-(X) = 0$ we get

$$z_1^2P - z_2^2R = 0; \qquad -z_1z_2P - z_2^2Q = 0; \qquad z_1^2Q + z_1z_2R = 0. \tag{11}$$

From the first equation in (11), we conclude that $P = z_2^2P_1$ and $R = z_1^2P_1$ for some polynomial P_1. The second and third equation of (11) now both become $Q = -z_1z_2P_1$. Substituting into (10), we get

$$(\partial_1^2 z_1^2 + 2\partial_1\partial_2 z_1z_2 + \partial_2^2 z_2^2)P_1 = 0.$$

This has no nonzero solutions, since the operator in parentheses acts on any monomial $z_1^nz_2^k$ by a positive scalar.

A solution of degree 3 with respect to the second variable is of the form $X = P \otimes v_1 v_2 v_3$. This is automatically killed by C, while $C^-(X) = 0$ gives

$$z_1^2 P = 0; \qquad z_1 z_2 P = 0; \qquad z_2^2 P = 0.$$

So we see that there are no nonzero solutions of degree 3.

So we obtained that the Dirac cohomology of W_0 is spanned by

$$1 \otimes 1 \qquad \text{and} \qquad z_2^2 \otimes v_1 - 2z_1 z_2 \otimes v_2 + z_1^2 \otimes v_3,$$

while the Dirac cohomology of W_1 is spanned by

$$z_1 \otimes 1, \qquad z_2 \otimes 1, \qquad z_2 \otimes v_1 - z_1 \otimes v_2 \qquad \text{and} \qquad -z_2 \otimes v_2 + z_1 \otimes v_3.$$

It remains to determine the $\tilde{K}$-action on the Dirac cohomology. (Recall from Section 2 that the spin double cover $\tilde{K}$ of K acts on the Dirac cohomology.) To do this, we first describe the diagonally embedded copy $\mathfrak{k}_\Delta$ of $\mathfrak{k}$. We will use the following formula for $\alpha : \mathfrak{k} \to C(\mathfrak{p})$:

$$\alpha(X) = \frac{1}{4} \sum_i ([X, v_i] v_i^* + [X, v_i^*] v_i). \tag{12}$$

This can easily be proved from formula (3) in Section 2 of [11]. (See also [9], Chapter 2, but note that the sign is opposite there, because of different conventions.) It is now clear from the commutation table in Section 3 that $\mathfrak{k}_\Delta$ is spanned by

$$\begin{aligned} h_{1\Delta} &= h_1 \otimes 1 - 1 \otimes (v_1^* v_1 + \frac{1}{2} v_2^* v_2) + \frac{3}{2} \otimes 1, \\ h_{2\Delta} &= h_2 \otimes 1 - 1 \otimes (\frac{1}{2} v_2^* v_2 + v_3^* v_3) + \frac{3}{2} \otimes 1, \\ u_{1\Delta} &= -\frac{1}{2} v_2^* v_1 - v_3^* v_2, \\ u_{1\Delta}^* &= 2 v_1^* v_2 + v_2^* v_3. \end{aligned} \tag{13}$$

Acting by these elements on the solutions obtained above, and denoting by H_D^0 resp. H_D^1 the even resp. odd Dirac cohomology (with respect to the spinor degree), we obtain the following result.

Theorem 1. *Dirac cohomology of the Weil representation consists of the following modules for the spin double cover $\tilde{K}$ of K:*

$H_D^0(\mathfrak{g}, \mathfrak{k}; W_0)$ *is the one-dimensional $\tilde{K}$-module with highest weight* $(-1, -1)$*, spanned by* $1 \otimes 1$.

$H_D^1(\mathfrak{g}, \mathfrak{k}; W_0)$ *is the one-dimensional $\tilde{K}$-module with highest weight* $(1, 1)$*, spanned by* $z_2^2 \otimes v_1 - 2z_1 z_2 \otimes v_2 + z_1^2 \otimes v_3$.

$H_D^0(\mathfrak{g}, \mathfrak{k}; W_1)$ *is the two-dimensional $\tilde{K}$-module with highest weight* $(0, -1)$*, spanned by* $z_1 \otimes 1$ *(of weight* $(0, -1)$*) and* $z_2 \otimes 1$ *(of weight* $(-1, 0)$*).*

$H_D^1(\mathfrak{g}, \mathfrak{k}; W_1)$ *is the two-dimensional $\tilde{K}$-module with highest weight* $(1, 0)$*, spanned by* $z_2 \otimes v_1 - z_1 \otimes v_2$ *(of weight* $(1, 0)$*) and* $-z_2 \otimes v_2 + z_1 \otimes v_3$ *(of weight* $(0, 1)$*).*

Remark 1. *One could also obtain the Dirac cohomology of W by using the result from [11] which says it is the same as $\mathfrak{p}^-$-cohomology up to a ρ-shift, and then using the results in [1] or [5] to determine the $\mathfrak{p}^-$-cohomology. Our present approach is however more direct and gives explicit (harmonic) representatives.*

It is also possible to use the approach of [10] to get the K-types of $H_D(W)$ in a rather direct fashion. Finding the explicit vectors in $H_D(W) \subset W \otimes S$ as we did above would however require additional work.

6. Dirac cohomology of W with respect to $D(\mathfrak{g}, \mathfrak{t})$

By the results of [11], we know that we can get $H_D(\mathfrak{g}, \mathfrak{t}; W)$ as the Dirac cohomology of the $\mathfrak{k}$-module $H_D(\mathfrak{g}, \mathfrak{k}; W)$ with respect to $D(\mathfrak{k}, \mathfrak{t})$. Since $H_D(\mathfrak{g}, \mathfrak{k}; W)$ is finite-dimensional, we can apply Kostant's formula ([14], Theorem 5.1), which says

$$H_D(\mathfrak{k}, \mathfrak{t}; V(\mu)) = \bigoplus_{w \in W_{\mathfrak{k}}} \mathbb{C}_{w(\mu+\rho_{\mathfrak{k}})},$$

where $V(\mu)$ denotes the finite-dimensional $\mathfrak{k}$-module with highest weight μ, and $\mathbb{C}_\lambda$ denotes the one-dimensional $\mathfrak{t}$-module of weight λ.

So we can start from the description of $H_D(\mathfrak{g}, \mathfrak{k}; W)$ from Theorem 1. Then we note that $\rho_{\mathfrak{k}} = (\frac{1}{2}, -\frac{1}{2})$ and that $W_{\mathfrak{k}} \cong \mathbb{Z}_2$, with the nontrivial element being $(x, y) \mapsto (y, x)$. Now a short calculation gives the following result.

Proposition 1. *$H_D(\mathfrak{g}, \mathfrak{t}; W_0)$ is a direct sum of four one-dimensional spaces, of respective weights $(-\frac{1}{2}, -\frac{3}{2})$, $(-\frac{3}{2}, -\frac{1}{2})$, $(\frac{3}{2}, \frac{1}{2})$ and $(\frac{1}{2}, \frac{3}{2})$.*

$H_D(\mathfrak{g}, \mathfrak{t}; W_1)$ is a direct sum of four one-dimensional spaces, of respective weights $(\frac{1}{2}, -\frac{3}{2})$, $(-\frac{3}{2}, \frac{1}{2})$, $(\frac{3}{2}, -\frac{1}{2})$ and $(-\frac{1}{2}, \frac{3}{2})$.

We now want to describe $H_D(W)$ more explicitly by finding the harmonic representatives in $W \otimes S$, where S is the space of spinors for $(\mathfrak{g}, \mathfrak{t})$. We start from the harmonic representatives for $H_D(\mathfrak{g}, \mathfrak{k}; W)$ from Theorem 1. We tensor these with the space of spinors for $(\mathfrak{k}, \mathfrak{t})$, $S_{\mathfrak{k}} = \bigwedge(\mathbb{C}u_1)$. Then we find $\operatorname{Ker} D_\Delta(\mathfrak{k}, \mathfrak{t}) = \operatorname{Ker} C_\Delta(\mathfrak{k}, \mathfrak{t}) \cap \operatorname{Ker} C_\Delta^-(\mathfrak{k}, \mathfrak{t})$ on thus obtained space. For this we first calculate $C_\Delta(\mathfrak{k}, \mathfrak{t})$ and $C_\Delta^-(\mathfrak{k}, \mathfrak{t})$. Using the formula (13), we get

$$\begin{aligned} C_\Delta(\mathfrak{k}, \mathfrak{t}) &= -2z_2\partial_1 \otimes u_1 + 2 \otimes u_1 v_1^* v_2 + 1 \otimes u_1 v_2^* v_3; \\ C_\Delta^-(\mathfrak{k}, \mathfrak{t}) &= z_1\partial_2 \otimes u_1^* - \frac{1}{2} \otimes u_1^* v_2^* v_1 - 1 \otimes u_1^* v_3^* v_2. \end{aligned}$$

Now acting upon our candidates for harmonic representatives, we see that all elements of $H_D(\mathfrak{g}, \mathfrak{k}; W_0) \otimes S_{\mathfrak{k}}$ are killed by both $C_\Delta(\mathfrak{k}, \mathfrak{t})$ and $C_\Delta^-(\mathfrak{k}, \mathfrak{t})$. That is, the space of harmonic representatives for $H_D(\mathfrak{g}, \mathfrak{t}; W_0)$ is spanned by

$$\begin{aligned} 1 \otimes 1, &\qquad 1 \otimes u_1; \\ z_2^2 \otimes v_1 - 2z_1z_2 \otimes v_2 + z_1^2 \otimes v_3, &\qquad z_2^2 \otimes u_1v_1 - 2z_1z_2 \otimes u_1v_2 + z_1^2 \otimes u_1v_3. \end{aligned}$$

In $H_D(\mathfrak{g}, \mathfrak{k}; W_1) \otimes S_{\mathfrak{k}}$, $\operatorname{Ker} C_\Delta(\mathfrak{k}, \mathfrak{t}) \cap \operatorname{Ker} C_\Delta^-(\mathfrak{k}, \mathfrak{t})$ is spanned by the elements

$$z_2 \otimes 1, \qquad z_1 \otimes u_1, \qquad -z_2 \otimes v_2 + z_1 \otimes v_3, \qquad z_2 \otimes u_1v_1 - z_1 \otimes u_1v_2;$$

so these elements span the space of harmonic representatives for $H_D(\mathfrak{g}, \mathfrak{t}; W_1)$.

We leave it to the reader to identify the weights of these eight elements. Of course, these are the weights appearing in Proposition 1.

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