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On (g, φ)-contraction in intuitionistic fuzzy metric spaces

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Abstract. In this paper, we give a generalization of Hicks type contractions and Golet type contractions in intuitionistic fuzzy metric spaces. We prove some fixed point theorems for this new type contraction mappings on intuitionistic fuzzy metric spaces. These results generalize some known results in fuzzy metric spaces and probabilistic metric spaces.

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1. Introduction

As a generalization of fuzzy sets introduced by Zadeh [29], Atanassov [2] introduced the idea of an intuitionistic fuzzy set. Recently, much work has been done with these concepts [3,4]. Coker and a coworker [5,6] introduced the idea of the topology of intuitionistic fuzzy sets. Samanta and Mondal [26, 27] introduced the definition of the intuitionistic gradation of openness. Using the idea of intuitionistic fuzzy sets, Park [25] defined the notion of intuitionistic fuzzy metric spaces with the help of continuous t-norms and continuous t-conorms as a generalization of a fuzzy metric space due to George and Veeramani [8]. Recently, Hicks [13] and Golet et al. [9] introduced H-contraction and g-contraction mappings in probabilistic and fuzzy metric spaces, respectively. In addition, many authors [1, 10, 11, 24] have proved fixed point theorems for contractions in fuzzy and intuitionistic fuzzy metric spaces.

In this paper, we shall give a generalization of Hicks type contractions and Golet type contractions on intuitionistic fuzzy metric spaces and prove some fixed point theorems for this new type contraction mappings on intuitionistic fuzzy metric spaces. Let N be the set of all positive integers. The structure of this paper is as follows. In Section 2, we recall some definitions and the uniform structure of intuitionistic fuzzy metric spaces. In Section 3, we give some concepts on contraction and prove two fixed point theorems in intuitionistic fuzzy metric spaces. Our results generalize and extend many known results in fuzzy metric spaces and probabilistic metric spaces.

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2. Preliminaries

In this section, some definitions and preliminary results are given which will be used in this paper.

Definition 1 (see [28]). *A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is a continuous t-norm if* $*$ *satisfies the following conditions:*

(a) $*$ *is commutative and associative;*
(b) $*$ *is continuous;*
(c) $a * 1 = a$ *for all* $a \in [0,1]$*;*
(d) $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$*, and* $a, b, c, d \in [0,1]$.

Example 1. $a * b = ab, a * b = \min(a, b)$*, for all* $a, b \in [0,1]$.

Definition 2 (see [28]). *A binary operation* $\diamond : [0,1] \times [0,1] \to [0,1]$ *is a continuous t-conorm if* $\diamond$ *satisfies the following conditions:*

(a) $\diamond$ *is commutative and associative;*
(b) $\diamond$ *is continuous;*
(c) $a \diamond 0 = a$ *for all* $a \in [0,1]$*;*
(d) $a \diamond b \leq c \diamond d$ *whenever* $a \leq c$ *and* $b \leq d$*, and* $a, b, c, d \in [0,1]$.

Example 2. $a \diamond b = a + b - ab, a \diamond b = \max(a, b)$*, for all* $a, b \in [0,1]$.

Remark 1. *The concepts of triangular norms (t-norms) and triangular conorms (t-conorms) are known as the axiomatic skeletons used for characterizing fuzzy intersections and unions, respectively. These concepts were originally introduced by Menger [22] in his study of statistical metric spaces. More examples of these concepts were proposed by many authors (see [7, 12, 17-19]).*

Definition 3 (see [1]). *A 5-tuple* $(X, M, N, *, \diamond)$ *is said to be an intuitionistic fuzzy metric space if* X *is an arbitrary set,* $*$ *a continuous* t*-norm,* $\diamond$ *a continuous* t*-conorm and* M, N *are fuzzy sets on* $X^2 \times [0, \infty)$ *satisfying the following conditions: for all* $x, y, z \in X, s, t > 0$,

(a) $M(x, y, t) + N(x, y, t) \leq 1$*;*
(b) $M(x, y, 0) = 0$*;*
(c) $M(x, y, t) = 1$ *for all* $t > 0$ *if and only if* $x = y$*;*
(d) $M(x, y, t) = M(y, x, t)$*;*
(e) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ *for all* $x, y, z \in X, s, t > 0$*;*
(f) $M(x, y, \cdot) : [0, \infty) \to [0,1]$ *is left continuous;*
(g) $\lim_{t \to \infty} M(x, y, t) = 1$ *for all* $x, y \in X$*;*
(h) $N(x, y, 0) = 1$*;*
(i) $N(x, y, t) = 0$ *for all* $t > 0$ *if and only if* $x = y$*;*
(j) $N(x, y, t) = N(y, x, t)$*;*
(k) $N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)$ *for all* $x, y, z \in X, s, t > 0$*;*
(l) $N(x, y, \cdot) : [0, \infty) \to [0,1]$ *is right continuous;*
(m) $\lim_{t \to \infty} N(x, y, t) = 0$ *for all* $x, y \in X$.

Then (M, N) *is called an intuitionistic fuzzy metric on* X*. The functions* $M(x, y, t)$ *and* $N(x, y, t)$ *denote a degree of nearness and a degree of non-nearness between* x *and* y *with respect to* t*, respectively.*

Remark 2. *Every fuzzy metric space $(X, M, *)$ is an intuitionistic fuzzy metric space of the form $(X, M, 1 - M, *, \diamond)$ such that t-norm $*$ and t-conorm $\diamond$ are associated [21], i.e. $x \diamond y = 1 - ((1-x) * (1-y))$ for all $x, y \in X$.*

Remark 3. *In intuitionistic fuzzy metric space X, $M(x, y, \cdot)$ is non-decreasing and $N(x, y, \cdot)$ is non-increasing for all $x, y \in X$.*

Example 3 (Induced intuitionistic fuzzy metric, see [25])**.** *Let (X, d) be a metric space. Denote $a * b = ab$ and $a \diamond b = \min\{1, a + b\}$ for all $a, b \in [0, 1]$ and let M_d and N_d be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:*

$$M_d(x, y, t) = \frac{ht^n}{ht^n + md(x, y)}, \quad N_d(x, y, t) = \frac{d(x, y)}{kt^n + md(x, y)}$$

for all $h, k, m, n \in N$. Then $(X, M_d, N_d, *, \diamond)$ is an intuitionistic fuzzy metric space.

Remark 4. *Note that the above example holds even with the t-norms $a * b = \min\{a, b\}$ and the t-conorms $a * b = \max\{a, b\}$ and hence (M, N) is an intuitionistic fuzzy metric with respect to any continuous t-norms and continuous t-conorms. In the above example, by taking $h = k = m = n = 1$, we get*

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}, \quad N_d(x, y, t) = \frac{d(x, y)}{t + d(x, y)}$$

This intuitionistic fuzzy metric induced by a metric d is called the standard intuitionistic fuzzy metric.

Lemma 1. *If $*$ is a continuous t-norm and $\diamond$ is a continuous t-conorm, then for each $r \in (0, 1)$, there is a $s \in (0, 1)$ such that $s * s \geq 1 - r$ and $s \diamond s \leq r$.*

Definition 4. *Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space, the mapping $f : X \to X$ is said to be intuitionistic fuzzy continuous at x_0 if for each $t > 0$, there exists $s > 0$ such that*

$$M(x_0, y, s) > 1 - s \Rightarrow M(fx_0, fy, t) > 1 - t \ \text{ and } \ N(x_0, y, s) < s \Rightarrow N(fx_0, fy, t) < t.$$

The mapping $f : X \to X$ is intuitionistic fuzzy continuous if and only if it is intuitionistic fuzzy continuous at every point in X.

Definition 5. *Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space, the mapping $f : X \to X$ is said to be sequentially intuitionistic fuzzy continuous at x_0 if for any sequence $\{x_n\}, x_n \in X$ with $x_n \to x_0$, implies $fx_n \to x_0$, i.e.*

$$\lim_{n\to\infty} M(x_0, x_n, t) = 1 \ \text{ and } \ \lim_{n\to\infty} N(x_0, x_n, t) = 0$$

for each $t > 0$ imply

$$\lim_{n\to\infty} M(fx_0, fx_n, t) = 1 \ \text{ and } \ \lim_{n\to\infty} N(fx_0, fx_n, t) = 0$$

for each $t > 0$.

The mapping $f : X \to X$ is sequentially intuitionistic fuzzy continuous if and only if it is sequentially intuitionistic fuzzy continuous at each point in X.

Proposition 1. *Self mappings f of an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is intuitionistic fuzzy continuous at x_0 if and only if f is sequentially intuitionistic fuzzy continuous at x_0.*

Definition 6. *Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space and A a nonempty subset of X. The intuitionistic fuzzy closure of A, denoted by $\overline{A}$ is the set*

$$\overline{A} = \{y \in X : \exists x \in A, M(x, y, \varepsilon) > 1 - \lambda, \quad \text{and} \quad N(x, y, \varepsilon) < \lambda, \varepsilon > 0, \lambda \in (0, 1)\}.$$

Definition 7 (see [1]). *Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. Then*

(a) a sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $t > 0$ and $p > 0$, $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$ and $\lim_{n\to\infty} N(x_{n+p}, x_n, t) = 0$;

(b) a sequence $\{x_n\}$ in X is said to be converged to x in X (written as $x_n \to x$) if for each $t > 0$, $\lim_{n\to\infty} M(x_n, x, t) = 1$ and $\lim_{n\to\infty} N(x_n, x, t) = 0$.

An intuitionistic fuzzy metric space is said to be complete if and only if every Cauchy sequence is convergent. It is called compact if every sequence contains a convergent subsequence.

3. Main results

In this section, we give some fixed point theorems in an intuitionistic fuzzy metric space. Before proceeding further, some definitions and lemmas are given.

Definition 8 (see [10]). *Let Φ be the class of all mappings $\varphi : R^+ \to R^+ (R^+ = [0, +\infty))$ with the following properties:*
(a) φ non-decreasing;
(b) φ is right continuous;
(c) $\lim_{n\to\infty} \varphi^n(t) = 0$ for each $t > 0$.

Remark 5 (see [10]). *(a) It is easy to see that under these conditions, the function φ also satisfies $\varphi(t) < t$ for each $t > 0$ and therefore $\varphi(0) = 0$.*

(b) By property (c) we mean that for each $\varepsilon > 0$ and $\lambda \in (0, 1)$ there exists an integer $N(\varepsilon, \lambda)$ such that $\varphi^n(t) \le \min(\varepsilon, \lambda)$ whenever $n \ge N(\varepsilon, \lambda)$.

In [9, 13], Hicks and Golet introduced H-contraction and g-contraction mappings in probabilistic metric spaces, respectively. In the following definition, we give the H-contraction and g-contraction in the intuitionistic fuzzy setting.

Definition 9. *Let f be a self mapping defined on an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$. The mapping f is called an intuitionistic fuzzy H-contraction if there exists a number $k \in (0, 1)$ such that*

$$M(x, y, t) > 1 - t \Rightarrow M(fx, fy, kt) > 1 - kt \quad \text{and} \quad N(x, y, t) < t \Rightarrow N(fx, fy, kt) < kt,$$

for all $x, y \in X, t > 0$.

Definition 10. *Let f and g be self mappings defined on an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ and suppose g is bijective. The mapping f is called an intuitionistic fuzzy g-contraction if there exists a number $k \in (0,1)$ such that*

$$M(gx, gy, t) > 1 - t \Rightarrow M(fx, fy, kt) > 1 - kt \tag{1}$$

and

$$N(gx, gy, t) < t \Rightarrow N(fx, fy, kt) < kt, \quad \text{for all } x, y \in X, t > 0.$$

By considering a mapping $\varphi \in \Phi$ as given in Definition 8, we generalize the condition (1) in Definition 11 as follow.

Definition 11. *Let f and g be self mappings defined on an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ and $\varphi \in \Phi$. We say that the mapping $f : X \to X$ is an intuitionistic fuzzy (g,φ)-contraction if there exists a bijective $g : X \to X$ such that*

$$M(gx, gy, t) > 1 - t \Rightarrow M(fx, fy, \varphi(t)) > 1 - \varphi(t) \tag{2}$$

and

$$N(gx, gy, t) < t \Rightarrow N(fx, fy, \varphi(t)) < \varphi(t), \quad \text{for all } x, y \in X, t > 0.$$

Note that, if $\varphi(t) = kt$ for $k \in (0,1), t > 0$, then condition (2) is actually the intuitionistic fuzzy g-contraction which generalizes the fuzzy g-contraction due to Golet [9]. If the function g is an identity function, then condition (2) represents an intuitionistic fuzzy (φ, H)-contraction which generalizes the fuzzy (φ, H)-contraction according to Mihet [23]. Hence the (g,φ)-contraction generalizes the Golet and Mihet's contraction principles respectively in intuitionistic fuzzy metric spaces.

The following lemma is reproduced from [9] to suit our purposes in intuitionistic fuzzy metric spaces.

Lemma 2. *Let g be an injective mapping on $(X, M, N, *, \diamond)$ and $X_1 = g(X)$.*

(a) If $M^g(x, y, t) = M(gx, gy, t)$ and $N^g(x, y, t) = N(gx, gy, t)$, then $(X, M^g, N^g, *, \diamond)$ is an intuitionistic fuzzy metric space;

(b) If $(X, M, N, *, \diamond)$ is a complete intuitionistic fuzzy metric space, then $(X, M^g, N^g, *, \diamond)$ is also a complete intuitionistic fuzzy metric space;

(c) If $(X_1, M, N, *, \diamond)$ is intuitionistic fuzzy compact, then $(X, M^g, N^g, *, \diamond)$ is also intuitionistic fuzzy compact.

Proof. The proofs of (a) and (b) are immediate. To prove (c), let $\{x_n\}$ be a sequence in X. Then, for $u_n = gx_n, \{u_n\}$ is a sequence in X_1 for which we can find a convergent subsequence $\{u_{n_k}\}$, say $u_{n_k} \to u \in X_1$ as $k \to \infty$. Suppose that the sequence $\{y_n\}$ and y in X, and set $y_{n_k} = g^{-1}u_{n_k}$ is a subsequence of $\{y_n\}$, and $y = g^{-1}u$. Then

$$M^g(y_{n_k}, y, t) = M(gy_{n_k}, gy, t) = M(u_{n_k}, u, t) \to 1$$
$$N^g(y_{n_k}, y, t) = N(gy_{n_k}, gy, t) = N(u_{n_k}, u, t) \to 0$$

as $k \to \infty$ for each $t > 0$. This implies that $(X, M^g, N^g, *, \diamond)$ is intuitionistic fuzzy compact. □

Lemma 3. *Let f be an intuitionistic fuzzy (g,φ)-contraction; then*

(a) f is an intuitionistic fuzzy continuous mapping on $(X, M^g, N^g, *, \diamond)$ with values in $(X, M, N, *, \diamond)$.

(b) $g^{-1} \circ f$ is a continuous mapping on $(X, M^g, N^g, *, \diamond)$ with values into itself.

Proof. (a) Let x_n be a sequence in X such that $x_n \to x$ in X. In $(X, M^g, N^g, *, \diamond)$, this implies that

$$\lim_{n\to\infty} M^g(x_n, x, t) = \lim_{n\to\infty} M(gx_n, gx, t) = 1, \forall t > 0.$$

and

$$\lim_{n\to\infty} N^g(x_n, x, t) = \lim_{n\to\infty} N(x_n, x, t) = 0, \forall t > 0.$$

By the intuitionistic fuzzy (g,φ)-contraction (2) and $\varphi(t) < t, \forall t > 0$ it follows that

$$\lim_{n\to\infty} M(fx_n, fx, t) \geq \lim_{n\to\infty} M(fx_n, fx, \varphi(t)) = 1, \forall t > 0.$$

and

$$\lim_{n\to\infty} N(fx_n, fx, t) \leq \lim_{n\to\infty} N(fx_n, fx, \varphi(t)) = 0, \forall t > 0.$$

This implies that f is intuitionistic fuzzy continuous.

(b) Note that, since

$$\lim_{n\to\infty} M(gg^{-1}fx_n, gg^{-1}fx, t) = \lim_{n\to\infty} M^g(g^{-1}fx_n, g^{-1}fx, \varphi(t)) = 1, \forall t > 0.$$

and

$$\lim_{n\to\infty} N(gg^{-1}fx_n, gg^{-1}fx, t) = \lim_{n\to\infty} N(g^{-1}fx_n, g^{-1}fx, t) = 0, \forall t > 0.$$

This shows that the mapping $g^{-1} \circ f$ defined on $(X, M^g, N^g, *, \diamond)$ with values into itself is an intuitionistic fuzzy continuous mapping. $\square$

Theorem 1. *Let f and g be two functions defined on a complete intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$. If g is bijective and f is an intuitionistic fuzzy (g,φ)-contraction, then there exists a unique coincidence point $z \in X$ such that $gz = fz$.*

Proof. It is obvious that $M^g(x, y, t) > 1 - t$ and $N^g(x, y, t) < t$ whenever $t > 1$. Hence, we have $M(gx, gy, t) > 1 - t$ and $N(gx, gy, t) < t$. By condition (2), we have $M(fx, fy, \varphi(t)) > 1 - \varphi(t)$ and $N(fx, fy, \varphi(t)) < \varphi(t)$.

But,

$$M(fx, fy, \varphi(t)) = M(gg^{-1}fx, gg^{-1}fy, \varphi(t)) = M^g(g^{-1}fx, g^{-1}fy, \varphi(t)) > 1 - \varphi(t)$$

and

$$N(fx, fy, \varphi(t)) = N(gg^{-1}fx, gg^{-1}fy, \varphi(t)) = N^g(g^{-1}fx, g^{-1}fy, \varphi(t)) < \varphi(t).$$

Now, putting $h = g^{-1}f$, we have

$$M^g(hx, hy, \varphi(t)) > 1 - \varphi(t) \text{ and } N^g(hx, hy, \varphi(t)) < \varphi(t).$$

By condition (2), we obtain

$$\begin{aligned} M(fhx, fhy, \varphi^2(t)) &= M(gg^{-1}fhx, gg^{-1}fhy, \varphi^2(t)) = M^g(g^{-1}fhx, g^{-1}fhy, \varphi^2(t)) \\ &= M^g(h^2x, h^2y, \varphi^2(t)) > 1 - \varphi^2(t) \end{aligned}$$

and

$$\begin{aligned} N(fhx, fhy, \varphi^2(t)) &= N(gg^{-1}fhx, gg^{-1}fhy, \varphi^2(t)) = N^g(g^{-1}fhx, g^{-1}fhy, \varphi^2(t)) \\ &= N^g(h^2x, h^2y, \varphi^2(t)) < \varphi^2(t). \end{aligned}$$

By repeating this process, we get

$$M^g(h^nx, h^ny, \varphi^n(t)) > 1 - \varphi^n(t) \text{ and } N^g(h^nx, h^ny, \varphi^n(t)) < \varphi^n(t).$$

Since $\lim_{n\to\infty} \varphi^n(t) = 0$, then for every $\varepsilon > 0$ and $\lambda \in (0,1)$, there exists an integer $N(\varepsilon,\lambda)$ such that $\varphi^n(t) \le \min\{\varepsilon,\lambda\}$ whenever $n \ge N(\varepsilon,\lambda)$. Furthermore, since M is non-decreasing and N is non-increasing with respect to the third variable we have

$$M^g(h^nx, h^ny, \varepsilon) \ge M^g(h^nx, h^ny, \varphi^n(t)) > 1 - \varphi^n(t) \to 1 \text{ as } n \to \infty$$

and

$$N^g(h^nx, h^ny, \varepsilon) \le N^g(h^nx, h^ny, \varphi^n(t)) < \varphi^n(t) \to 0 \text{ as } n \to \infty$$

for each $t > 0$.

Let x_0 in X be fixed and let the sequence $\{x_n\}$ in X be defined recursively by $x_{n+1} = hx_n$, or equivalently by $gx_{n+1} = fx_n$ Now, considering $x = x_p$ and $y = x_0$, then from the above inequalities we have

$$\lim_{n\to\infty} M^g(x_{n+p}, x_n, \varepsilon) = \lim_{n\to\infty} M^g(h^nx_p, h^nx_0, \varepsilon) = 1$$

and

$$\lim_{n\to\infty} N^g(x_{n+p}, x_n, \varepsilon) = \lim_{n\to\infty} N^g(h^nx_p, h^nx_0, \varepsilon) = 0$$

for each $\varepsilon > 0$ and $p > 0$.

By Definition 7 (a), $\{x_n\}$ is a Cauchy sequence in X. Since $(X, M, N, *, \diamond)$ is complete, by Lemma 2(b) $(X, M^g, N^g, *, \diamond)$ is complete and there exists a point $z \in X$ such that $x_n \to z$ under (M^g, N^g). Since by Lemma 3(b) h is continuous, we have $z = hz$, i.e. $z = g^{-1}fz$, or equivalently $gz = fz$.

For the uniqueness, assume $gw = fw$ for some $w \in X$. Then, for any $t > 0$ and using (2) repeatedly, we can show that after n iterates, we have

$$M(gz, gw, t) > 1 - t \Rightarrow M(fz, fw, \varphi^n(t)) > 1 - \varphi^n(t)$$

and

$$N(gz, gw, t) < t \Rightarrow N(fz, fw, \varphi^n(t)) < \varphi^n(t).$$

Thus, we have $\lim_{n\to\infty} M(fz, fw, \varphi^n(t)) = 1$ and $\lim_{n\to\infty} N(fz, fw, \varphi^n(t)) = 0$ which implies that $fz = fw$. $\square$

As a multi-valued generalization of the notion of g-contraction (Definition 10), we shall introduce the notion of an intuitionistic fuzzy (g,φ)-contraction where $\varphi \in \Phi$ for a multi-valued mapping.

Let 2^X be the family of all nonempty subsets of X.

Definition 12. *Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space, A a nonempty subset of X and $T : A \to 2^X$. The mapping T is called an intuitionistic fuzzy (g,φ)-contraction, where $\varphi \in \Phi$ if there exists a bijective function $g : X \to X$ such that for each $x, y \in X$ and each $t > 0$*

$$M^g(x,y,t) > 1-t \quad and \quad N^g(x,y,t) < t \tag{3}$$

$\Rightarrow \forall u \in (T\circ g)(x), \exists v \in (T\circ g)(y) : M(u,v,\varphi(t)) > 1-\varphi(t)$ *and* $N(u,v,\varphi(t)) < \varphi(t)$.

Definition 13. *Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space, A a nonempty subset of X. We say that $T : A \to 2^X$ is weakly intuitionistic fuzzy demicompact if for each sequence $\{x_n\}$ from A such that $x_{n+1} \in T(x_n), n \in N$ and*

$$\lim_{n\to\infty} M(x_{n+1}, x_n, t) = 1 \quad and \quad \lim_{n\to\infty} N(x_{n+1}, x_n, t) = 0,$$

for each $t > 0$ there exists an intuitionistic fuzzy convergent subsequence $\{x_{n_k}\}$.

By $cl(X)$ we shall denote the family of all nonempty closed subsets of X.

Theorem 2. *Let $(X, M, N, *, \diamond)$ be a complete intuitionistic fuzzy metric space, $g : X \to X$ a bijective function and $T : A \to cl(A)$ where $A \in cl(A)$ is an intuitionistic fuzzy (g,φ)-contraction, where $\varphi \in \Phi$. If T is weakly fuzzy demicompact, then there exists at least one element $x \in A$ such that $x \in Tx$.*

Proof. Let $x_0 \in A, x_1 \in (T \circ g)(x_0)$ and $t > 0$ such that

$$M^g(x_1, x_0, t) > 1-t \quad \text{and} \quad N^g(x_1, x_0, t) < t.$$

The mapping T is an intuitionistic fuzzy (g,φ)-contraction. Therefore, by Definition 12, there exists $x_2 \in (T \circ g)(x_1)$ such that

$$M(x_2, x_1, \varphi(t)) > 1-\varphi(t) \quad \text{and} \quad N(x_2, x_1, \varphi(t)) < \varphi(t).$$

Since

$$M(x_2, x_1, \varphi(t)) = M^g(g^{-1}x_2, g^{-1}x_1, \varphi(t))$$

and

$$N(x_2, x_1, \varphi(t)) = N^g(g^{-1}x_2, g^{-1}x_1, \varphi(t)),$$

we have

$$M^g(g^{-1}x_2, g^{-1}x_1, \varphi(t)) > 1-\varphi(t) \quad \text{and} \quad N^g(g^{-1}x_2, g^{-1}x_1, \varphi(t)) < \varphi(t).$$

Similarly, it follows that there exists $x_3 \in (T \circ g)(x_2)$ such that

$$M(x_3, x_2, \varphi^2(t)) > 1-\varphi^2(t) \quad \text{and} \quad N(x_3, x_2, \varphi^2(t)) < \varphi^2(t).$$

By repeating the above process, there exists $x_n \in (T\circ g)(x_{n-1}), (n \geq 4)$ such that

$$M(x_n, x_{n-1}, \varphi^{n-1}(t)) > 1-\varphi^{n-1}(t) \quad \text{and} \quad N(x_n, x_{n-1}, \varphi^{n-1}(t)) < \varphi^{n-1}(t)$$

By Remark 5 (b) and letting $n \to \infty$, we have

$$\lim_{n\to\infty} M(x_n, x_{n-1}, \varepsilon) = 1 \quad \text{and} \quad \lim_{n\to\infty} N(x_n, x_{n-1}, \varepsilon) = 0, \forall \varepsilon > 0.$$

Since T is weakly fuzzy demicompact from the above limit, there exists a convergent fuzzy subsequence $\{x_{n_k}\}$ such that $\lim_{k\to\infty} x_{n_k} = x$ for some $x \in X$. Now, we show that $x \in (T\circ g)(x)$. Since $(T\circ g)(x) = \overline{(T\circ g)(x)}$, we shall prove that $x \in \overline{(T\circ g)(x)}$, i.e. for each $\varepsilon > 0$ and $\lambda \in (0,1)$ there exists a $y \in (T\circ g)(x)$ such that

$$M(x,y,\varepsilon) > 1-\lambda \quad \text{and} \quad N(x,y,\varepsilon) < \lambda.$$

Note that, since $*$ is a continuous t-norm and $\diamond$ a continuous t-conorm, by Lemma 1, for $\lambda \in (0,1)$ there is a $\delta \in (0,1)$ such that

$$(1-\delta)*(1-\delta) \geq 1-\lambda \quad \text{and} \quad \delta \diamond \delta \leq \lambda.$$

Further, there is a $\delta_1 \in (0,1)$ such that

$$(1-\delta_1)*(1-\delta_1) \geq 1-\delta \quad \text{and} \quad \delta_1 \diamond \delta_1 \leq \delta,$$

and $\delta_2 = \min\{\delta, \delta_1\}$, we have

$$(1-\delta)*[(1-\delta_2)*(1-\delta_2)] \geq (1-\delta)*[(1-\delta_1)*(1-\delta_1)] \geq (1-\delta)*(1-\delta) > 1-\delta$$

and

$$\delta \diamond (\delta_2 \diamond \delta_2) \leq \delta \diamond (\delta_1 \diamond \delta_1) \leq \delta \diamond \delta < \delta.$$

Since $\lim_{k\to\infty} x_{n_k} = x$, there exists an integer k_1 such that

$$M(x, x_{n_k}, \varepsilon/3) > 1-\delta_2 \quad \text{and} \quad N(x, x_{n_k}, \varepsilon/3) < \delta_2, \forall k \geq k_1.$$

Let k_2 be an integer such that

$$M(x_{n_k}, x_{n_k+1}, \varepsilon/3) > 1-\delta_2 \quad \text{and} \quad N(x_{n_k}, x_{n_k+1}, \varepsilon/3) < \delta_2, \forall k \geq k_2.$$

Let $s > 0$ be such that $\varphi(s) < \min\{\varepsilon/3, \delta_2\}$ and let k_3 be an integer such that

$$M^g(x_{n_k}, x, s) > 1-s \quad \text{and} \quad N^g(x_{n_k}, x, s) < s, \forall k \geq k_3.$$

Since T is an intuitionistic fuzzy (g,φ)-contraction, there exists a $y \in (T\circ g)(x)$ such that

$$M(x_{n_k+1}, y, \varphi(s)) > 1-\varphi(s) \quad \text{and} \quad N(x_{n_k+1}, y, \varphi(s)) < \varphi(s), \forall k \geq k_3.$$

and so

$$M(x_{n_k+1}, y, \varepsilon/3) \geq M(x_{n_k+1}, y, \varphi(s)) > 1-\varphi(s) > 1-\delta_2$$

and

$$N(x_{n_k+1}, y, \varepsilon/3) \leq N(x_{n_k+1}, y, \varphi(s)) < \varphi(s) < \delta_2$$

for each $k \geq k_3$. If $k \geq \max\{k_1, k_2, k_3\}$, we have

$$\begin{aligned} M(x, y, \varepsilon) &\geq M(x, x_{n_k}, \varepsilon/3) * M(x_{n_k}, x_{n_k+1}, \varepsilon/3) * M(x_{n_k+1}, y, \varepsilon/3) \\ &> (1-\delta_2) * (1-\delta_2) * (1-\delta_2) > 1-\lambda \end{aligned}$$

and

$$\begin{aligned} N(x, y, \varepsilon) &\leq N(x, x_{n_k}, \varepsilon/3) \diamond N(x_{n_k}, x_{n_k+1}, \varepsilon/3) \diamond N(x_{n_k+1}, y, \varepsilon/3) \\ &< \delta_2 \diamond \delta_2 \diamond \delta_2 < \lambda. \end{aligned}$$

Hence $x \in \overline{(T \circ g)(x)}$. The proof is completed. □

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