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Dirac operators on Weil representations II*

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Abstract. Let G be a metaplectic double cover of the group G of four-by-four real symplectic matrices. Let $\mathfrak{g}$ be the complexified Lie algebra of G. Denote by W_0 and W_1 the Harish-Chandra modules of the even and odd Weil representations of G, respectively. We find the Dirac cohomology of W_0 and W_1 with respect to a noncompact Levi subalgebra $\mathfrak{l}$ of a θ-stable parabolic subalgebra of $\mathfrak{g}$. The results can be considered as counterexamples to certain generalizations of the main results of [9].

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1. Introduction

In this note we continue the work of [13] with treating a much more difficult case of Dirac cohomology with respect to a noncompact subalgebra $\mathfrak{l}$ of $\mathfrak{g} = \mathfrak{sp}(4,\mathbb{C})$. The modules we consider are the same as in [13]: the even and odd Weil representations of $\mathfrak{g}$, W_0 and W_1. These are also called Segal-Shale-Weil representations. See [17] or [2], Ch. VIII. For the approach we adopt see [1, 3, 5] and [15].

The subalgebra $\mathfrak{l}$ is a Levi subalgebra of a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ of $\mathfrak{g}$, and it is isomorphic to $\mathfrak{gl}(2,\mathbb{C})$, just as $\mathfrak{k}$, the complexified Lie algebra of the maximal compact subgroup K of $G = \mathrm{Mp}(4,\mathbb{R})$. The obtained results are however in sharp contrast with those for $\mathfrak{k}$. None of the results of [9], that would apply if $\mathfrak{l}$ were contained in $\mathfrak{k}$, works here. For example, even though the modules we study decompose discretely under $\mathfrak{l}$, the square of the Dirac operator does not act semisimply. The Dirac cohomology is not the same as the $\mathfrak{u}$-homology or the $\bar{\mathfrak{u}}$-cohomology. The differentials for the $\mathfrak{u}$-homology and the $\bar{\mathfrak{u}}$-cohomology, C and C^-, are not disjoint. Finally, Dirac cohomology cannot be calculated in stages with respect to the compact Cartan subalgebra $\mathfrak{t} \subset \mathfrak{l}$. See Section 6 for more details.

We will keep the notation of [13]. We refer to Introduction and Section 2 of [13] (and to other references) for the definitions of Dirac operators and Dirac cohomology and for their basic properties, as well as for facts about the structure of $\mathfrak{g}$.

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2. A Levi subalgebra of $\mathfrak{g}$

Let $\mathfrak{l} \subset \mathfrak{g}$ be the subalgebra spanned by $\mathfrak{t}$, and the short noncompact root vectors $v_2 = z_1 z_2$ and $v_2^* = 2\partial_1\partial_2$. It is clear from the commutation relations of $\mathfrak{g}$ that $\mathfrak{l}$ is a subalgebra isomorphic to $\mathfrak{gl}(2,\mathbb{C})$. Note that $h_1 - h_2$ is central in $\mathfrak{l}$, while $h_1 + h_2$ is in $[\mathfrak{l},\mathfrak{l}]$ and satisfies $[h_1 + h_2, v_2] = 2v_2$ and $[h_1 + h_2, v_2^*] = -2v_2^*$. Therefore, we change the basis of $\mathfrak{t}$ to h_1 and $\tilde{h}_2 = -h_2$ to get the standard $\mathfrak{gl}(2)$-coordinates.

Moreover, $\mathfrak{l}$ is a noncompact Levi subalgebra of $\mathfrak{g}$: a corresponding parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ can be defined by setting $\mathfrak{u}$ to be the span of v_1, u_1 and v_3^*. (Note the change of the positive root system compared to [13], where v_3 was a positive root vector and not v_3^*.) In particular, Kostant's cubic Dirac operator $D(\mathfrak{g},\mathfrak{l})$ is defined (but since $\mathfrak{l}$ is a symmetric subalgebra of $\mathfrak{g}$, the cubic term actually vanishes).

Recall the pictures representing bases of the irreducible $(\mathfrak{g},K)$-submodules W_0 and W_1 of the Weil representation W of $\mathfrak{g}$ given in equation (4.1) of [13]. It is clear that the rows of each picture are invariant and irreducible for $\mathfrak{l}$. Note that all these rows are lowest weight $\mathfrak{gl}(2,\mathbb{C})$-modules (we are fixing a choice of positive roots for $\mathfrak{l}$ such that $v_2 = z_1 z_2$ corresponds to the positive root). The lowest weights appearing in W are

$$(n + \frac{1}{2}, -\frac{1}{2}); \qquad (\frac{1}{2}, -n - \frac{1}{2}), \tag{1}$$

for each positive integer n, with the corresponding lowest weight vectors z_1^n and z_2^n, respectively, and

$$(\frac{1}{2}, -\frac{1}{2}), \tag{2}$$

with the corresponding weight vector 1. These belong to W_0 if n is even, and to W_1 if n is odd. (The weight $(\frac{1}{2}, -\frac{1}{2})$ corresponds to $1 \in W_0$.)

Remark 1. *If we denote by $\mathfrak{g}_1$ the subalgebra of $\mathfrak{g}$ spanned by $h_1 + \tilde{h}_2$, and by $\mathfrak{g}_2$ the subalgebra of $\mathfrak{g}$ spanned by $h_1 - \tilde{h}_2$, v_2 and v_2^*, then it is well known (and easy to check) that $\mathfrak{g}_1 \times \mathfrak{g}_2 \cong \mathfrak{o}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$ is a (complexified) compact dual pair in $\mathfrak{g}$. Therefore, our decomposition is a very special case of general results of Howe [5, 6].*

3. Tensoring some $\mathfrak{gl}(2,\mathbb{C})$-modules

To calculate the Dirac cohomology of W with respect to $D(\mathfrak{g},\mathfrak{l})$, we first need to understand the $\mathfrak{l}$-module $W \otimes S$. Here $S = \bigwedge \mathfrak{u}$ is the spin module for $C(\mathfrak{u} \oplus \bar{\mathfrak{u}})$. Since $\bigwedge^0 \mathfrak{u}$ and $\bigwedge^3 \mathfrak{u}$ are one-dimensional, it is obvious how to tensor any module with them. So what we need to study are the tensor products of the lowest weight $\mathfrak{l}$-modules appearing in W with three-dimensional $\mathfrak{l}$-modules $\bigwedge^1 \mathfrak{u}$ and $\bigwedge^2 \mathfrak{u}$. In this section we do the analysis of such tensor products for slightly more general modules.

Let $V_{(a,b)}^{\text{low}}$ be a lowest weight $\mathfrak{l} = \mathfrak{gl}(2,\mathbb{C})$-module with lowest weight (a,b), on which v_2 acts freely (i.e., v_2 is an injective linear operator). Then $V_{(a,b)}^{\text{low}}$ is spanned by the weight vectors $x_{(a,b)}$, $x_{(a+1,b-1)}$, $x_{(a+2,b-2)}, \dots$, of respective weights $(a,b), (a+1,b-1), (a+2,b-2), \dots$, such that

$$x_{(a+k,b-k)} = v_2^k\, x_{(a,b)}, \qquad k \in \mathbb{Z}_+.$$

It follows by induction on k that

$$v_2^*\, x_{(a+k,b-k)} = 2k(a-b+k-1)x_{(a+k-1,b-k+1)}.$$

We see that if $a-b \geq 1$, then $V_{(a,b)}^{\mathrm{low}}$ is an irreducible $\mathfrak{l}$-module. In particular, the $\mathfrak{l}$-module W is a direct sum of such modules, for (a,b) as in (1) and (2).

If $a-b \leq 0$, then $V_{(a,b)}^{\mathrm{low}}$ is reducible. Namely, the above formula implies that $v_2^*\, x_{b+1,a-1} = 0$. It follows that $V_{(a,b)}^{\mathrm{low}}$ contains $V_{(b+1,a-1)}^{\mathrm{low}}$ as an irreducible submodule, and the quotient is equal to the finite-dimensional module $F(b,a)$ with highest weight (b,a). We are mostly interested in the irreducible case, but as we shall see, the reducible $V_{(a,b)}^{\mathrm{low}}$ will also appear along the way.

We now want to describe $V_{(a,b)}^{\mathrm{low}} \otimes F(c,c-2)$, where $F(c,c-2)$ is a three-dimensional $\mathfrak{l}$-module with highest weight $(c,c-2)$. The weights of $F(c,c-2)$ are $(c,c-2)$, $(c-1,c-1)$ and $(c-2,c)$. We denote by $y_{(c,c-2)}$, $y_{(c-1,c-1)}$ and $y_{(c-2,c)}$ the corresponding weight vectors, normalized so that

$$v_2^*\, y_{(c,c-2)} = y_{(c-1,c-1)}, \qquad v_2^*\, y_{(c-1,c-1)} = y_{(c-2,c)}.$$

It follows that

$$v_2\, y_{(c-2,c)} = -4y_{(c-1,c-1)}, \qquad v_2\, y_{(c-1,c-1)} = -4y_{(c,c-2)}.$$

A short calculation gives

Lemma 1. *Assume that $a-b \geq 1$, so that $V_{(a,b)}^{\mathrm{low}}$ is irreducible. Then the lowest weight vectors of $V_{(a,b)}^{\mathrm{low}} \otimes F(c,c-2)$ form a three-dimensional space spanned by the vectors*

$$\begin{aligned}
c_1 &= x_{(a,b)} \otimes y_{(c-2,c)};\\
c_2 &= x_{(a+1,b-1)} \otimes y_{(c-2,c)} - 2(a-b)x_{(a,b)} \otimes y_{(c-1,c-1)};\\
c_3 &= x_{(a+2,b-2)} \otimes y_{(c-2,c)} - 4(a-b+1)x_{(a+1,b-1)} \otimes y_{(c-1,c-1)}\\
&\quad +8(a-b)(a-b+1)x_{(a,b)} \otimes y_{(c,c-2)}.
\end{aligned}$$

Their weights are $(a+c-2,b+c)$, $(a+c-1,b+c-1)$ and $(a+c,b+c-2)$, respectively. Furthermore, the action of v_2 on $V_{(a,b)}^{\mathrm{low}} \otimes F(c,c-2)$ is injective.

One now checks that for $a-b \geq 3$, vectors $v_2^2\, c_1$, $v_2\, c_2$ and c_3 are linearly independent. Combined with injectivity of v_2, this immediately implies

Corollary 1. *If $a-b \geq 3$, then*

$$V_{(a,b)}^{\mathrm{low}} \otimes F(c,c-2) = V_{(a+c-2,b+c)}^{\mathrm{low}} \oplus V_{(a+c-1,b+c-1)}^{\mathrm{low}} \oplus V_{(a+c,b+c-2)}^{\mathrm{low}},$$

where the lowest weight vectors of the three summands are c_1, c_2 and c_3.

Here is a picture of the weight vectors in this simplest case:

$$\begin{array}{ccccccc} & & c_3 & v_2\,c_3 & v_2^2\,c_3 & \dots \\ & c_2 & v_2\,c_2 & v_2^2\,c_2 & v_2^3\,c_2 & \dots \\ c_1 & v_2\,c_1 & v_2^2\,c_1 & v_2^3\,c_1 & v_2^4\,c_1 & \dots \end{array}$$

If $a - b = 1$, then $v_2^2\,c_1 = c_3$, while $v_2\,c_2$ is linearly independent of c_3. We introduce another vector of weight $(a+c, b+c-2) = (a+c, a+c-3)$,

$$\tilde{c}_3 = x_{(a+1,a-2)} \otimes y_{(c-1,c-1)} - 6x_{(a,a-1)} \otimes y_{(c,c-2)}. \tag{3}$$

The choice is such that $\tilde{c}_3$ is independent of $v_2\,c_2$ and c_3, and $v_2^*\,\tilde{c}_3 = v_2\,c_1$. It follows:

Corollary 2. *The module $V^{\text{low}}_{(a,a-1)} \otimes F(c, c-2)$ is a direct sum of a copy of $V^{\text{low}}_{(a+c-1,a+c-2)}$ and a module $\tilde{V}_{(a+c-1,a+c-2)}$. The lowest weight vector of $V^{\text{low}}_{(a+c-1,a+c-2)}$ is c_2. The Jordan-Hölder series of $\tilde{V}_{(a+c-1,a+c-2)}$ is*

$$\tilde{V}_{(a+c-1,a+c-2)} \supset V^{\text{low}}_{(a+c-2,a+c-1)} \supset V^{\text{low}}_{(a+c,a+c-3)} \supset 0,$$

with subquotients $V^{\text{low}}_{(a+c,a+c-3)}$, $F(a+c-1, a+c-2)$ and $V^{\text{low}}_{(a+c,a+c-3)}$, respectively. The lowest weight vectors for these subquotients are the image of $\tilde{c}_3$, the image of c_1 and c_3, respectively.

Finally, if $a - b = 2$, then $v_2\,c_1 = c_2$, while $v_2\,c_2$ and c_3 are linearly independent. We introduce another vector of weight $(a+c-1, b+c-1) = (a+c-1, a+c-3)$,

$$\tilde{c}_2 = x_{(a,a-2)} \otimes y_{(c-1,c-1)}.$$

Then $v_2\,c_2$, $v_2\,\tilde{c}_2$ and c_3 are independent, while $v_2^*\,\tilde{c}_2 = c_1$. It follows:

Corollary 3. *The module $V^{\text{low}}_{(a,a-2)} \otimes F(c, c-2)$ is a direct sum of a copy of $V^{\text{low}}_{(a+c,a+c-4)}$ and a module $\tilde{V}_{(a+c-1,a+c-3)}$. The lowest weight vector of $V^{\text{low}}_{(a+c,a+c-4)}$ is c_3. The Jordan-Hölder series of $\tilde{V}_{(a+c-1,a+c-3)}$ is*

$$\tilde{V}_{(a+c-1,a+c-3)} \supset V^{\text{low}}_{(a+c-2,a+c-2)} \supset V^{\text{low}}_{(a+c-1,a+c-3)} \supset 0,$$

with subquotients $V^{\text{low}}_{(a+c-1,a+c-3)}$, $F(a+c-2, a+c-2)$ and $V^{\text{low}}_{(a+c-1,a+c-3)}$, respectively. The lowest weight vectors for these subquotients are the image of $\tilde{c}_2$, the image of c_1 and c_2, respectively.

4. Describing the $\mathfrak{l}$-module $W \otimes S$

We now apply the results of Section 3 to $W \otimes S$. As we know, W is a direct sum of the irreducible $\mathfrak{l}$-modules $V^{\text{low}}_{(a,b)}$ for (a, b) as in (1) and (2). The corresponding basis as in Section 3 is

$$x_{a+k,b-k} = z_1^{n+k} z_2^k, \qquad k \geq 0, \tag{4}$$

if $(a,b)=(n+\frac{1}{2},-\frac{1}{2})$ for some integer $n\geq 0$ and

$$x_{a+k,b-k}=z_1^k z_2^{n+k},\qquad k\geq 0, \tag{5}$$

if $(a,b)=(\frac{1}{2},-n-\frac{1}{2})$ for some integer $n>0$, respectively.

We now describe the spin module $S=\bigwedge^{\cdot}\mathfrak{u}$ in more detail. Using the formula (5.7) from [13] for $\alpha:\mathfrak{l}\to C(\mathfrak{u}\oplus\bar{\mathfrak{u}})$, one gets

$$\begin{aligned}
\alpha(h_1) &= -t_1^*t_1-\tfrac{1}{2}t_2^*t_2+\tfrac{3}{2};\\
\alpha(\tilde{h}_2) &= -\tfrac{1}{2}t_2^*t_2-t_3^*t_3+\tfrac{3}{2};\\
\alpha(v_2) &= \tfrac{1}{2}t_2^*t_1+t_3^*t_2;\\
\alpha(v_2^*) &= -2t_1^*t_2-t_2^*t_3.
\end{aligned}$$

From this, it is easily calculated that the irreducible $\mathfrak{l}$-submodules of S are

$$\begin{aligned}
&\textstyle\bigwedge^0\mathfrak{u}=F(-\frac{3}{2},-\frac{3}{2}); &&\textstyle\bigwedge^1\mathfrak{u}=F(\frac{1}{2},-\frac{3}{2});\\
&\textstyle\bigwedge^2\mathfrak{u}=F(\frac{3}{2},-\frac{1}{2}); &&\textstyle\bigwedge^3\mathfrak{u}=F(\frac{3}{2},\frac{3}{2}).
\end{aligned}\tag{6}$$

Moreover, the spin action of v_2 and v_2^* on $\bigwedge^1\mathfrak{u}$ and $\bigwedge^2\mathfrak{u}$ is given on the weight vectors t_i and $t_i\wedge t_j$ by

$$\begin{aligned}
v_2:\quad & t_3\mapsto -2t_2,\quad t_2\mapsto -t_1,\quad t_1\mapsto 0;\\
& t_2\wedge t_3\mapsto -t_1\wedge t_3,\quad t_1\wedge t_3\mapsto -2t_1\wedge t_2,\quad t_1\wedge t_2\mapsto 0;\\
v_2^*:\quad & t_1\mapsto 4t_2,\quad t_2\mapsto 2t_3,\quad t_3\mapsto 0;\\
& t_1\wedge t_2\mapsto 2t_1\wedge t_3,\quad t_1\wedge t_3\mapsto 4t_2\wedge t_3,\quad t_2\wedge t_3\mapsto 0.
\end{aligned}$$

So we can pick a basis for $\bigwedge^1\mathfrak{u}=F(c,c-2)=F(\frac{1}{2},-\frac{3}{2})$ normalized as in Section 3 by setting

$$y_{c,c-2}=t_1,\qquad y_{c-1,c-1}=4t_2,\qquad y_{c-2,c}=8t_3, \tag{7}$$

and for $\bigwedge^1\mathfrak{u}=F(c,c-2)=F(\frac{3}{2},-\frac{1}{2})$ by setting

$$y_{c,c-2}=t_1\wedge t_2,\qquad y_{c-1,c-1}=2t_1\wedge t_3,\qquad y_{c-2,c}=8t_2\wedge t_3, \tag{8}$$

It is now not difficult to describe the complete decomposition of $W\otimes S$ as an $\mathfrak{l}$-module. There are a lot of irreducible direct summands of the form $V^{\mathrm{low}}_{(k,m)}$ for various integers k and m, most of them with multiplicity 2. To list them all, one would first tensor the $V^{\mathrm{low}}_{(a,b)}\subset W$ with one-dimensional modules $\bigwedge^0\mathfrak{u}$ and $\bigwedge^3\mathfrak{u}$. In this way we get

$$V^{\mathrm{low}}_{(n-1,-2)};\qquad V^{\mathrm{low}}_{(n+2,1)},\qquad n\geq 0$$

for $(a,b)=(n+\frac{1}{2},-\frac{1}{2})$, and

$$V^{\mathrm{low}}_{(-1,-n-2)};\qquad V^{\mathrm{low}}_{(2,-n+1)},\qquad n\geq 1$$

for $(a,b)=(\frac{1}{2},-n-\frac{1}{2})$.

Further components come from tensoring $V^{\mathrm{low}}_{(a,b)}\subset W$, $a-b\geq 3$, with $\bigwedge^1\mathfrak{u}$, which gives

$$V^{\mathrm{low}}_{(n-1,0)};\qquad V^{\mathrm{low}}_{(n,-1)};\qquad V^{\mathrm{low}}_{(n+1,-2)};\qquad V^{\mathrm{low}}_{(-1,-n)};\qquad V^{\mathrm{low}}_{(0,-n-1)};\qquad V^{\mathrm{low}}_{(1,-n-2)}$$

for $n\geq 2$. Tensoring the same $V^{\mathrm{low}}_{(a,b)}$ with $\bigwedge^2\mathfrak{u}$ gives

$$V^{\mathrm{low}}_{(n,1)};\qquad V^{\mathrm{low}}_{(n+1,0)};\qquad V^{\mathrm{low}}_{(n+2,-1)};\qquad V^{\mathrm{low}}_{(0,-n+1)};\qquad V^{\mathrm{low}}_{(1,-n)};\qquad V^{\mathrm{low}}_{(2,-n-1)}$$

for $n\geq 2$. These are all described in Corollary 1.

We next tensor the $V^{\mathrm{low}}_{(a,b)}\subset W$ with $a-b=1$, and that is only $V^{\mathrm{low}}_{(\frac{1}{2},-\frac{1}{2})}$, with $\bigwedge^1\mathfrak{u}$ and $\bigwedge^2\mathfrak{u}$, which gives

$$V^{\mathrm{low}}_{(0,-1)};\qquad \tilde{V}_{(0,-1)};\qquad V^{\mathrm{low}}_{(1,0)};\qquad \tilde{V}_{(1,0)},$$

as described in Corollary 2. Finally, we tensor $V^{\mathrm{low}}_{(a,b)}\subset W$ with $a-b=2$, and these are $V^{\mathrm{low}}_{(\frac{3}{2},-\frac{1}{2})}$ and $V^{\mathrm{low}}_{(\frac{1}{2},-\frac{3}{2})}$, with $\bigwedge^1\mathfrak{u}$ to get

$$V^{\mathrm{low}}_{(2,-2)};\qquad \tilde{V}_{(1,-1)};\qquad V^{\mathrm{low}}_{(1,-3)};\qquad \tilde{V}_{(0,-2)},$$

and with $\bigwedge^2\mathfrak{u}$ to get

$$V^{\mathrm{low}}_{(3,-1)};\qquad \tilde{V}_{(2,0)};\qquad V^{\mathrm{low}}_{(2,-2)};\qquad \tilde{V}_{(1,-1)},$$

as described in Corollary 3. All this can be summarized as follows:

Proposition 1. *The $\mathfrak{l}$-module $W\otimes S$ decomposes into a direct sum of the following modules:*

1. *the irreducible lowest weight module $V^{\mathrm{low}}_{(2,-2)}$ with multiplicity four;*
2. *the irreducible lowest weight modules $V^{\mathrm{low}}_{(k,m)}$ for (k,m) of the form*

   $(n,0)$ *or* $(0,-n)$ *for* $n\geq 1$; $\qquad (n,1)$, $(n,-1)$ $(1,-n)$ *or* $(-1,-n)$ *for* $n\geq 2$;
   $(n,-2)$ *or* $(2,-n)$ *for* $n\geq 3$,

   each of them with multiplicity two;
3. *the indecomposable but reducible modules*

   $$\tilde{V}_{(1,0)},\quad \tilde{V}_{(0,-1)},\quad \tilde{V}_{(2,0)},\quad \text{and}\quad \tilde{V}_{(0,-2)}$$

   with multiplicity one, and $\tilde{V}_{(1,-1)}$ with multiplicity two.

The results from this and the previous section in fact contain more information, including all the irreducible subquotients of the indecomposable reducible modules that appear, as well as the lowest weight generators for all the modules that appear as irreducible subquotients of $W\otimes S$. We will however see in the next section that only a few modules from the above long list have a chance to contribute to cohomology with respect to any of the operators $D(\mathfrak{g},\mathfrak{l})$, $C(\mathfrak{g},\mathfrak{l})$, or $C^-(\mathfrak{g},\mathfrak{l})$. The condition is to belong to the generalized 0-eigenspace of D^2. For these modules, we will write out (and use) all the available information.

5. Dirac cohomology of W with respect to $D(\mathfrak{g}, \mathfrak{l})$

Before applying the results of Section 4 to find $H_D(W)$ with respect to $D = D(\mathfrak{g}, \mathfrak{l})$, let us examine which of the modules appearing in Section 4 contribute to the generalized 0-eigenspace for D^2. We will see that D^2 acts finitely on every indecomposable submodule of $W \otimes S$. Therefore, the following lemma shows that in calculating cohomology of D, and also $C = C(\mathfrak{g}, \mathfrak{l})$ and $C^- = C^-(\mathfrak{g}, \mathfrak{l})$, we can restrict our attention to the generalized 0-eigenspace for D^2.

Lemma 2. *Suppose D^2 acts finitely on an $\mathfrak{l}_\Delta$-invariant direct summand X of $W \otimes S$ with only one eigenvalue, $\lambda \neq 0$. Then X does not contribute to cohomology with respect to D, C or C^-.*

Proof. The statement is obvious for D: $\operatorname{Ker} D \cap X = 0$, since 0 is not an eigenvalue for D^2 on X. To prove the statement for C, note that any $x \in X$ is annihilated by

$$(D^2 - \lambda)^n = (-\lambda)^n + \sum_{k=1}^{n} (-\lambda)^{n-k} D^{2k}$$

for some n. Furthermore, by [9], Remark 2.2, $C^2 = (C^-)^2 = 0$. It follows that

$$D^{2k} = ((C + C^-)^2)^k = (CC^- + C^-C)^k = (CC^-)^k + (C^-C)^k.$$

Consequently, if $Cx = 0$, it follows that

$$(-\lambda)^n x = -\sum_{k=1}^{n} (-\lambda)^{n-k} (CC^-)^k x,$$

so x is in the image of C and C has no cohomology on X. Analogously, C^- has no cohomology on X. $\square$

Note that with our choice of invariant form on $\mathfrak{g}$, we have

$$D^2 = \operatorname{Cas}_{\mathfrak{g}} \otimes 1 - \operatorname{Cas}_{\mathfrak{l}_\Delta} - 9.$$

$\operatorname{Cas}_{\mathfrak{g}} = -2h_1^2 - 2\tilde{h}_2^2 + 8h_1 - 4\tilde{h}_2 + 2u_1u_1^* + 2\sum_i v_i v_i^*$ is easily seen to act on W as 5. Namely,

$$\operatorname{Cas}_{\mathfrak{g}} 1 = -2(\frac{1}{2})^2 - 2(-\frac{1}{2})^2 + 8\frac{1}{2} - 4(-\frac{1}{2}) = 5.$$

Furthermore, $\operatorname{Cas}_{\mathfrak{l}_\Delta}$ is a diagonal version of $\operatorname{Cas}_{\mathfrak{l}} = -2h_1^2 - 2\tilde{h}_2^2 + 2h_1 - 2\tilde{h}_2 + 2v_2v_2^*$. So

$$D^2 = -\operatorname{Cas}_{\mathfrak{l}_\Delta} - 4 = \Delta(2h_1^2 + 2\tilde{h}_2^2 - 2h_1 + 2\tilde{h}_2 - 4 - 2v_2v_2^*).$$

We are interested in subrepresentations of $W \otimes S$ on which D^2 acts nilpotently. We saw in Section 4 that $W \otimes S$ is a direct sum of submodules which are all either irreducible of the form $V^{\text{low}}_{(k,m)}$ for some $(k, m) \in \mathbb{Z}^2$, or of the form $\tilde{V}_{(k,k-1)}$ or $\tilde{V}_{(k,k-2)}$ for some $k \in \mathbb{Z}$. On any $V^{\text{low}}_{(k,m)}$, $-\operatorname{Cas}_{\mathfrak{l}} - 4$ can be calculated on the lowest weight vector, where it is equal to

$$\alpha_{k,m} = 2k^2 + 2m^2 - 2k + 2m - 4 = \frac{1}{2}((2k-1)^2 + (2m+1)^2 - 10). \tag{9}$$

It follows that for integer values of k and m, $\alpha_{k,m}$ is zero precisely for

$$(k,m) \in \{(2,0),(2,-1),(-1,0),(-1,-1),(1,1),(1,-2),(0,1),(0,-2)\}. \qquad (10)$$

If $V^{\mathrm{low}}_{(k,m)}$ appear as a direct summand in $W \otimes S$, then, by Proposition 1, k and m must be integers with $k > m$. Therefore the following lemma holds.

Lemma 3. *For any $V^{\mathrm{low}}_{(k,m)}$ appearing as a direct summand in $W \otimes S$, the action of D^2 is the scalar $\alpha_{k,m}$ defined by (9). This scalar is zero precisely when (k,m) equals $(2,0)$, $(2,-1)$, $(1,-2)$, or $(0,-2)$.*

Now examining the list given in Proposition 1 gives

Corollary 4. *The irreducible direct summands $V^{\mathrm{low}}_{(k,m)}$ of $W \otimes S$ in the generalized 0-eigenspace of D^2 are:*

1. *$V^{\mathrm{low}}_{(2,-1)}$, appearing in $V^{\mathrm{low}}_{(\frac{1}{2},-\frac{5}{2})} \otimes F(\frac{3}{2},\frac{3}{2}) \subset W_0 \otimes S$, generated by $z_2^2 \otimes t_1 \wedge t_2 \wedge t_3$, and in $V^{\mathrm{low}}_{(\frac{5}{2},-\frac{1}{2})} \otimes F(\frac{1}{2},-\frac{3}{2}) \subset W_0 \otimes S$, generated by c_2.*

2. *$V^{\mathrm{low}}_{(1,-2)}$, appearing in $V^{\mathrm{low}}_{(\frac{5}{2},-\frac{1}{2})} \otimes F(-\frac{3}{2},-\frac{3}{2}) \subset W_0 \otimes S$, generated by $z_1^2 \otimes 1$, and in $V^{\mathrm{low}}_{(\frac{1}{2},-\frac{5}{2})} \otimes F(\frac{3}{2},-\frac{1}{2}) \subset W_0 \otimes S$, generated by c_2.*

3. *$V^{\mathrm{low}}_{(2,0)}$, appearing in $V^{\mathrm{low}}_{(\frac{1}{2},-\frac{3}{2})} \otimes F(\frac{3}{2},\frac{3}{2}) \subset W_1 \otimes S$, generated by $z_2 \otimes t_1 \wedge t_2 \wedge t_3$, and in $V^{\mathrm{low}}_{(\frac{7}{2},-\frac{1}{2})} \otimes F(\frac{1}{2},-\frac{3}{2}) \subset W_1 \otimes S$, generated by c_1.*

4. *$V^{\mathrm{low}}_{(0,-2)}$, appearing in $V^{\mathrm{low}}_{(\frac{3}{2},-\frac{1}{2})} \otimes F(-\frac{3}{2},-\frac{3}{2}) \subset W_1 \otimes S$, generated by $z_1 \otimes 1$, and in $V^{\mathrm{low}}_{(\frac{1}{2},-\frac{7}{2})} \otimes F(\frac{3}{2},-\frac{1}{2}) \subset W_1 \otimes S$, generated by c_1.*

On the module $\tilde{V}_{(a+c-1,a+c-2)}$ of Corollary 2, the action of D^2 is not by a scalar, or even semisimple. Namely, since the generator $\tilde{c}_3$ of (3) is of weight $(a+c,a+c-3)$, and since

$$v_2 v_2^* \tilde{c}_3 = v_2^2 c_1 = c_3,$$

we see that

$$D^2 \tilde{c}_3 = \alpha_{a+c,a+c-3} \tilde{c}_3 - 2c_3.$$

On the other hand, on $V^{\mathrm{low}}_{(a+c-2,a+c-1)} \subset \tilde{V}_{(a+c-1,a+c-2)}$, D^2 is the scalar $\alpha_{a+c-2,a+c-1}$ which is easily seen to be the same as $\alpha_{a+c,a+c-3}$.

Similarly, the action of D^2 on the module $\tilde{V}_{(a+c-1,a+c-3)}$ of Corollary 3 is given by the scalar $\alpha_{a+c-2,a+c-2}$ on $V^{\mathrm{low}}_{(a+c-2,a+c-2)} \subset \tilde{V}_{(a+c-1,a+c-2)}$, while

$$D^2 \tilde{c}_2 = \alpha_{a+c-1,a+c-3} \tilde{c}_2 - 2c_2.$$

Since one can check that $\alpha_{a+c-2,a+c-2} = \alpha_{a+c-1,a+c-3}$, we get

Lemma 4. *The operator D^2 acts finitely on the modules $\tilde{V}_{(a+c-1,a+c-2)}$ of Corollary 2, and on the modules $\tilde{V}_{(a+c-1,a+c-3)}$ of Corollary 3. The only eigenvalue of D^2 on $\tilde{V}_{(a+c-1,a+c-2)}$ is $\alpha_{(a+c,a+c-3)}$, and the only eigenvalue of D^2 on $\tilde{V}_{(a+c-1,a+c-3)}$ is $\alpha_{(a+c-1,a+c-3)}$. These numbers are defined in (9).*

Using (10), it now follows that the relevant $\tilde{V}_{(a+c-1,a+c-2)}$ are those with $(a+c, a+c-3)$ equal to $(2,-1)$ or $(1,-2)$, i.e., $\tilde{V}_{(1,0)}$ and $\tilde{V}_{(0,-1)}$, while the relevant $\tilde{V}_{(a+c-1,a+c-3)}$ are $\tilde{V}_{(2,0)}$ and $\tilde{V}_{(0,-2)}$. These are precisely the indecomposable but reducible modules from Proposition 1 that appears with multiplicity one. The only such module appearing with multiplicity two, $\tilde{V}_{(1,-1)}$, does not contribute to cohomology. So we have

Corollary 5. *The indecomposable direct summands $\tilde{V}_{k,m}$ of $W\otimes S$ in the generalized 0-eigenspace of D^2 are:*

1. *$\tilde{V}_{(1,0)}$, appearing in $V^{\text{low}}_{(\frac{1}{2},-\frac{1}{2})}\otimes F(\frac{3}{2},-\frac{1}{2})\subset W_0\otimes S$. Its irreducible subquotients are two copies of $V^{\text{low}}_{(2,-1)}$, one generated by the image of $\tilde{c}_3$ and the other by c_3, and the two-dimensional module $F(1,0)$ generated by the image of c_1.*

2. *$\tilde{V}_{(0,-1)}$, appearing in $V^{\text{low}}_{(\frac{1}{2},-\frac{1}{2})}\otimes F(\frac{1}{2},-\frac{3}{2})\subset W_0\otimes S$. Its irreducible subquotients are two copies of $V^{\text{low}}_{(1,-2)}$, one generated by the image of $\tilde{c}_3$ and the other by c_3, and the two-dimensional module $F(0,-1)$ generated by the image of c_1.*

3. *$\tilde{V}_{(2,0)}$, appearing in $V^{\text{low}}_{(\frac{3}{2},-\frac{1}{2})}\otimes F(\frac{3}{2},-\frac{1}{2})\subset W_1\otimes S$. Its irreducible subquotients are two copies of $V^{\text{low}}_{(2,0)}$, one generated by the image of $\tilde{c}_2$ and the other by c_2, and the one-dimensional module $F(1,1)$ generated by the image of c_1.*

4. *$\tilde{V}_{(0,-2)}$, appearing in $V^{\text{low}}_{(\frac{1}{2},-\frac{3}{2})}\otimes F(\frac{1}{2},-\frac{3}{2})\subset W_1\otimes S$. Its irreducible subquotients are two copies of $V^{\text{low}}_{(0,-2)}$, one generated by the image of $\tilde{c}_2$ and the other by c_2, and the one-dimensional module $F(-1,-1)$ generated by the image of c_1.*

Combining Corollary 4 and Corollary 5, we see that the irreducible subquotients of the generalized zero-eigenspace of D^2 in $W\otimes S$ are:

1. $V^{\text{low}}_{(2,-1)}$, appearing twice as a direct summand as described in Corollary 4 (1), and twice as a subquotient of $\tilde{V}_{(1,0)}$ as described in Corollary 5 (1);

2. $V^{\text{low}}_{(1,-2)}$, appearing twice as a direct summand as described in Corollary 4 (2), and twice as a subquotient of $\tilde{V}_{(0,-1)}$ as described in Corollary 5 (2);

3. $V^{\text{low}}_{(2,0)}$, appearing twice as a direct summand as described in Corollary 4 (3), and twice as a subquotient of $\tilde{V}_{(2,0)}$ as described in Corollary 5 (3);

4. $V^{\text{low}}_{(0,-2)}$, appearing twice as a direct summand as described in Corollary 4 (4), and twice as a subquotient of $\tilde{V}_{(0,-2)}$ as described in Corollary 5 (4);

5. The remaining irreducible subquotients $F(1,0)$, $F(0,-1)$, $F(1,1)$ and $F(-1,-1)$, described in Corollary 5 (1)-(4).

Note that cases (1) and (2) correspond to the even Weil representation W_0, the cases (3) and (4) correspond to the odd Weil representation W_1, and in case (5),

$F(1,0)$ and $F(0,-1)$ correspond to W_0 while $F(1,1)$ and $F(-1,-1)$ correspond to W_1.

The easiest case to handle is case (5). Namely, each of these four finite-dimensional modules appears only once, and so it has to be killed by C, C^- and D, and it cannot be in the image of any of them. (Namely, none of these operators can act as a nonzero scalar, since the only possible eigenvalue for each of them is zero.) Hence each of the four finite-dimensional modules contributes to cohomology with respect to either of the three operators C, C^- or D. Their generators, each of the form c_1, are easy to read off in each case (see the statements of the theorems below).

For cases (1)-(4), we have to see the action of C, C^- and D on the generators of the four isomorphic modules appearing in each of the cases, and then read off the cohomology with respect to C, C^- and D from the obtained table.

For case (1), i.e., for the four copies of the module $V^{\text{low}}_{(2,-1)}$, using (4), (5), (7) and (4), we see that the generators are

$$\begin{aligned} d_1 &= z_2^2 \otimes t_1 \wedge t_2 \wedge t_3; \\ c_2 &= 8z_1^3 z_2 \otimes t_3 - 24 z_1^2 \otimes t_2; \\ \tilde{c}_3 &= 2z_1 z_2 \otimes t_1 \wedge t_3 - 6 \otimes t_1 \wedge t_2; \\ c_3 &= 8z_1^2 z_2^2 \otimes t_2 \wedge t_3 - 16 z_1 z_2 \otimes t_1 \wedge t_3 + 16 \otimes t_1 \wedge t_2. \end{aligned}$$

Applying the operators

$$C = \partial_1^2 \otimes t_1 - 2z_2\partial_1 \otimes t_2 + z_2^2 \otimes t_3; \qquad C^- = z_1^2 \otimes t_1^* + z_1\partial_2 \otimes t_2^* + \partial_2^2 \otimes t_3^*,$$

we get after a short calculation:

$$\begin{aligned} Cd_1 &= 0; & C^-d_1 &= \frac{1}{4}c_3; \\ Cc_2 &= -3c_3; & C^-c_2 &= 0; \\ C\tilde{c}_3 &= -2d_1; & C^-\tilde{c}_3 &= \frac{1}{2}c_2; \\ Cc_3 &= C^-c_3 = 0. \end{aligned}$$

We can now conclude that the four subquotients isomorphic to $V^{\text{low}}_{(2,-1)}$ contribute nothing to cohomology with respect to C or C^-. Furthermore, we see that

$$D: \quad \tilde{c}_3 \mapsto -2d_1 + \frac{1}{2}c_2 \mapsto -2c_3 \mapsto 0,$$

while

$$D(c_2 + 12d_1) = 0,$$

and $c_2 + 12d_1$ is not in the image of D. So the contribution of the four subquotients isomorphic to $V^{\text{low}}_{(2,-1)}$ to cohomology with respect to D is equal to a copy of $V^{\text{low}}_{(2,-1)}$ generated by $c_2 + 12d_1$.

A very similar analysis applies to case (2). The generators of the four copies of $V^{\text{low}}_{(1,-2)}$ are

$$\begin{aligned} d_1 &= z_1^2 \otimes 1; \\ c_2 &= 8z_1z_2^3 \otimes t_2 \wedge t_3 - 12z_2^2 \otimes t_1 \wedge t_3; \\ \tilde{c}_3 &= 4z_1z_2 \otimes t_2 - 6 \otimes t_1; \\ c_3 &= 8z_1^2z_2^2 \otimes t_3 - 32z_1z_2 \otimes t_2 + 16 \otimes t_1. \end{aligned}$$

The action of C and C^- is as follows:

$$\begin{aligned} Cd_1 &= \frac{1}{8}c_3; && C^- d_1 = 0; \\ Cc_2 &= 0; && C^- c_2 = 3c_3; \\ C\tilde{c}_3 &= -\frac{1}{2}c_2; && C^- \tilde{c}_3 = -4d_1; \\ Cc_3 &= C^- c_3 = 0. \end{aligned}$$

Hence the four subquotients isomorphic to $V^{\text{low}}_{(1,-2)}$ contribute nothing to cohomology with respect to C or C^-. Since the action of D is

$$\begin{aligned} &D: \quad \tilde{c}_3 \mapsto -\frac{1}{2}c_2 - 4d_1 \mapsto -2c_3 \mapsto 0; \\ &D(c_2 - 24d_1) = 0, \end{aligned}$$

we see that the contribution of the four subquotients isomorphic to $V^{\text{low}}_{(1,-2)}$ to cohomology with respect to D is a copy of $V^{\text{low}}_{(1,-2)}$ generated by $c_2 - 24d_1$. So far we have proved:

Theorem 1. *The cohomology of C on $W_0 \otimes S$ is equal to $F(1,0) \oplus F(0,-1)$, where the two summands are generated by (the images of) $1 \otimes t_2 \wedge t_3$ and $1 \otimes t_3$, respectively. The same is true for the cohomology of C^- on $W_0 \otimes S$.*

The cohomology of D on $W_0 \otimes S$ is

$$F(1,0) \oplus F(0,-1) \oplus V^{\text{low}}_{(2,-1)} \oplus V^{\text{low}}_{(1,-2)},$$

where the first two summands are the same as above, the third summand is generated by $2z_1^3z_2 \otimes t_3 - 6z_1^2 \otimes t_2 + 3z_2^2 \otimes t_1 \wedge t_2 \wedge t_3$, and the fourth summand is generated by $2z_1z_2^3 \otimes t_2 \wedge t_3 - 3z_2^2 \otimes t_1 \wedge t_3 - 6z_1^2 \otimes 1$.

The calculations are completely analogous for the odd Weil representation W_1. For case (3), the generators are

$$\begin{aligned} d_1 &= z_2 \otimes t_1 \wedge t_2 \wedge t_3; \\ c_1 &= 8z_1^3 \otimes t_3; \\ \tilde{c}_2 &= 2z_1 \otimes t_1 \wedge t_3; \\ c_2 &= 8z_1^2z_2 \otimes t_2 \wedge t_3 - 8z_1 \otimes t_1 \wedge t_3. \end{aligned}$$

The action of C and C^- on these generators is given by

$$\begin{aligned} Cd_1 &= 0, & C^-d_1 &= \frac{1}{4}c_2; \\ Cc_1 &= -6c_2, & C^-c_1 &= 0; \\ C\tilde{c}_2 &= 4d_1, & C^-\tilde{c}_2 &= \frac{1}{2}c_1; \\ Cc_2 &= C^-c_2 = 0. \end{aligned}$$

For the case (4), the generators are

$$\begin{aligned} d_1 &= z_1 \otimes 1; \\ c_1 &= 8z_2^3 \otimes t_2 \wedge t_3; \\ \tilde{c}_2 &= 4z_2 \otimes t_2; \\ c_2 &= 8z_1z_2^2 \otimes t_3 - 16z_2 \otimes t_2. \end{aligned}$$

The action of C and C^- on these generators is given by

$$\begin{aligned} Cd_1 &= \frac{1}{8}c_2, & C^-d_1 &= 0; \\ Cc_1 &= 0, & C^-c_1 &= 6c_2; \\ C\tilde{c}_2 &= -\frac{1}{2}c_1, & C^-\tilde{c}_2 &= 8d_1; \\ Cc_2 &= C^-c_2 = 0. \end{aligned}$$

The final result is

Theorem 2. *The cohomology of C on $W_1 \otimes S$ is equal to $F(1,1) \oplus F(-1,-1)$, where the two summands are generated by (the images of) $z_1 \otimes t_2 \wedge t_3$ and $z_2 \otimes t_3$, respectively. The same is true for the cohomology of C^- on $W_1 \otimes S$.*

The cohomology of D on $W_1 \otimes S$ is

$$F(1,1) \oplus F(-1,-1) \oplus V_{(2,0)}^{\mathrm{low}} \oplus V_{(0,-2)}^{\mathrm{low}},$$

where the first two summands are the same as above, the third summand is generated by $z_1^3 \otimes t_3 + 3z_2 \otimes t_1 \wedge t_2 \wedge t_3$, and the fourth summand is generated by $z_2^3 \otimes t_2 \wedge t_3 - 6z_1 \otimes 1$.

6. Concluding remarks

The results of this note provide counterexamples to generalization of several results proved in [9] for the case of a compact Levi subalgebra:

1. The Dirac cohomology is not equal to the $\bar{\mathfrak{u}}$-cohomology or the $\mathfrak{u}$-homology up to a modular twist. In fact, the Dirac cohomology is larger than the $\bar{\mathfrak{u}}$-cohomology or the $\mathfrak{u}$-homology, and the latter two are equal to each other.

2. The operators C and C^- are not disjoint, i.e., it is possible that $CC^-x = 0$ without $C^-x = 0$ and also that $C^-Cy = 0$ without $Cy = 0$. Moreover, the

images of C and C^- actually intersect. In particular, there can be no positive definite Hermitian form on $W_0 \otimes S$ or on $W_1 \otimes S$ so that C and C^- are adjoint to each other with respect to this form.

3. Even though W_0 and W_1 are discretely decomposable under $\mathfrak{l}$, the operator $D(\mathfrak{g}, \mathfrak{l})^2$ is not semisimple.

4. Calculating Dirac cohomology in stages fails for $\mathfrak{t} \subset \mathfrak{l} \subset \mathfrak{g}$. Indeed, one sees right away that the Dirac cohomology with respect to $D(\mathfrak{l}, \mathfrak{t})$ of the $\mathfrak{l}$-module $H_D(\mathfrak{g}, \mathfrak{l}; W_0)$ is of dimension 6, so it cannot be equal to $H_D(\mathfrak{g}, \mathfrak{t}; W_0)$ which was seen to be 4-dimensional in [13], Proposition 6.1.

Finally, let us note that the part of the Dirac cohomology which does not come from the cohomology with respect to C or C^- seems to be related to the "bottom part" of the decomposition of the $\mathfrak{l}$-module W, with some shift.

References

[1] J. ADAMS, *Nilpotent cohomology of the oscillator representation,* J. Reine Angew. Math. **449**(1994), 1–7.

[2] A. BOREL, N. WALLACH, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, Second edition, American Math. Soc., Providence, 2000.

[3] P. CARTIER, *Quantum mechanical commutation relations and θ-functions*, Proc. Symp. Pure Math. 9, American Math. Soc., Providence, 1966.

[4] C. CHEVALLEY, *The algebraic theory of spinors*, Columbia University Press, New York, 1954.

[5] R. HOWE, *Remarks on classical invariant theory*, Trans. Amer. Math. Soc. **313**(1989), 539–570.

[6] R. HOWE, *Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond*, in: *Israel Math. Conf. Proc.*, (I. Piatetski-Shapiro, ed.), Bar-Ilan University, 1995, 1–182.

[7] J.-S. HUANG, P. PANDŽIĆ, *Dirac cohomology, unitary representations and a proof of a conjecture of Vogan*, J. Amer. Math. Soc. **15**(2002), 185–202.

[8] J.-S. HUANG, P. PANDŽIĆ, *Dirac Operators in Representation Theory*, Birkhäuser, Boston, 2006.

[9] J.-S. HUANG, P. PANDŽIĆ, D. RENARD, *Dirac operators and Lie algebra cohomology*, Represent. Theory **10**(2006), 299–313.

[10] B. KOSTANT, *Clifford algebra analogue of the Hopf-Koszul-Samelson theorem, the ρ-decomposition $C(\mathfrak{g}) = \operatorname{End} V_\rho \otimes C(P)$, and the $\mathfrak{g}$-module structure of $\bigwedge \mathfrak{g}$*, Adv. Math. **125**(1997), 275–350.

[11] B. KOSTANT, *A cubic Dirac operator and the emergence of Euler number multiplets of representations for equal rank subgroups*, Duke Math. J. **100**(1999), 447–501.

[12] B. KOSTANT, *Dirac cohomology for the cubic Dirac operator*, Progr. Math. **210**(2003), 69–93.

[13] P. PANDŽIĆ, *Dirac operators and unitarizability of Harish-Chandra modules*, Math. Commun. **15**(2010), 273–279.

[14] R. PARTHASARATHY, *Dirac operator and the discrete series*, Ann. of Math. **96**(1972), 1–30.

[15] T. TILGNER, *Graded generalizations of Weyl and Clifford algebras*, J. Pure Appl. Algebra **10**(1977), 163–168.

[16] D. A. VOGAN, *Dirac operators and unitary representations*, Three talks at MIT Lie groups seminar, Fall 1997.
[17] A. WEIL, *Sur certains groupes d'opérateurs unitaires*, Acta Math. **111**(1964), 143–211.