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Estimates for the spectral condition number of cardinal B-spline collocation matrices*,†

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Abstract. The famous de Boor conjecture states that the condition of the polynomial B-spline collocation matrix at the knot averages is bounded independently of the knot sequence, i.e., it depends only on the spline degree.
For highly nonuniform knot meshes, like geometric meshes, the conjecture is known to be false. As an effort towards finding an answer for uniform meshes, we investigate the spectral condition number of cardinal B-spline collocation matrices. Numerical testing strongly suggests that the conjecture is true for cardinal B-splines.

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1. Introduction

We consider a classical Lagrange function interpolation problem in the following "discrete" setting. Let $\tau_1, \ldots, \tau_\nu$ be a given set of mutually distinct interpolation nodes, and let $f_1, \ldots, f_\nu$ be a given set of "basis" functions.

For any given function g we seek a linear combination of the basis functions that interpolates g at all interpolation nodes,

$$\sum_{j=1}^{\nu} y_j f_j(\tau_i) = g(\tau_i), \quad i = 1, \ldots, \nu.$$

The coefficients y_j can be computed by solving the linear system

$$Ay = g,$$

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†For interpretation of color in all figures, the reader is referred to the web version of this article available at www.mathos.hr/mc.
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where A is the so-called *collocation matrix* containing the values of the basis functions at the interpolation nodes

$$a_{ij} = f_j(\tau_i).$$

If the basis functions are linearly independent on $\{\tau_1, \ldots, \tau_\nu\}$, the matrix A is nonsingular, and the interpolation problem has a unique solution for all functions g.

The sensitivity of the solution is then determined by the condition number $\kappa_p(A)$ of the collocation matrix

$$\kappa_p(A) = \|A\|_p \|A^{-1}\|_p, \tag{1}$$

where $\| \ \|_p$ denotes a standard operator p-norm, with $1 \leq p \leq \infty$.

The polynomial B-splines of a fixed degree d are frequently used as the basis functions in practice. Such a basis is uniquely determined by a given multiset of knots that defines the local smoothness of the basis functions. The corresponding collocation matrix is always totally nonnegative (see [6, 16]), regardless of the choice of the interpolation nodes.

A particular choice of nodes is of special interest, both in theory and in practice, for shape preserving approximation. When the nodes are located at the so-called Greville sites, i.e., at the *knot averages*, the interpolant has the variation diminishing property. Moreover, for a low order B-spline interpolation, it can be shown that $\kappa_\infty(A)$ is bounded *independently* of the knot sequence [6].

On these grounds, in 1975, Carl de Boor [5] conjectured that the interpolation by B-splines of degree d at knot averages is bounded by a function that depends only on d, regardless of the knots themselves. In our terms, the conjecture says that $\kappa_\infty(A)$ or, equivalently, $\|A^{-1}\|_\infty$ is bounded by a function of d only. The conjecture was disproved by Rong–Qing Jia [11] in 1988. He proved that, for geometric meshes, the condition number $\kappa_\infty(A)$ is not bounded independently of the knot sequence, for degrees $d \geq 19$.

Therefore, it is a natural question whether there exists any class of meshes for which de Boor's conjecture is valid. Since geometric meshes are highly nonuniform, the most likely candidates for the validity are uniform meshes.

Here we discuss the problem of interpolation at knot averages by B-splines with *equidistant simple* knots. The corresponding B-splines are symmetric on the support, and have the highest possible smoothness. It is easy to see that the condition of this interpolation does not depend on the knot spacing h, and we can take $h = 1$. So, just for simplicity, we shall consider only the cardinal B-splines, i.e., B-splines with simple knots placed at successive integers. It should be stressed that the only free parameters in this problem are the *degree* d of the B-splines and the *size* ν of the interpolation problem. Our aim is to prove that the condition of A can be bounded independently of its order.

The corresponding collocation matrices A are symmetric, positive definite, and most importantly, Töplitz. But, it is not easy to compute the elements of A^{-1}, or even reasonably sharp estimates of their magnitudes. So, the natural choice of norm in (1) is the spectral norm

$$\kappa_2(A) = \frac{\sigma_{\max}(A)}{\sigma_{\min}(A)}, \tag{2}$$

where $\sigma_{\max}(A)$, and $\sigma_{\min}(A)$ denote the largest and the smallest singular value of A, respectively.

For low spline degrees $d \leq 6$, the collocation matrices are also strictly diagonally dominant, and it is easy to bound $\kappa_2(A)$ by a constant. Consequently, de Boor's conjecture is valid for $d \leq 6$.

For higher degrees, the condition number can be estimated by embedding the Töplitz matrix A into circulant matrices of higher orders. The main advantage of this technique, developed by Davis [4] and Arbenz [2], lies in the fact that the eigenvalues of a circulant matrix are easily computable. The final bounds for $\kappa_2(A)$ are obtained by using the Cauchy interlace theorem for singular values (see [10] for details), to bound both singular values in (2).

The paper is organized as follows. In Section 2 we briefly review some basic properties of cardinal B-splines. The proof of de Boor's conjecture for low degree ($d \leq 6$) cardinal B-splines is given in Section 3. In Section 4 we describe the embedding technique and derive the estimates for $\kappa_2(A)$.

Despite all efforts, we are unable to prove de Boor's conjecture in this, quite probably, the easiest case. The final section contains the results of numerical testing that strongly support the validity of the conjecture, as well as some additional conjectures based on these test results.

2. Properties of cardinal splines

Let $x_i = x_0 + ih$, for $i = 0, \ldots, n$, be a sequence of simple uniformly spaced knots. This sequence determines a unique sequence of normalized B-splines $N_0^d, \ldots, N_{n-d-1}^d$ of degree d, such that the spline N_i^d is non-trivial only on the interval $\langle x_i, x_{i+d+1} \rangle$.

Each of these B-splines can be obtained by translation and scaling from the basic B-spline Q^d with knots $i = 0, \ldots, d+1$,

$$Q^d(x) = \frac{1}{(d+1)!} \sum_{i=0}^{d+1} (-1)^i \binom{d+1}{i} (x-i)_+^d. \tag{3}$$

Here, $(x-i)_+^d$ denotes the truncated powers $(x-i)_+^d = (x-i)^d (x-i)_+^0$, for $d > 0$, while

$$(x-i)_+^0 = \begin{cases} 0, & x < i, \\ 1, & x \geq i. \end{cases}$$

The normalized version of the basic spline is defined as

$$N^d(x) = (d+1) Q^d(x). \tag{4}$$

From (3) and (4), we obtain the *normalized* B-spline basis $\{N_i^d\}$:

$$N_i^d(x) = N^d\left(\frac{x - x_i}{h}\right).$$

If the interpolation nodes τ_i are located at the knot averages, i.e.,

$$x_i^* = \frac{x_{i+1} + \cdots + x_{i+d}}{d} = x_i + h\frac{d+1}{2}, \quad i = 0, \ldots, n-d-1, \tag{5}$$

then

$$x_i < x_i^* < x_{i+d+1},$$

and the Schönberg–Whitney theorem [6] guarantees that the collocation matrix is nonsingular. Moreover, this matrix is totally nonnegative [12], i.e., all of its minors are nonnegative. Due to symmetry of B-splines on uniform meshes, the collocation matrices are also symmetric and Töplitz. So, we can conclude that B-spline collocation matrix is Töplitz, symmetric and positive definite.

It is easy to show that the elements of the collocation matrix do not depend on the step-size h of the uniform mesh, so we take the simplest one with $h = 1$ and $x_i = i$. Such B-splines are called cardinal. The interpolation nodes (5) are integers for odd degrees, while for even degrees, the interpolation nodes are in the middle of the two neighbouring knots of the cardinal B-spline.

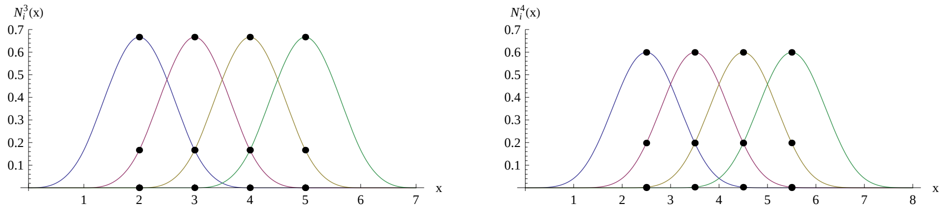

Figure 1. *The first four cardinal splines N_i^d of degree $d = 3$ and 4, respectively. Big black dots denote the spline values at the knot averages*

The normalized basic cardinal spline N^d suffices to determine all basis function values at the interpolation nodes

$$N_i^d(x_j^*) = N^d(x_{j-i}^*).$$

The general de Boor–Cox reccurence relation [6], written in terms of the degree of a spline is:

$$N^d(x) = \frac{xN^{d-1}(x) + (d+1-x)N^{d-1}(x-1)}{d}. \tag{6}$$

Note that the elements of a collocation matrix are rational, because the interpolation nodes are rational, and the de Boor–Cox recurrence formula (6) involves only basic arithmetic operations on rational coefficients. These elements are therefore exactly computable in (arbitrary precise) rational arithmetic.

3. Low degree cardinal B-splines

Let $t_i^d = N^d(x_i^*)$ denote the values of cardinal B-splines at knot averages (see (5)). Then, the cardinal B-spline collocation matrix A with interpolation nodes x_i^* is a banded symmetric Töplitz matrix of order $n - d$, to be denoted by

$$T_n^d = \begin{bmatrix} t_0^d & \cdots & t_r^d & & 0 \\ \vdots & \ddots & & \ddots & \\ t_r^d & & \ddots & & t_r^d \\ & \ddots & & \ddots & \vdots \\ 0 & & t_r^d & \cdots & t_0^d \end{bmatrix}, \quad r = \left\lfloor \frac{d}{2} \right\rfloor. \tag{7}$$

The matrix T_n^d is represented by its first row, usually called the symbol,

$$t = (t_0^d, \dots, t_r^d, 0, \dots, 0), \quad t \in \mathbb{R}^{n-d}.$$

It is useful to note that each B-spline of degree $d > 0$ is a unimodal function, i.e., it has only one local maximum on the support. In the case of cardinal B-splines, we have already concluded that the splines are symmetric, and therefore the maximum value of N^d is attained at the middle of the support, for $x = (d{+}1)/2$. The maximum value is

$$N^d\left(\frac{d+1}{2}\right) = N^d(x_0^*) = t_0^d.$$

Furthermore, unimodality implies that the values of the spline N^d are decreasing in the interval $[(d+1)/2, d+1]$, so

$$t_0^d > t_1^d > \cdots > t_r^d. \tag{8}$$

To estimate the condition number of a cardinal B-spline we need to bound both the minimal and the maximal singular value of T_n^d. For a symmetric and positive definite matrix, the singular values are eigenvalues. Therefore, the bounds for the eigenvalues of T_n^d are sought for. From the Geršgorin bound for the eigenvalues, and the partition of unity of the B-spline basis, we obtain an upper bound for $\lambda_{\max}(T_n^d)$

$$\lambda_{\max}(T_n^d) \le t_0^d + 2(t_1^d + \cdots + t_r^d) = 1. \tag{9}$$

Similarly, we also obtain a lower bound for $\lambda_{\min}(T_n^d)$,

$$\lambda_{\min}(T_n^d) \ge t_0^d - 2(t_1^d + \cdots + t_r^d) = 2t_0^d - 1, \tag{10}$$

which is sensible only if T_n^d is strictly diagonally dominant. Strict diagonal dominance is achieved only for B-spline degrees $d = 1, \dots, 6$ (easily verifiable by a computer). The corresponding Geršgorin bounds are presented in Table 1. This directly proves de Boor's conjecture for low order B-splines.

$n\backslash d$	2	3	4	5	6
64	1.998136	2.994873	4.785918	7.466648	11.727897
128	1.999541	2.998757	4.796641	7.492176	11.785901
256	1.999886	2.999694	4.799180	7.498105	11.799106
512	1.999971	2.999924	4.799797	7.499534	11.802256
1024	1.999993	2.999981	4.799950	7.499884	11.803026
2048	1.999998	2.999995	4.799987	7.499971	11.803216
GB(d)	2	3	$\frac{96}{19} \approx 5.052632$	10	$\frac{5760}{127} \approx 45.354331$

Table 1. *Comparison of the actual condition numbers $\kappa_2(T_n^d)$, for $d = 2, \dots, 6$, $n = 64, \dots, 2048$, and the bounds GB(d) for $\kappa_2(T_n^d)$, obtained by the Geršgorin circle theorem*

Note that in the case of tridiagonal Töplitz matrices, i.e. for $d = 2, 3$, and, thus, $r = 1$ in (7), the exact eigenvalues are also known (see Böttcher–Grudsky [3])

$$\lambda_k(T_n^d) = t_0^d + 2t_1^d \cos \frac{\pi k}{n-d+1}, \quad d = 2, 3, \quad k = 1, \dots, n-d.$$

The largest and the smallest eigenvalue can then be uniformly bounded by

$$\begin{aligned}
\lambda_{\max}(T_n^d) &= t_0^d + 2t_1^d \cos \frac{\pi}{n-d+1} < t_0^d + 2t_1^d, \\
\lambda_{\min}(T_n^d) &= t_0^d - 2t_1^d \cos \frac{\pi}{n-d+1} > t_0^d - 2t_1^d > 0.
\end{aligned}$$

These uniform bounds are somewhat better than those obtained by the Geršgorin circles.

4. Embeddings of Töplitz matrices into circulants

When the degree of a cardinal B-spline is at least 7, the eigenvalue bounds for Töplitz matrices can be computed by circulant embeddings. First, we shall introduce the smallest possible circulant embedding and give its properties. Then we shall present some other known embeddings, with positive semidefinite circulants.

To obtain a bound for $\lambda_{\min}(T_n^d)$, the collocation matrix T_n^d is to be embedded into a circulant

$$C_m^d = \left[\begin{array}{ccccc|ccc}
 & & & & & t_r^d & \cdots & t_1^d \\
 & & & & & & \ddots & \vdots \\
 & & & & & & & t_r^d \\
 & & T_n^d & & & & & \\
 & & & & & t_r^d & & \\
 & & & & & \vdots & \ddots & \\
 & & & & & t_1^d & & t_r^d \\ \hline
t_r^d & & & t_r^d \cdots & t_1^d & t_0^d & \ddots & \vdots \\
\vdots & \ddots & & \ddots & & \ddots & \ddots & t_1^d \\
t_1^d & \cdots & t_r^d & & t_r^d & \cdots & t_1^d & t_0^d
\end{array}\right], \quad m = n - d + r. \tag{11}$$

It is obviously a Töplitz matrix with the following symbol

$$t = (t_0^d, \dots, t_r^d, 0, \dots, 0, t_r^d, \dots, t_1^d), \quad t \in \mathbb{R}^{n-d+r}.$$

This circulant C_m^d is called a periodization of T_n^d by Böttcher and Grudsky [3].

The bounds (9)–(10) for the eigenvalues of T_n^d are also valid for C_m^d. Moreover, C_m^d is doubly stochastic, always having $\lambda_{\max}(C_m^d) = 1$ as its largest eigenvalue. Interestingly enough, the upper bound (9) is attained here (the Geršgorin bounds are rarely so sharp).

The symmetry of T_n^d immediately implies the symmetry of C_m^d, and we can conclude that the eigenvalues of C_m^d are real, but not necessarily positive. For symmetric matrices, the singular values are, up to a sign, equal to the eigenvalues, so

$$\sigma_i(C_m^d) = |\lambda_i(C_m^d)|. \tag{12}$$

If the eigenvalues of the circulant C_m^d are known, the spectrum of embedded T_n^d can be bounded by the Cauchy interlace theorem for singular values, applied to C_m^d.

Theorem 1 (Cauchy interlace theorem)**.** *Let $C \in \mathbb{C}^{m\times n}$ be given, and let C_ℓ denote a submatrix of C obtained by deleting a total of ℓ rows and/or ℓ columns of C. Then*

$$\sigma_k(C) \geq \sigma_k(C_\ell) \geq \sigma_{k+\ell}(C), \quad k = 1, \ldots, \min\{m, n\},$$

where we set $\sigma_j(C) \equiv 0$ if $j > \min\{m, n\}$.

The proof can be found, for example, in [10, p. 149].

If we delete the last r rows and columns of C_m^d, we obtain T_n^d. The Cauchy interlace theorem will then give useful bounds for $\sigma_{\min}(T_n^d) = \lambda_{\min}(T_n^d)$, provided that C_m^d is nonsingular. Moreover, if we delete more than r last rows and columns of C_m^d, we obtain bounds for Töplitz matrices T_k^d, of order $k-d$, for $k \leq n$,

$$\kappa_2(T_k^d) = \frac{\sigma_{\max}(T_k^d)}{\sigma_{\min}(T_k^d)} \leq \frac{\sigma_{\max}(C_m^d)}{\sigma_{\min}(C_m^d)} = \frac{1}{\min_j |\lambda_j(C_m^d)|}. \tag{13}$$

Now we need to calculate the smallest singular value of C_m^d, and show that it is non-zero.

The eigendecomposition of a circulant matrix is well-known (see [4, 2]). A circulant C of order m, defined by the symbol $(c_0, \ldots, c_{m-1})$, can be written as

$$C = \sum_{j=0}^{m-1} c_j \Pi^j,$$

where

$$\Pi = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{bmatrix}.$$

The spectral decomposition of Π is $\Pi = F\Omega F^*$, where

$$\Omega = \operatorname{diag}(1, \omega, \omega^2, \ldots, \omega^{m-1}), \quad \omega = \frac{2\pi i}{m}, \quad i = \sqrt{-1},$$

while

$$F_{j,k} = \frac{1}{\sqrt{m}} \omega^{kj}, \quad 0 \leq k, j \leq m-1.$$

Hence, C can be decomposed as

$$C = F\Lambda F^*, \quad \Lambda = \operatorname{diag}(\lambda_0, \ldots, \lambda_{m-1}) = \sum_{j=0}^{m-1} c_j \Omega^j.$$

The eigenvalues of a real symmetric circulant C are real and given by

$$\lambda_k(C) = c_0 + \sum_{j=1}^{m-1} c_j \cos\frac{2\pi kj}{m}, \quad k = 0, \dots, m-1. \tag{14}$$

They can also be viewed as the discrete Fourier transform (DFT) of the symbol $(c_0, \dots, c_{m-1})$.

For real and symmetric C, i.e., when $c_k = c_{m-k}$, for $k = 1, \dots, m-1$, from (14) it also follows that

$$\lambda_k(C) = c_0 + \sum_{j=1}^{m-1} c_j \cos\frac{2\pi kj}{m} = c_0 + \sum_{j=1}^{m-1} c_{m-j} \cos\frac{2\pi k(m-j)}{m} = \lambda_{m-k}(C).$$

So, all the eigenvalues, except $\lambda_0(C)$, and possibly $\lambda_{\frac{m}{2}}(C)$, for even m, are multiple.

Therefore, the eigenvalues of the circulant C_m^d from (11) are

$$\lambda_k(C_m^d) = t_0^d + 2\sum_{j=1}^{r} t_j^d \cos\frac{2\pi kj}{m}, \quad k = 0, \dots, m-1. \tag{15}$$

For prime orders m, the nonsingularity of C_m^d is a consequence of the following theorem from [9].

Theorem 2 (Geller, Kra, Popescu and Simanca)**.** *Let m be a prime number. Assume that the circulant C of order m has entries in $\mathbb{Q}$. Then* $\det C = 0$ *if and only if*

$$\lambda_0 = \sum_{j=0}^{m-1} c_i = 0,$$

or all the symbol entries c_i are equal.

If m is prime, then we must have $\det C_m^d \neq 0$, since (8) implies that c_i's are not equal, and from (15) we get

$$\lambda_0 = c_0 + 2\sum_{j=1}^{r} c_j = 1 \neq 0.$$

Theorem 2 suggests how to get the nonsingular embedding of T_n^d. First, T_n^d should be embedded into the Töplitz matrix T_p^d, of order $p-d$, where $p \geq n$ is chosen so that $m = p-d+r$ is a prime number. Then, T_p^d is embedded into the circulant C_m^d.

The other possibility is to embed T_n^d into the smallest circulant matrix C_m^d, as in (11), and calculate its eigenvalues from (15), hoping that C_m^d is nonsingular. In this case, extensive numerical testing suggests that C_m^d is always positive definite, but we have not been able to prove it.

There are also several other possible embeddings that guarantee the positive semidefiniteness of the circulant matrix C.

The first one, constructed by Dembo, Mallows and Shepp in [7], ensures that the positive definite Töplitz matrix T of order n can be embedded in the positive semidefinite circulant C of order m, where

$$m \geq 2\left(n + \kappa_2(T)\frac{n^2}{\sqrt{6}}\right). \tag{16}$$

A few years later, Newsam and Dietrich [14] reduced the size of embedding to

$$m \geq 2\sqrt{6n^2 + \kappa_2(T)\frac{3 \cdot 2^{11/2}\, n^{5/2}}{5^{5/2}}}. \tag{17}$$

Note that among all positive semidefinite matrices C of order greater than or equal to m, we can choose one of prime order. This embedding will be positive definite according to Theorem 2. It is obvious that embeddings (16)–(17) are bounded by a function of the condition number of T, i.e., the quantity we are trying to bound.

Ferreira in [8] embeds a Töplitz matrix T of order n, defined by the symbol $t = (t_0, \dots, t_r, 0, \dots, 0) \in \mathbb{R}^n$, into the circulant C of order $m = 2n$,

$$C = \begin{bmatrix} T & S \\ S & T \end{bmatrix}, \tag{18}$$

where the symbol of the Töplitz matrix S is $s = (0, \dots, 0, t_r, \dots, t_1) \in \mathbb{R}^n$.

If we take $T = T_n^d$ from (7), the only difference between embeddings (11) and (18) is in exactly $n - d - r$ zero diagonals, added as the first diagonals of S. A sufficient condition for positive semidefiniteness of C is given by the next result.

Theorem 3 (Ferreira)**.** *Let C be defined as in (18), and let $b^T = [t_0, \dots, t_{n-1}]$, $c^T = [t_{n-1}, \dots, t_1]$. If T is positive definite, and $|b^T T^{-1} c| < 1$, then C is positive semidefinite.*

Once again, there is no obvious efficient way to verify whether the condition $|b^T T^{-1} c| < 1$ is fullfiled or not.

5. Conjecture about the minimal eigenvalues

Extensive numerical testing has been conducted by using *Mathematica* 7 from Wolfram Research for symbolic, arbitrary-precision rational, and machine-precision floating-point computations. Whenever feasible, full accuracy was maintained. Owing mostly to the elegance and the accuracy of these results, an insight into and the following conjecture about the spectral properties of the collocation matrices and the corresponding periodizations were obtained.

Conjecture 1 (The smallest eigenvalue of a circulant)**.** *The circulant C_m^d from (11) is always positive definite, and the index μ of its smallest eigenvalue $\lambda_\mu(C_m^d)$ is always the integer nearest to $m/2$, i.e.,*

$$\lambda_\mu(C_m^d) = \begin{cases} \lambda_{\frac{m\pm1}{2}}(C_m^d) = t_0^d + 2\displaystyle\sum_{j=1}^{r}(-1)^j t_j^d \cos\frac{\pi j}{m}, & m \text{ odd}, \\ \lambda_{\frac{m}{2}}(C_m^d) = t_0^d + 2\displaystyle\sum_{j=1}^{r}(-1)^j t_j^d, & m \text{ even}. \end{cases} \tag{19}$$

Figure 2 illustrates both cases of Conjecture 1.

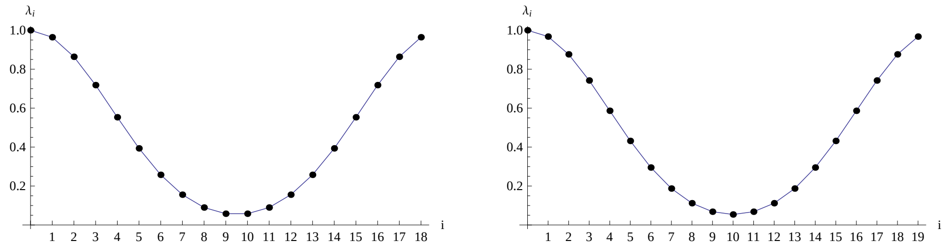

Figure 2. *The eigenvalues (black dots)* $\lambda_k(C_m^d)$ *for spline of degree* $d = 7$ *with* $n = 23, 24$*, respectively. The associated circulants have order* $m = n - 7 + 3$*, i.e.,* 19 *and* 20*. Note that for* $m = 20$ *there is only one minimal eigenvalue, while for* $m = 19$ *we have two minimal eigenvalues*

For even m, $\lambda_\mu(C_m^d)$ (and, therefore, $\kappa_2(C_m^d)$) depends solely on d, i.e., the order m of a circulant is irrelevant here. Moreover, for m odd and even alike, the limiting value of $\lambda_\mu(C_m^d)$ is the same:

$$\lambda_\infty^d := \lim_{m\to\infty} \lambda_\mu(C_m^d) = t_0^d + 2\sum_{j=1}^{r}(-1)^j t_j^d. \tag{20}$$

Hence, the notation λ_∞^d is justified, since that value is determined uniquely by the degree d of the chosen cardinal splines. This is consistent with de Boor's conjecture.

Equations (19) and (20) provide us with efficiently and exactly computable estimates of the spectral condition numbers of large collocation matrices T_n^d. As demonstrated in Figure 3 and Table 2, the smallest eigenvalues of the collocation matrices converge rapidly and monotonically to the smallest eigenvalues of the corresponding circulant periodizations C_m^d, as well as to the limiting value (20).

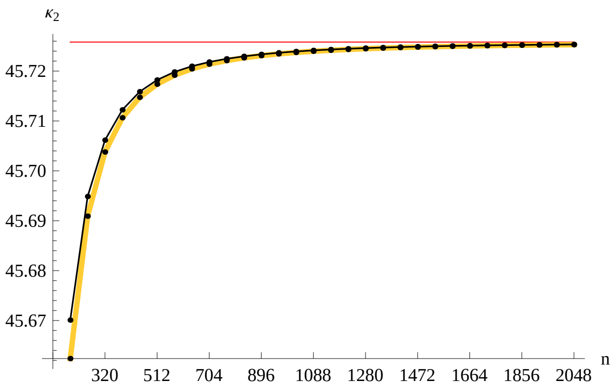

Figure 3. *Spectral condition numbers of Töplitz matrices* T_n^9 *(lower, brighter line), and the circulant periodizations* C_m^9 *(solid black line). The constant function denotes* $1/\lambda_\infty^9$

It is worth noting that the spectral bounds obtained in such a way for lower degrees ($d = 2, \ldots, 6$) of cardinal B-splines are quite sharper than those established by the Geršgorin circle theorem (cf. Table 1 and Table 2), at no additional cost.

$n\backslash d$	T_n^2	C_m^2	T_n^5	C_m^5	T_n^6	C_m^6
64	1.998137	1.998758	7.466648	7.472749	11.72790	11.74214
128	1.999541	1.999694	7.492176	7.493492	11.78590	11.78866
256	1.999886	1.999924	7.498105	7.498410	11.79911	11.79971
512	1.999971	1.999981	7.499534	7.499607	11.80226	11.80240
1024	1.999993	1.999995	7.499884	7.499902	11.80303	11.80306
2048	1.999998	1.999999	7.499971	7.499975	11.80322	11.80322
$1/\lambda_\infty^d$		2.000000		7.500000		11.80328

$n\backslash d$	T_n^9	C_m^9	T_n^{21}	C_m^{21}	T_n^{30}	C_m^{30}
64	45.04067	45.17179	9012.21	9543.49	371000.6	502472.1
128	45.57648	45.59721	10100.96	10150.47	569223.5	579852.3
256	45.69092	45.69486	10273.67	10279.58	594976.6	596037.0
512	45.71737	45.71822	10308.14	10309.00	599497.1	599628.0
1024	45.72373	45.72393	10315.86	10316.01	600450.4	600469.5
2048	45.72529	45.72534	10317.69	10317.72	600669.7	600673.0
$1/\lambda_\infty^d$		45.72581		10318.28		600739.5

Table 2. *Comparison of the spectral condition numbers* $\kappa_2(T_n^d)$ *and* $\kappa_2(C_m^d)$*, for* $d = 2, 5, 6, 9, 21, 30$*,* $n = 64, \ldots, 2048$*,* $m = n - d + r$*, and* $1/\lambda_\infty^d$

Since t_j^d are rational numbers, (20) is useful for the exact computation of λ_∞^d. But, in floating-point arithmetic, the direct computation of λ_∞^d from (20) is numerically unstable, as it certainly leads to severe cancellation.

It can be easily shown from (3) or (6) that the smallest non-zero value of the cardinal B-spline of degree d at an interpolation node is:

$$t_r^d = \begin{cases} N^d(1) = \frac{1}{d!}, & \text{for odd } d, \\ N^d\left(\frac{1}{2}\right) = \frac{1}{2^d \cdot d!}, & \text{for even } d. \end{cases}$$

Moreover, all other values t_j^d in (20) and, consequently, λ_∞^d are integer multiples of t_r^d. With that in mind, yet another, somewhat surprising conjecture emerged from the test results:

$$\lambda_\infty^d = \begin{cases} t_r^d \cdot T_d = \frac{1}{d!} T_d, & d \text{ odd}, \\ t_r^d \cdot 2^d E_d = \frac{1}{d!} E_d, & d \text{ even}, \end{cases} \tag{21}$$

where, as in [13], T_n are the tangent numbers, and E_n are the Euler numbers, defined by the Taylor expansions of $\tan t$ and $\sec t$, respectively,

$$\tan t = \sum_{n=0}^{\infty} T_n \frac{t^n}{n!}, \quad \sec t = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}.$$

These numbers are also related to the sequences A000182 (the tangent or "zag" numbers), A000364 (the Euler or "zig" numbers) and A002436, from [17].

If true, (21) would be of significant practical merit, for there exist very stable and elegant algorithms for calculation of T_n and E_n by Knuth and Buckholtz [13]. So, it deserved an effort to find the proof.

A unifying framework for handling both cases is provided by the Euler polynomials $E_n(x)$, defined by the following exponential generating function (see [1, 23.1.1, p. 804])

$$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \tag{22}$$

which is valid for $|t| < \pi$.

First, note that $T_{2k} = E_{2k+1} = 0$, for all $k \geq 0$. The remaining nontrivial values can be expressed in terms of special values of Euler polynomials. For the tangent numbers, we have

$$T_{2k+1} = (-1)^k 2^{2k+1} E_{2k+1}(1), \quad k \geq 0. \tag{23}$$

This follows easily, by comparing the Taylor expansion of $1 + \tanh t$

$$1 + \tanh t = \frac{2e^{2t}}{e^{2t}+1} = 1 + \sum_{k=0}^{\infty} (-1)^k T_{2k+1} \frac{t^{2k+1}}{(2k+1)!}$$

and (22), with $x = 1$ and $2t$, instead of t. Similarly, by comparing the Taylor expansion of $\operatorname{sech} t$

$$\operatorname{sech} t = \frac{2e^t}{e^{2t}+1} = \sum_{k=0}^{\infty} (-1)^k E_{2k} \frac{t^{2k}}{(2k)!}$$

and (22), with $x = 1/2$ and $2t$, instead of t, we get

$$E_{2k} = (-1)^k 2^{2k} E_{2k}\left(\frac{1}{2}\right), \quad k \geq 0. \tag{24}$$

The following identities will also be needed in the proof of (21).

Lemma 1. *Let $d \geq 0$ be a non-negative integer. Then*

$$\sum_{\ell=0}^{d+1} (-1)^\ell \binom{d+1}{\ell} E_n(\ell) = 0, \tag{25}$$

$$\sum_{\ell=0}^{d+1} (-1)^\ell \binom{d+1}{\ell} E_n(\ell+1) = 0, \tag{26}$$

for all $n = 0, \ldots, d$.

Proof. Consider the function g_d defined by

$$g_d(t) := \frac{2(1-e^t)^{d+1}}{e^t+1} = \sum_{\ell=0}^{d+1} (-1)^\ell \binom{d+1}{\ell} \frac{2e^{\ell t}}{e^t+1}.$$

From (22) with $x = \ell$, the Taylor expansion of g_d can be written as

$$g_d(t) = \sum_{n=0}^{\infty} \left[\sum_{\ell=0}^{d+1} (-1)^\ell \binom{d+1}{\ell} E_n(\ell) \right] \frac{t^n}{n!},$$

so

$$D^n g_d(t)\big|_{t=0} = \sum_{\ell=0}^{d+1} (-1)^\ell \binom{d+1}{\ell} E_n(\ell), \quad n \geq 0.$$

On the other hand, the Leibniz rule gives

$$D^n g_d(t) = \sum_{m=0}^{n} \binom{n}{m} D^m \left[(1-e^t)^{d+1} \right] D^{n-m} \left[\frac{2}{e^t+1} \right].$$

If $n \leq d$, then $D^m \left[(1-e^t)^{d+1} \right]$ is always divisible by $(1-e^t)$. Hence,

$$D^n g_d(t)\big|_{t=0} = 0, \quad n = 0, \ldots, d,$$

which proves the first identity (25).

The second one follows similarly, by considering

$$h_d(t) := g_d(t) - g_{d+1}(t) = \frac{2e^t(1-e^t)^{d+1}}{e^t+1} = \sum_{\ell=0}^{d+1} (-1)^\ell \binom{d+1}{\ell} \frac{2e^{(\ell+1)t}}{e^t+1}.$$

The Taylor expansion of h_d is then given by

$$h_d(t) = \sum_{n=0}^{\infty} \left[\sum_{\ell=0}^{d+1} (-1)^\ell \binom{d+1}{\ell} E_n(\ell+1) \right] \frac{t^n}{n!}.$$

If $n \leq d$, from the first part of the proof, it follows immediately that

$$D^n h_d(t)\big|_{t=0} = D^n g_d(t)\big|_{t=0} - D^n g_{d+1}(t)\big|_{t=0} = 0,$$

which proves (26). □

Finally, we are ready to prove the conjecture (21).

Theorem 4 (Relation to integer sequences)**.** *The following holds for all cardinal B-spline degrees $d \geq 0$*

$$\lambda_\infty^d = \frac{1}{d!} \cdot \begin{cases} T_d, & d \text{ odd}, \\ E_d, & d \text{ even}. \end{cases}$$

Proof. To simplify the notation, let $L_d := d!\,\lambda_\infty^d$. Due to the symmetry of interpolation nodes, the sum in (20) can be written as

$$\lambda_\infty^d = \sum_{j=-r}^{r} (-1)^j t_j^d, \quad t_j^d = N^d\left(j + \frac{d+1}{2}\right), \quad j = -r, \ldots, r,$$

where $r = \lfloor d/2 \rfloor$. From (3) and (4), it follows that

$$t_j^d = \frac{1}{d!}\sum_{\ell=0}^{d+1}(-1)^\ell \binom{d+1}{\ell}\left(j + \frac{d+1}{2} - \ell\right)_+^d .$$

Then

$$L_d = \sum_{j=-r}^{r}(-1)^j \sum_{\ell=0}^{d+1}(-1)^\ell\binom{d+1}{\ell}\left(j - \ell + \frac{d+1}{2}\right)_+^d . \tag{27}$$

Let d be odd, $d = 2k+1$, with $k \geq 0$. Then $r = k$ and $(d+1)/2 = k+1$, so (27) becomes

$$L_{2k+1} = \sum_{j=-k}^{k}(-1)^j \sum_{\ell=0}^{2k+2}(-1)^\ell\binom{2k+2}{\ell}(j - \ell + k + 1)_+^{2k+1} .$$

From the definition of truncated powers with positive exponents, the second sum contains only the terms with $j - \ell + k + 1 > 0$, i.e., for $l \leq j + k$. By changing the order of summation, we get

$$L_{2k+1} = \sum_{\ell=0}^{2k}(-1)^\ell\binom{2k+2}{\ell}\sum_{j=\ell-k}^{k}(-1)^j\,(j - \ell + k + 1)^{2k+1} .$$

Then we shift j by $k - \ell + 1$, so that j starts at 1, to obtain

$$L_{2k+1} = (-1)^k\sum_{\ell=0}^{2k}(-1)^\ell\binom{2k+2}{\ell}\sum_{j=1}^{2k+1-\ell}(-1)^{(2k+1-\ell)-j} j^{2k+1}.$$

The second sum can be simplified as (see [1, 23.1.4, p. 804])

$$\sum_{j=1}^{2k+1-\ell}(-1)^{(2k+1-\ell)-j} j^{2k+1} = \frac{1}{2}\left(E_{2k+1}(2k+2-\ell) + (-1)^{2k+2-\ell}E_{2k+1}(1)\right).$$

Hence

$$L_{2k+1} = \frac{(-1)^k}{2}\left[\sum_{\ell=0}^{2k}(-1)^\ell\binom{2k+2}{\ell}E_{2k+1}(2k+2-\ell) + E_{2k+1}(1)\sum_{\ell=0}^{2k}\binom{2k+2}{\ell}\right].$$

By reversing the summation, from (25) with $d = 2k+1$ and $n = d$, we conclude that

$$\begin{aligned}\sum_{\ell=0}^{2k}(-1)^\ell\binom{2k+2}{\ell}E_{2k+1}(2k+2-\ell) &= \sum_{\ell=2}^{2k+2}(-1)^\ell\binom{2k+2}{\ell}E_{2k+1}(\ell)\\ &= -\sum_{\ell=0}^{1}(-1)^\ell\binom{2k+2}{\ell}E_{2k+1}(\ell).\end{aligned}$$

Since $E_{2k+1}(0) = -E_{2k+1}(1)$, by using (23), we have

$$L_{2k+1} = \frac{(-1)^k}{2} E_{2k+1}(1) \sum_{\ell=0}^{2k+2} \binom{2k+2}{\ell} = (-1)^k 2^{2k+1} E_{2k+1}(1) = T_{2k+1}.$$

This proves the claim for odd values of d.

Let d be even, $d = 2k$, with $k \geq 0$. For $d = 0$, it is obvious that $L_0 = t_0^0 = 1 = E_0$, so we may assume that $k > 0$. Then $r = k$ and $(d+1)/2 = k + 1/2$, so (27) becomes

$$L_{2k} = \sum_{j=-k}^{k} (-1)^j \sum_{\ell=0}^{2k+1} (-1)^\ell \binom{2k+1}{\ell} \left(j - \ell + k + \frac{1}{2}\right)_+^{2k}.$$

The second sum contains only the terms with $j - \ell + k + 1/2 > 0$, i.e., for $l \leq j + k$. By exactly the same transformation as before, we arrive at

$$L_{2k} = (-1)^k \sum_{\ell=0}^{2k} (-1)^\ell \binom{2k+1}{\ell} \sum_{j=1}^{2k+1-\ell} (-1)^{(2k+1-\ell)-j} \left(j - \frac{1}{2}\right)^{2k}.$$

Now we expand the last factor in terms of powers of j. Then L_{2k} can be written as

$$L_{2k} = (-1)^k \sum_{n=0}^{2k} \binom{2k}{n} \left(-\frac{1}{2}\right)^{2k-n} S_{2k,n}, \tag{28}$$

with

$$S_{2k,n} = \sum_{\ell=0}^{2k} (-1)^\ell \binom{2k+1}{\ell} \sum_{j=1}^{2k+1-\ell} (-1)^{(2k+1-\ell)-j} j^n, \quad n = 0, \ldots, 2k.$$

Like before, the second sum can be simplified as

$$\sum_{j=1}^{2k+1-\ell} (-1)^{(2k+1-\ell)-j} j^n = \frac{1}{2} \left(E_n(2k+2-\ell) + (-1)^{2k+2-\ell} E_n(1)\right),$$

which gives

$$S_{2k,n} = \frac{1}{2} \left[\sum_{\ell=0}^{2k} (-1)^\ell \binom{2k+1}{\ell} E_n(2k+2-\ell) + E_n(1) \sum_{\ell=0}^{2k} \binom{2k+1}{\ell}\right].$$

By reversing the summation, from (26) with $d = 2k$, for $n = 0, \ldots, d$, we see that

$$\sum_{\ell=0}^{2k} (-1)^\ell \binom{2k+1}{\ell} E_n(2k+2-\ell) = -\sum_{\ell=1}^{2k+1} (-1)^\ell \binom{2k+1}{\ell} E_n(\ell+1) = E_n(1).$$

Therefore,

$$S_{2k,n} = \frac{1}{2} E_n(1) \sum_{\ell=0}^{2k+1} \binom{2k+1}{\ell} = 2^{2k} E_n(1).$$

From (28) we obtain

$$L_{2k} = (-1)^k 2^{2k} \sum_{n=0}^{2k} \binom{2k}{n} E_n(1) \left(-\frac{1}{2}\right)^{2k-n}.$$

Finally, by using [1, 23.1.7, p. 804])

$$\sum_{n=0}^{2k} \binom{2k}{n} E_n(1) \left(-\frac{1}{2}\right)^{2k-n} = E_{2k}\left(\frac{1}{2}\right).$$

Together with (24), this gives

$$L_{2k} = (-1)^k 2^{2k} E_{2k}\left(\frac{1}{2}\right) = E_{2k}.$$

This completes the proof for even values of d. □

The proof of Theorem 4 also follows by using exponential splines (see [15]).

We would like to conclude with an observation that, to the best of our knowledge, scarcely any result could be found about sufficient conditions for the non-negativeness of the DFT in terms of its coefficients, apart from the classical result of Young and Kolmogorov (cited in Zygmund [18, page 109]):

Theorem 5. *For a convex sequence* $(a_n, n \in \mathbb{N})$*, where* $\lim_{n\to\infty} a_n = 0$*, the sum*

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

converges (save for $x = 0$*), and is non-negative.*

Here, a sequence is convex if $\Delta^2 a_n \geq 0$ for all n, with $\Delta a_n = a_n - a_{n+1}$.

Convexity is not fulfilled in the case of cardinal B-spline coefficients, since there is always one inflection point on each slope of the spline. And yet, our numerical experiments strongly suggest that the class of series with a positive DFT is worth investigating further, for the theoretical and practical reasons alike.

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