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Thebault circles of the triangle in an isotropic plane

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Abstract. In this paper the existence of three circles, which touch the circumscribed circle and Euler circle of an allowable triangle in an isotropic plane, is proved. Some relations between these three circles and elements of a triangle are investigated. Formulae for their radii are also given.

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In [6] V. Thébault considered the following figure in Euclidean geometry.

Let ABC be any triangle and A_m, B_m, C_m the midpoints of its sides BC, CA, AB. If the medians AA_m, BB_m, CC_m of that triangle meet its Euler circle $\mathcal{K}_e = A_mB_mC_m$ (or the nine–point circle, i.e., the circle through the feet of the three altitudes, the midpoints of the three sides, and the midpoints of the segments from the three vertices to the orthocenter [2]) again at the points L, M, N, then there exist three circles $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$, which touch the circumscribed circle $\mathcal{K}_o = ABC$ of the triangle ABC at the points A, B, C, and they touch the Euler circle $\mathcal{K}_e$ successively at the points L, M, N. Radii of these circles are d, e, f, respectively, where for example

$$d = \frac{R}{2}\left(1 - \frac{BC^2}{CA^2 + AB^2}\right) = \frac{2R \triangle \cot A}{CA^2 + AB^2},$$

while R denotes the radius of the circumscribed circle and $\triangle$ the area of the triangle ABC. It is shown that the equation

$$\frac{1}{R-d} + \frac{1}{R-e} + \frac{1}{R-f} = \frac{4}{R}$$

holds.

Besides the circles $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$, there are three more circles $\mathcal{D}'$, $\mathcal{E}'$, $\mathcal{F}'$, which touch the circumscribed circle $\mathcal{K}_o$ at the points A, B, C, and they touch Euler circle $\mathcal{K}_e$

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at the feet A_h, B_h, C_h of the altitudes of the triangle ABC.

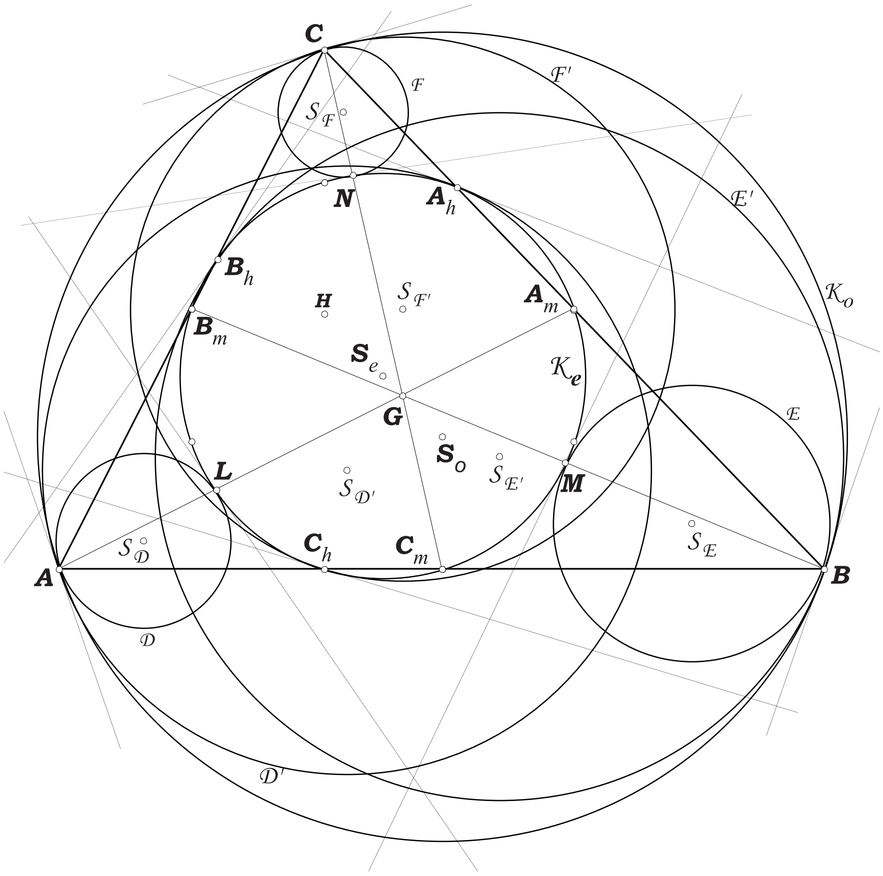

Figure 1. *Thebault circles of the triangle in Euclidean plane*

In this paper we shall investigate an analogous figure in an isotropic plane. Let $\mathbf{P}_2(\mathbb{R})$ be a real projective plane, $\mathbf{f}$ a real line in $\mathbf{P}_2$, and $\mathbf{A}_2 = \mathbf{P}_2 \setminus \mathbf{f}$ the associated affine plane. The isotropic plane $\mathbf{I}_2(\mathbb{R})$ is a real affine plane $\mathbf{A}_2$ where the metric is introduced with a real line $\mathbf{f} \subset \mathbf{P}_2$ and a real point F incidental with it. All straight lines through the point F are called isotropic lines. Isotropic circle in an isotropic plane is a conic which touches the absolute line $\mathbf{f}$ at the absolute point F.

Each allowable triangle in an isotropic plane can be set, by a suitable choice of coordinates, in the so-called standard position, i.e., that its circumscribed circle has the equation $y = x^2$, and its vertices are of the form $A = (a, a^2)$, $B = (b, b^2)$, $C = (c, c^2)$, where $a + b + c = 0$. With the labels $p = abc$, $q = bc + ca + ab$ it can be shown that the equality $a^2 = bc - q$ holds.

If A_h is the foot of the isotropic altitude through A, then A and A_h always lie on the same isotropic line, and a circle meets any isotropic line at only one proper point, it immediately follows that there are no analogous circles to the circles $\mathcal{D}'$, $\mathcal{E}'$, $\mathcal{F}'$ from the introduction.

In [3] it is shown that the median AA_m of the standard triangle ABC has the equation

$$y = \frac{3bc - q}{3a}x - \frac{2q}{3},$$

and in [1] it is obtained that Euler circle $\mathcal{K}_e$ of that triangle has the equation $y = -2x^2 - q$. The point

$$L = \left(-\frac{q}{3a}, -\frac{2q^2}{9a^2} - q\right) \tag{1}$$

obviously lies on the circle $\mathcal{K}_e$, and because of

$$\frac{3bc - q}{3a} \cdot \left(-\frac{q}{3a}\right) - \frac{2q}{3} = \frac{(3a^2 + 2q)(-q)}{9a^2} - \frac{2q}{3} = -\frac{2q^2}{9a^2} - q,$$

it also lies on the median AA_m.

Let us consider the circle $\mathcal{D}$ with the equation

$$(q + 3a^2)y = (3a^2 - 2q)x^2 + 6aqx - 3a^2q. \tag{2}$$

From this equation and the equation $y = x^2$ of circumscribed circle $\mathcal{K}_o$ of the triangle ABC it follows that for the abscissas of their intersection the equation

$$(q + 3a^2)x^2 = (3a^2 - 2q)x^2 + 6aqx - 3a^2q$$

holds, i.e., the equation $3q(x^2 - 2ax + a^2) = 0$ with the double solution $x = a$, so the circles $\mathcal{K}_o$ and $\mathcal{D}$ touch each other at the point A.

From equation (2) and the equation $y = -2x^2 - q$ of the circle $\mathcal{K}_e$ there follows the equation

$$(q + 3a^2)(-2x^2 - q) = (3a^2 - 2q)x^2 + 6aqx - 3a^2q,$$

i.e., the equation $9a^2x^2 + 6aqx + q^2 = 0$ with the double solution $x = -\frac{q}{3a}$. It means the circles $\mathcal{D}$ and $\mathcal{K}_e$ touch each other at the point L. So, we have proved

Theorem 1. *If medians of the allowable triangle ABC in an isotropic plane meet its Euler circle again (except midpoints of the sides BC, CA, AB) at the points L, M, N, then there are three circles $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$, which touch the circumscribed circle of the triangle ABC successively at the points A, B, C, and its Euler circle at the points L, M, N.*

The circle with the equation of the form $2\rho y = x^2 + ux + v$ has the radius ρ. Because of that, in the standard triangle ABC the circumscribed circle $\mathcal{K}_o$ with the equation $y = x^2$ has the radius $R = \frac{1}{2}$, and circle $\mathcal{D}$ with equation (2) has the radius

$$d = \frac{1}{2} \cdot \frac{q + 3a^2}{3a^2 - 2q} = \frac{q + 3a^2}{3a^2 - 2q} R.$$

From this it follows

$$R - d = \frac{3q}{2q - 3a^2} R$$

and with analogous equalities for the radii e and f of the circles $\mathcal{E}$ and $\mathcal{F}$ we get

$$\frac{1}{R-d}+\frac{1}{R-e}+\frac{1}{R-f}=\frac{1}{3qR}\left(6q-3a^2-3b^2-3c^2\right)=\frac{12q}{3qR}=\frac{4}{R}.$$

Now we have:

Theorem 2. *For the radii d, e, f of the circles $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$ from Theorem 1 there follows the equality*

$$\frac{1}{R-d}+\frac{1}{R-e}+\frac{1}{R-f}=\frac{4}{R},$$

where R is the radius of the circumscribed circle of the triangle ABC.

In [5] it is shown that the power of the point (x,y) with respect to the circle with the equation $2\rho y=x^2+ux+v$, has the value $x^2+ux+v-2\rho y$. In [4] it is shown that the standard triangle ABC has the centroid $G=\left(0,-\frac{2q}{3}\right)$. The power Π of that centroid with respect to the circle $\mathcal{D}$ with equation (2) is given by the formula

$$(3a^2-2q)\Pi=-3a^2q-(q+3a^2)\cdot\left(-\frac{2q}{3}\right)=\frac{q}{3}(2q-3a^2),$$

so $\Pi=-\frac{q}{3}$. The point G also has the same power with respect to the circles $\mathcal{E}$ and $\mathcal{F}$. The point whose powers with respect to the three circles are all equal is called the potential center for these three circles. So we get:

Theorem 3. *The centroid of the triangle ABC is the potential center for the circles $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$ from Theorem 1.*

The points M and N given by the equalities

$$M=\left(-\frac{q}{3b},-\frac{2q^2}{9b^2}-q\right),\quad N=\left(-\frac{q}{3c},-\frac{2q^2}{9c^2}-q\right)$$

are analogous to the point L from (1). The points L, M, N lie on the medians AG, BG, CG of the triangle ABC, i.e., the triangles ABC and LMN are homologic, and the center of this homology is the centroid G of the triangle ABC. What is the axis of this homology?

The line

$$\mathcal{L}\quad\ldots\quad y=-\frac{2aq}{3bc}x-q+\frac{2q^2}{9bc}\tag{3}$$

passes through the points M and N since, for example, for the first of them we get

$$-\frac{2aq}{3bc}\cdot\left(-\frac{q}{3b}\right)-q+\frac{2q^2}{9bc}=\frac{2q^2}{9b^2c}(a+b)-q=-\frac{2q^2}{9b^2}-q.$$

Because of that $\mathcal{L}$ is the line MN. In [4] it is shown that the line BC has the equation $y=-ax-bc$. If we set $-ax=y+bc$ in (3), then for ordinate y of the intersection $D=BC\cap MN$ we get the equation

$$y=\frac{2q}{3bc}\left(y+bc\right)-q+\frac{2q^2}{9bc},$$

i.e.,

$$\frac{1}{3bc}(3bc - 2q)y = -\frac{q}{3} + \frac{2q^2}{9bc} = \frac{q}{9bc}(2q - 3bc),$$

whence $y = -\frac{q}{3}$. The corresponding sides of the triangle ABC and its orthic triangle $A_hB_hC_h$ intersect at three points which lie on the same line. This line is called, by analogy with the Euclidean case, an orthic line of the observed triangle. In [4] it is shown that the orthic line $\mathcal{H}$ of the standard triangle ABC has the equation $y = -\frac{q}{3}$. So the point D lies on that orthic line, and the same thing is valid for the points $E = CA \cap NL$ and $F = AB \cap LM$. So we have proved the statement.

Theorem 4. *With the labels from Theorem 1, the axis of homology of the triangles ABC and LMN is the orthic line of the triangle ABC.*

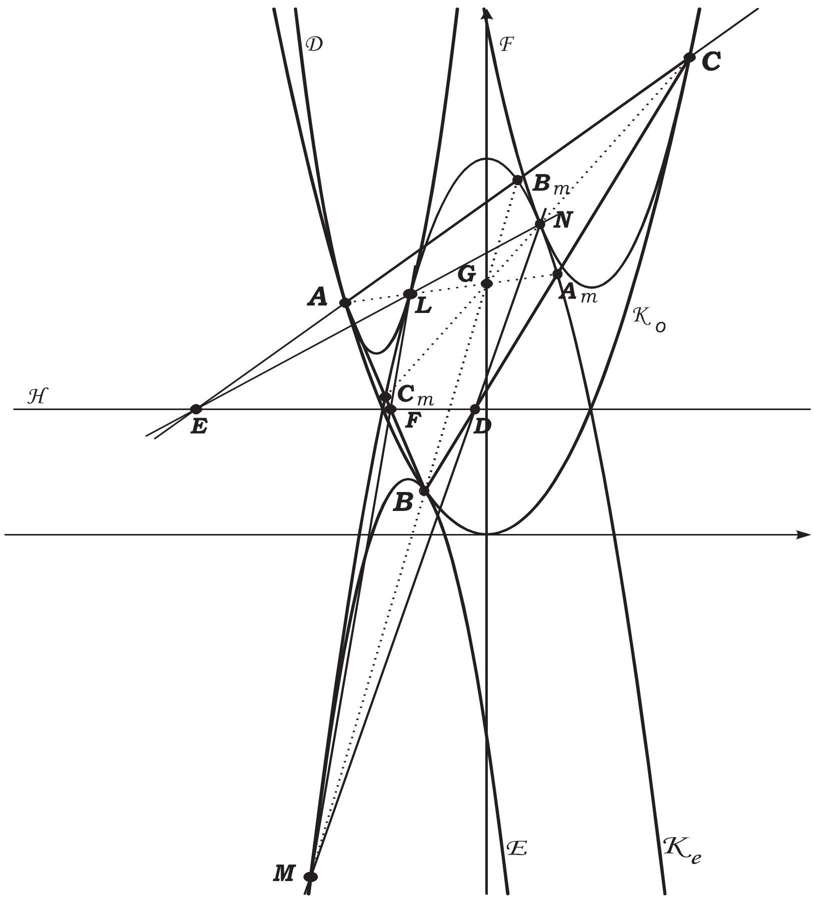

Figure 2. *Thebault circles of the triangle in an isotropic plane*

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References

[1] J. Beban–Brkić, R. Kolar–Šuper, Z. Kolar–Begović, V. Volenec, *On Feuerbach's Theorem and a Pencil of Circles in the Isotropic Plane*, J. Geom. Graphics **10**(2006), 125–132.

[2] H. S. M. Coxeter, S. L. Greitzer, *Geometry revisited*, The Matematical Association of America, Washington, 1967.

[3] Z. Kolar–Begović, R. Kolar–Šuper, J. Beban–Brkić, V. Volenec, *Symmedians and the symmedian center of the triangle in an isotropic plane*, Math. Pannon. **17**(2006), 287–301.

[4] R. Kolar–Šuper, Z. Kolar–Begović, V. Volenec, J. Beban–Brkić, *Metrical relationships in a standard triangle in an isotropic plane*, Math. Commun. **10**(2005), 149–157.

[5] R. Kolar–Šuper, Z. Kolar–Begović, V. Volenec, *Apollonius Circles of the Triangle in an Isotropic Plane*, Taiwanese J. Math. **12**(2008), 1239–1249.

[6] V. Thébault, *Cercles et sphères associés au triangle et au tétraèdre*, Ann. Soc. Sci. Bruxelles **57**(1937), 54–59.