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Artin-Rees property and Artin-Rees rings*

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Abstract. We study the problem asking which ideals in Noetherian rings have the Artin-Rees property, and want to obtain new ideals with this property from those for which we know that satisfy it. We show that the class $\mathcal{AR}$-$\mathbf{Ring_c}$, of all (Noetherian) Artin-Rees rings in which every prime ideal is moreover completely prime, is closed under localization at an arbitrary denominator set. We discuss and illustrate our results on two particular examples: the enveloping algebra of $\mathfrak{sl}(2)$ and the three-dimensional Heisenberg Lie algebra.

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1. Introduction

Given a commutative Noetherian ring A, every ideal $\mathfrak{A}$ of A satisfies the following condition: (AR) For any ideal I of A there exists a positive integer n, depending on I, such that $I \cap \mathfrak{A}^n \subseteq I\mathfrak{A}$. This famous result, the Artin-Rees property, was proved more than fifty years ago independently by E. Artin (unpublished) and Rees [14]. It is a cornerstone for dealing with $\mathfrak{A}$-adic topologies on finitely generated A-modules M.

Let R be a Noetherian not necessarily commutative ring. Similarly to the above, we define the (left, right) Artin-Rees property. An ideal I of R has the *left* (resp. *right*) AR (Artin-Rees) *property* if for every left (resp. right) ideal X of R there exists $n \in \mathbb{N}$ such that $X \cap I^n \subseteq IX$ (resp. $X \cap I^n \subseteq XI$). We say that R is an *AR ring* if every ideal of R has both left and right AR property. Dealing with the question as to which ideals I of R satisfy the AR property, it turns out that the complexity of the structure of R is in a certain sense measured by the number of those I's which satisfy this property. Precisely, the more ideals will have it, the easier the structure of R will be.

Concerning the localization technique, the class of rings satisfying the second layer condition is very interesting; see [8], [12, Sect. 4.3]. Recall that some examples of rings within this class are enveloping algebras of solvable Lie algebras, and group rings $\mathcal{A}G$, where $\mathcal{A}$ is a commutative Noetherian ring and G is a polycyclic by finite

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group. The AR property is useful in dealing with many rings satisfying the second layer condition, but which are not FBN rings. Its main use in ring theory is for localization at semiprime ideals. For example, a well-known result of Smith [18] states that every semiprime ideal of an AR ring R is localizable; see also [19]. At the same time, if R is not such a ring, that is, if it has at least one (prime) ideal which fails to have the AR property, one can localize at many ideals which have the property (cf. [12, 4.2.12]). On the other hand, if we will be able to localize at a particular ideal I of a non-AR ring, it will often be the case that this I has the AR property; see [3], [4], [5], [16, Sect. 3].

Here we provide a method to deduce new ideals with the AR property from ideals known to satisfy it. Also, new AR rings can be obtained from old ones. That is, we point out at an interesting class of (noncommutative) AR rings which is closed under localization at an *arbitrary* denominator set. More precisely, we have this theorem which is the main result of the paper.

Theorem 1. *Let R be a Noetherian Artin-Rees ring all of whose prime ideals are completely prime. Then, for any denominator set $\mathcal{S}$ in R, the localization $R_{\mathcal{S}}$ is also a (Noetherian) Artin-Rees ring all of whose prime ideals are completely prime.*

Let us now formulate the following property of Noetherian rings: $(\diamondsuit)$ *Every prime ideal of a ring is completely prime.* Define $\mathcal{AR}$-**Ring** to be the class of Noetherian AR rings, and

$$\mathcal{AR}\text{-}\mathbf{Ring}_{\boldsymbol{c}} := \text{subclass of } \mathcal{AR}\text{-}\mathbf{Ring} \text{ consisting of those rings satisfying } (\diamondsuit).$$

In order to point out at certain important examples of rings within the defined subclass, recall the following non-trivial and well-known fact. This was obtained in [11]. See also [7, 3.7.2 Thm.], [10], [13]; and [12, Sect. 4.2], [17, Sect. 5].

Fact. *Let $U(\mathfrak{n})$ be the universal enveloping algebra of a finite-dimensional nilpotent Lie algebra $\mathfrak{n}$ defined over a field of characteristic zero. Then any factor ring of $U(\mathfrak{n})$ belongs to the class $\mathcal{AR}$-$\mathbf{Ring}_{\boldsymbol{c}}$.*

Being closed under taking homomorphic images for the AR property is a known fact, and for prime ideals being completely prime is quite clear. Thus the following corollary is an immediate consequence of the previous theorem; see also its proof. Roughly speaking, it says that $\mathcal{AR}$-$\mathbf{Ring}_{\boldsymbol{c}}$ is closed under taking homomorphic images and localizing.

Corollary 1. *Suppose that a ring Λ is obtained from a ring $\Lambda_0 \in \mathcal{AR}$-$\mathbf{Ring}_{\boldsymbol{c}}$ via finitely many procedures of taking homomorphic images or localizing. That is, we have*

$$\Lambda_0 \xrightarrow{\text{proc-1}} \Lambda_1 \xrightarrow{\text{proc-2}} \cdots \quad \cdots \xrightarrow{\text{proc-}n} \Lambda_n = \Lambda,$$

where proc-i *means that either $\Lambda_i = \theta(\Lambda_{i-1})$ for some ring homomorphism θ, or Λ_i is the localization of Λ_{i-1} at some denominator set $\mathcal{S}$. Then*

$$\Lambda \in \mathcal{AR}\text{-}\mathbf{Ring}_{\boldsymbol{c}}\,.$$

The paper is organized as follows. Section 2 contains a proof of Theorem 1; see also Proposition 1. In Section 3 we discuss and illustrate our results on two particular examples: the enveloping algebra of $\mathfrak{sl}(2,\mathbb{C})$ (semisimple case), and the three-dimensional Heisenberg Lie algebra (nilpotent case).

2. Results

Throughout this paper all rings are associative with identity and, if not otherwise said, all ideals are two-sided. For a ring R by $\operatorname{Id} R$, $\operatorname{Spm} R$, $\operatorname{Spec} R$ and $\operatorname{Spec}_c R$ we denote the set of all ideals, the set of all maximal ideals, the set of all prime ideals and the set of all completely prime ideals of $\mathcal{R}$, respectively. Recall that an ideal I of R is completely prime if R/I is a domain. For $r \in R$, by $\langle r \rangle$ we denote the ideal generated by r. By $\sqrt{I}$ we denote the prime radical of I, that is, the intersection of all prime ideals containing it.

Part (i) of this lemma is well-known (cf. [12, Prop. 4.2.4(i)]); for (ii) see [16, Lemma 3.1].

Lemma 1. *Suppose that I is an ideal of a ring R satisfying the left (resp. right) AR property.*

(i) If $\theta : R \to S$ is a ring epimorphism, then $\theta(I)$ has the left (resp. right) AR property in S.

(ii) I^k has the left (resp. right) AR property, for every $k \in \mathbb{N}$.

(iii) For any ideal J of R, $I + J/J$ has the left (resp. right) AR property in R/J.

Proof. (iii) Consider the ring $\overline{R} := R/I \cap J$, its ideal $\overline{J} := J/I \cap J$, and the epimorphism $\theta : \overline{R} \to R/J$ defined as $\theta := \imath \circ \varphi$, where $\varphi : \overline{R} \to \overline{R}/\overline{J}$ is the canonical epimorphism and $\imath : \overline{R}/\overline{J} \to R/J$ is the isomorphism obtained via the third isomorphism theorem. Clearly, $\theta(I/I \cap J) = I + J/J$. Thus, by (i), we have the claim. □

Part of the next result concerning sums of ideals is new and somewhat surprising. We provide two proofs of it. One is via the well-known characterization of the AR property [12, Thm. 4.2.2], and the other is direct and related to what follows.

Proposition 1. *Let R be a left (resp. right) Noetherian ring. The set $\mathcal{AR}$-$\operatorname{Id} R$ of all ideals satisfying the AR property is distinguished in the following sense. If $I_1, \ldots, I_k \in \mathcal{AR}$-$\operatorname{Id} R$, then*

$$I_1 \cdots I_k, \;\; I_1 \cap \cdots \cap I_k, \;\; I_1 + \cdots + I_k \in \mathcal{AR}\text{-}\operatorname{Id} R.$$

First proof. We only have to prove that $I_1 + \cdots + I_k \in \mathcal{AR}$-$\operatorname{Id} R$; for the other two claims see [16, Lemma 3.3]. Let $k = 2$ and define $J := I_1 + I_2$. Suppose that M is a finitely generated R-module and $N \leq M$ is an essential submodule such that $JN = 0$. Then in particular $I_i N = 0$ for $i = 1, 2$. As I_i satisfy the left AR property, we know that $I_i^{n_i} M = 0$, for some n_i. Hence $J^{n_1+n_2} M = 0$, and so J has the left AR property, as claimed. □

Second proof. Let, again, $k = 2$ and $J := I_1 + I_2$. For any left ideal L of R, we have to show that $L \cap J^n \subseteq JL$, for some n. To see this first note that $L \cap I_2^p \subseteq I_2 L$, for some p. Then define $A := I_1 + I_2^p$. By the previous lemma, A/I_2^p has the left AR property in R/I_2^p, and so there exists some t such that

$$L/I_2^p \cap (A/I_2^p)^t \subseteq (A/I_2^p)(L/I_2^p). \tag{1}$$

Let $n \geq pt$ be arbitrary. Then take any $\omega \in L \cap J^n$. Since $J^n \subseteq J^{pt} \subseteq A^t$, we particularly have that $\overline{\omega} := \omega + I_2^p \in R/I_2^p$ belongs to the left-hand side of (1). It immediately follows that $\omega - x = y$, for some $x \in AL \subseteq JL$ and $y \in I_2^p$, and furthermore $\omega - x = y \in I_2 L \subseteq JL$. We conclude that $\omega \in JL$, as desired. $\square$

Remark 1. *Let R be a (left) Noetherian ring and $I \subseteq J$ are ideals of R such that both I has the (left) AR property in R and J/I has the same property in R/I. One might ask the following question concerning the second proof of the above proposition:* Does then follow that J has this property in R? *Unfortunately, the answer to this question is negative for the whole class of (left) Noetherian rings. Thus it is reasonable to look for some 'nice' subclass of rings where the answer is positive. To see this, take R to be the enveloping algebra of a two-dimensional complex (solvable non-nilpotent) Lie algebra with basis $\{x, y\}$ and relation $[x, y] = y$. Let then $I := Ry$ and $J := Rx + Ry$. Since the element y is normal, I has the AR property in R. Also, since $R/I \cong \mathbb{C}[x]$ is commutative, any ideal of it has the same property. But J does not have it.*

The next result follows in the same vein as the above discussion.

Lemma 2. *Let $I \subseteq J$ be ideals of a Noetherian ring R, and suppose that I has the AR property. Then the following are equivalent:*

(a) J has the AR property;

(b) J/I^k has the AR property, for every $k \in \mathbb{N}$.

Proof. Given a left ideal L of R, we have $L \cap I^k \subseteq IL$, for some k. Next, supposing (b), for some t we have $L/I^k \cap J^t/I^k \subseteq JL/I^k$. Let now $\omega \in L \cap J^n$, with $n \geq t$. Clearly, there exists some $y \in JL$ such that $\omega - y \in I^k$. Hence, $\omega - y \in JL$, and therefore $\omega \in JL$. This proves (b)$\Rightarrow$(a); the opposite is clear. $\square$

Remark 2. *Suppose that R, I and J are as above, with the additional assumption that J does not have the AR property. Note that then there always exists certain $k_0 \in \mathbb{N}$ such that J/I^k has the AR property, for all $k < k_0$, and fails to have this property, for all $k \geq k_0$.*

The next result was first noticed by Smith [19, Lemma 4.4]. After I finished writing the first version of this present paper, I became aware of Smith's result. That is why I did not cite it in [16], where Corollary 3.5 is now just a slight improvement of Smith's result.

Lemma 3. *Suppose $\mathcal{R}$ is a left (resp. right) Noetherian ring. The following are equivalent:*

(a) Every ideal $\mathcal{I}$ of $\mathcal{R}$ has the left (resp. right) AR property;

(b) Every prime ideal $\mathcal{P}$ of $\mathcal{R}$ has the left (resp. right) AR property.

We need one more auxiliary result, which is interesting in its own right. But first, for the convenience of the reader, recall some terminology and well-known facts; more details can be found, e.g., in [12, Chap. 2]. Let $\emptyset \neq \mathcal{S} \subseteq R$ be a multiplicatively closed set. Suppose that $\mathcal{S}$ is a left Ore set. This means that $\mathcal{S}$ satisfies the left Ore condition; that is, we have $\mathcal{S}r \cap Rs \neq \emptyset$, for any $r \in R$ and $s \in \mathcal{S}$. Next, define a set

$$\operatorname{ass} \mathcal{S} := \{r \in R \mid rs = 0 \text{ for some } s \in \mathcal{S}\}.$$

It turns out that $\operatorname{ass} \mathcal{S}$ is an ideal of R, and thus we can define a quotient ring $\overline{R} := R/ \operatorname{ass} \mathcal{S}$. By $\overline{\mathcal{S}}$ denote the image of $\mathcal{S}$ in $\overline{R}$. A left Ore set $\mathcal{S}$, for which the elements of $\overline{\mathcal{S}}$ are regular in $\overline{R}$, is called a left denominator set. The main feature of this notion is the following: Given a multiplicatively closed subset $\mathcal{S}$ of R, one can define the left localization $R_{\mathcal{S}}$ if and only if $\mathcal{S}$ is a left denominator set. (The notions of a right Ore set, a right denominator set and a right localization, are defined analogously.)

Lemma 4. *Suppose that $\mathcal{R}$ is a Noetherian ring, Σ is a left denominator set, $\mathcal{P}$ is a completely prime ideal of $\mathcal{R}$ satisfying $\mathcal{P} \cap \Sigma = \emptyset$, and $\mathcal{L}$ is a left ideal of $\mathcal{R}$. Then*

$$(\mathcal{P}\mathcal{L})_\Sigma = \mathcal{P}_\Sigma \mathcal{L}_\Sigma,$$

and also

$$(\mathcal{P}^n)_\Sigma = (\mathcal{P}_\Sigma)^n, \qquad \text{for } n \in \mathbb{N}.$$

Proof. We only show the inclusion $\mathcal{P}_\Sigma \mathcal{L}_\Sigma \subseteq (\mathcal{P}\mathcal{L})_\Sigma$; the rest is then clear. For that, consider some $x = s^{-1}a$ and $y = t^{-1}b$, where $s, t \in \Sigma$ and $a \in \mathcal{P}$, $b \in \mathcal{L}$. Analogously to (i) of the theorem below, we have $\rho := ct = ua$ for some $(u, c) \in \Sigma \times \mathcal{R}$. Since $\rho \in \mathcal{P}$, by the assumptions on $\mathcal{P}$ we have $c \in \mathcal{P}$. Thus $cb \in \mathcal{P}\mathcal{L}$, and therefore $xy \in (\mathcal{P}\mathcal{L})_\Sigma$. This proves the lemma. □

For a ring R and $\mathcal{S}$ a left denominator set in R, by $R_{\mathcal{S}}$ we denote the left localization of R at $\mathcal{S}$, and by $\imath : R \to R_{\mathcal{S}}$ the canonical localization homomorphism. The elements of $R_{\mathcal{S}}$ will be written as $s^{-1}r$. Now we are ready to proceed with the main result of the paper.

Theorem 2. *Let R be a Noetherian ring such that every prime ideal of it is completely prime, that is,* $\operatorname{Spec} R = \operatorname{Spec}_c R$. *Let $\mathcal{S}$ be any denominator set in R. Then we have the following.*

(i) Every prime of $R_{\mathcal{S}}$ is completely prime, that is, $\operatorname{Spec} R_{\mathcal{S}} = \operatorname{Spec}_c R_{\mathcal{S}}$.

(ii) If R is an AR ring, then so is $R_{\mathcal{S}}$.

Proof. First some preliminary remarks (cf. [12, Prop. 2.1.16]). The left localization of a left ideal X of R at a denominator set $\mathcal{S}$ is denoted by $X_{\mathcal{S}}$; that is, $X_{\mathcal{S}} = \{s^{-1}x \mid s \in \mathcal{S}, x \in X\}$. This is the left ideal of $R_{\mathcal{S}}$ generated by $\imath(X)$. If X is moreover a (two-sided) ideal, then so is $X_{\mathcal{S}}$. The map

$$\phi : \{P \in \operatorname{Spec} R \mid P \cap \mathcal{S} = \emptyset\} \longrightarrow \operatorname{Spec} R_{\mathcal{S}}, \qquad \phi(P) := P_{\mathcal{S}},$$

is bijective; and its inverse is $\phi^{-1}(\widetilde{P}) = \imath^{-1}(\widetilde{P})$, that is, $\phi^{-1}(\widetilde{P}) = \widetilde{P} \cap R$. For later use also note the following: If X and $\widetilde{X}$ are left ideals of R and $R_{\mathcal{S}}$, respectively, then

$$\imath^{-1}(\widetilde{X})_{\mathcal{S}} = (\widetilde{X} \cap R)_{\mathcal{S}} = \widetilde{X} \qquad \text{and} \qquad X \subseteq \imath^{-1}(X_{\mathcal{S}});$$

where the last inclusion may be strict.

(i) Let $\widetilde{P} \in \operatorname{Spec} R_{\mathcal{S}}$, and define $P := \phi^{-1}(\widetilde{P})$. Suppose that $x = s^{-1}a$ and $y = t^{-1}b$ are from $R_{\mathcal{S}}$ such that $xy \in \widetilde{P}$. Since $xy = (us)^{-1}(cb)$, where $(u, c) \in \mathcal{S} \times R$ satisfies $ct = ua$, we have $cb \in P$. If $b \in P$, then $y \in \phi(P) = \widetilde{P}$. If $c \in P$, then $ct = ua \in P$; hence $a \in P$ and thus $x \in \widetilde{P}$.

(ii) By Lemma 3, it suffices to show that every prime $\widetilde{P}$ of $R_{\mathcal{S}}$ satisfies, for example, the left AR property. To do this take any left ideal $\widetilde{L}$ of $R_{\mathcal{S}}$, and define $L := \widetilde{L} \cap R$. By the assumption that R is an AR ring, there exists some n such that $L \cap P^n \subseteq PL$, where $P := \phi^{-1}(\widetilde{P})$. We claim that also

$$\widetilde{L} \cap \widetilde{P}^n \subseteq \widetilde{P}\widetilde{L}.$$

Using Lemma 4, we argue as follows. Let $\omega \in \widetilde{L} \cap \widetilde{P}^n = L_{\mathcal{S}} \cap (P^n)_{\mathcal{S}}$, where $\omega = s^{-1}l = s^{-1}p$ with $l \in L$ and $p \in P^n$. Hence, $s\omega \in PL$, and thus $\omega \in \widetilde{P}\widetilde{L}$, what we had to show. □

Remark 3. *Suppose $R, \mathcal{S}$ are as above and I is an ideal of R which has the AR property. Note that the argument as given in the above proof would not work while trying to prove that $I_{\mathcal{S}}$ has the AR property as well.*

3. Examples

Recall that the height and the max-height of an ideal I of a Noetherian ring R are defined by

$$\begin{aligned} \operatorname{ht}(I) &:= \min\{\operatorname{ht}(\mathfrak{p}) \mid \mathfrak{p} \in \mathsf{Min}(I)\}, \\ \text{max-ht}(I) &:= \max\{\operatorname{ht}(\mathfrak{p}) \mid \mathfrak{p} \in \mathsf{Min}(I)\}, \end{aligned}$$

respectively. Here $\mathsf{Min}(I)$ denotes the set of minimal primes over I and $\operatorname{ht}(\mathfrak{p})$ the height of the prime $\mathfrak{p}$ defined in the usual way.

Suppose R is a commutative Noetherian ring. Then, by the principal ideal theorem, it is clear that for any ideal I of R we can find (finitely many) ideals, say $I_1, \ldots, I_k$ of R, such that $I = I_1 + \cdots + I_k$ and $\text{max-ht}(I_i) \leq 1$, for every i. This suggests an obvious method of proving, via Proposition 1, that a particular (prime) ideal I of a Noetherian ring R has the AR property. But, among other things, the following two instructive examples show that in the noncommutative case the situation is different. Roughly speaking; although in various settings we will be able to obtain what we want, the approach is in general of limited nature.

Example 1. *Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, and let $U(\mathfrak{g})$ be its universal enveloping algebra. Recall the description of* $\operatorname{Spec} U(\mathfrak{g})$*; see [13, Sect. 6.5], and [20, Ex. 1.27], [15, Ex. 3.14]. For that purpose first take $\{y, h, x\}$ to be a standard basis of $\mathfrak{g}$, that is,*

$[x,y]=h$, $[h,x]=2x$ and $[y,h]=2y$. Let then $\mathfrak{h}:=\mathbb{C}h$ be a Cartan subalgebra. For any $\lambda\in\mathfrak{h}^*$, let $M(\lambda)$ be the corresponding Verma module and $L(\lambda)$ its (unique) simple quotient. Define the following ideals:

$$I_z := \operatorname{ann} M(\lambda) \qquad \text{for } \lambda(h)=z\in\mathbb{C},$$
$$I'_z := \operatorname{ann} L(\lambda) \qquad \text{for } \lambda(h)=z\in\mathbb{N};$$

note that $I_z\subset I'_z$ for $z\in\mathbb{N}$. It is easy to see that $I_z=U(\mathfrak{g})(\Omega-z^2+1)$, where Ω denotes the Casimir element. Also, for $z\in\mathbb{N}$, $L(\lambda)$ is the (unique) z-dimensional $\mathfrak{g}$-module. Now we have

$$\operatorname{Spec} U(\mathfrak{g})=\{\langle 0\rangle\}\cup\{I_z\mid z\in\mathbb{C}\}\cup\{I'_z\mid z\in\mathbb{N}\};$$

note that $\operatorname{ht}(I_z)=1$ and $\operatorname{ht}(I'_z)=2$.

Claim 1. *There are no (finitely many) ideals $A_1,\dots,A_k$ of $U(\mathfrak{g})$ such that, for $z\in\mathbb{N}$, $I'_z=A_1+\dots+A_k$ and at the same time $\text{max-ht}(A_i)\le 1$ for every i (cf. (2) below).*

Proof. *Suppose the contrary. Then for some $a_1,\dots,a_k\in\mathbb{N}$ we would have $\sqrt{A_i}=I_{z(1,i)}\cap\dots\cap I_{z(a_i,i)}$ for certain $z(j,i)\in\mathbb{C}$. Hence $(I_{z(1,i)}\cdots I_{z(a_i,i)})^t\subseteq A_i\subseteq I'_z$ for some $t\in\mathbb{N}$, and therefore $I_{z(j,i)}\subseteq I'_z$ for some j. But then clearly $z(j,i)=z$. This implies that $A_i\subseteq I_z$ for every i, and thus $I'_z\subseteq I_z$; a contradiction.* □

Our second remark concerns the AR property. Namely, every I_z satisfies this property; and moreover (see [16, Cor. 3.11])

$$\{I\in\operatorname{Id} U(\mathfrak{g})\mid \text{max-ht}(I)\le 1\}\subseteq\mathcal{AR}\text{-}\operatorname{Id} U(\mathfrak{g}). \tag{2}$$

But the fact is that at least one of the ideals I'_z fails to have it. More precisely, we have the following claim which is not new; cf. the diagram on p. 226 in [8]. For the reader's convenience we give an argument. (Note that the same argument also shows that $U(\mathfrak{g})$ is not an FBN ring.)

Claim 2. *$U(\mathfrak{g})$ is not an AR ring.*

Proof. *If we had that $U(\mathfrak{g})$ is an AR ring, then it would satisfy the second layer condition. But the enveloping algebra $U(\mathfrak{G})$ of a finite-dimensional complex Lie algebra $\mathfrak{G}$ satisfies that condition if and only if it is solvable.* □

Example 2. *Let $\mathfrak{H}=\mathbb{C}x+\mathbb{C}y+\mathbb{C}z$ be the three-dimensional Heisenberg Lie algebra over $\mathbb{C}$, where z is central and $[x,y]=z$. Let $\mathcal{U}=U(\mathfrak{H})$ be its enveloping algebra.*

For any $\alpha,\beta,\gamma\in\mathbb{C}$ and $F\in\operatorname{Irr}\mathbb{C}[X,Y]$, where the latter denotes the set of all irreducible polynomials, define the following left ideals:

$$M_{\alpha,\beta}:=\mathcal{U}z+\mathcal{U}(x-\alpha)+\mathcal{U}(y-\beta),\qquad Q_\gamma:=\mathcal{U}(z-\gamma),\qquad P^{F(x,y)}:=\mathcal{U}z+\mathcal{U}F(x,y).$$

By inspection, these are actually two-sided ideals. The maximal and prime spectra were described in a precise way in [15, App. B] and [16, Sect. 5]; cf. also [2, *5.7 Beispiel]. (This is certainly known, but we are not aware of any appropriate reference.)*

$$\begin{aligned}\mathrm{Spm}\,\mathcal{U} &= \{M_{\alpha,\beta} \mid \alpha,\beta \in \mathbb{C}\} \cup \{Q_\gamma \mid \gamma \in \mathbb{C}^*\},\\ \mathrm{Spec}\,\mathcal{U} &= \{\langle 0\rangle\} \cup \{Q_0\} \cup \{P^{F(x,y)} \mid F \in \mathrm{Irr}\,\mathbb{C}[X,Y]\} \cup \mathrm{Spm}\,\mathcal{U}.\end{aligned}$$

Note that the classical Krull dimension $\dim\mathcal{U}$ *and Krull dimension* $\boldsymbol{\mathcal{K}}(\mathcal{U})$ *satisfy the following:* $\dim\mathcal{U} = \boldsymbol{\mathcal{K}}(\mathcal{U}) = 3$*. Indeed, by the chain of primes*

$$\langle 0\rangle \subset \mathcal{U}z \subset \mathcal{U}z + \mathcal{U}x \subset \mathcal{U}z + \mathcal{U}x + \mathcal{U}y,$$

we have

$$3 \le \dim\mathcal{U} \le \boldsymbol{\mathcal{K}}(\mathcal{U}) \le \boldsymbol{\mathcal{K}}(\mathrm{gr}\,\mathcal{U}) = 3;$$

cf. [12, Thm. 6.6.2].

For localization theoretic reasons in [16] we introduced the notion of a weakly normal element of a Noetherian ring. Namely, given such an element w*, one can localize at* w*. There are a lot of weakly normal elements in* $\mathcal{U}$ *which are not normal. For example,* x*,* $1 + xz$ *and* $x^2 + xz$ *are such elements. An easy exercise shows that for these elements we have*

$$\langle x\rangle = P^x, \qquad \langle 1 + xz\rangle = \mathcal{U}, \qquad \langle x^2 + xz\rangle = (P^x)^2.$$

In particular, it is worthy to note the following: As $\mathrm{ht}(P^x) = 2$*, we see that the minimal prime over the ideal generated by a weakly normal element can be of height* > 1 *(cf. [12, Thm. 4.1.11]).*

Although the ring $\mathcal{U}$ *is in a sense of more simple structure than* $U(\mathfrak{sl}(2))$*, we have the following analog of Claim 1 in the previous example.*

Claim 3. *There are no ideals* $A_1, \ldots, A_k$ *of* $\mathcal{U}$ *of* $\text{max-ht}(A_i) \le 1$ *such that* $P^x = A_1 + \cdots + A_k$*.*

Proof. *Supposing the contrary and noting that* Q_γ*'s are the only primes of height* 1*, a simple argument as in the proof of the above mentioned claim gives that we would have* $A_i \subseteq Q_0$ *for every* i*, and therefore* $P^x \subseteq Q_0$*, yielding to a contradiction.* □

Concerning the structure of $\mathcal{U}$*, it is also interesting to note the following fact; we include a direct short argument. For more on FBN rings we refer the reader to [12, Sect. 6.4].*

Claim 4. $\mathcal{U}$ *is not an FBN ring.*

Proof. *First recall the following simple and very useful result. (A quick proof is via the famous 'Lesieur-Croisot trick' [9].) Given any left (resp. right) Noetherian ring* R *and any* $x \in R$ *which is not a right (resp. left) zero divisor, the left (resp. right) ideal* Rx *(resp.* xR*) is essential.*

Consider now the left ideal $L := \mathcal{U}x$ *of* $\mathcal{U}$*. By the above,* L *is essential. Suppose that* L *contains a nonzero ideal. The same argument as in Claims 1 and 3 enables us to conclude that then* L *contains some nonzero polynomial* $f(z)$*. But this is obviously impossible, and so the claim follows.* □

Let us conclude by an illuminating remark.

Remark 4. *It is known that $U(\mathfrak{sl}(2))$ has the following interesting property, first observed by Bavula ([1]; see also [6]):*

(∗) $IJ = JI$ for any ideals I and J.

It is useful to note that although $U(\mathfrak{H})$ is an AR ring, it does not have the property (∗); cf. [16, Lemma 3.10]. Namely, we leave to the reader to see that, for example,

$$P^x P^y \neq P^y P^x.$$

(At the same time we have $Q_\gamma \mathfrak{p} = \mathfrak{p} Q_\gamma = \mathfrak{p} \cap Q_\gamma$ for any γ and $\mathfrak{p} \in \operatorname{Spec} U(\mathfrak{H})$, and also $MM' = M'M = M \cap M'$ for any $M = M_{\alpha,\beta}$ and $M' = M_{\alpha',\beta'}$ such that $(\alpha,\beta) \neq (\alpha',\beta')$).

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