MATHEMATICAL COMMUNICATIONS 521
Math. Commun., Vol. **15**, No. 2, pp. 521-537 (2010)

Common fixed points of self maps satisfying an integral type contractive condition in fuzzy metric spaces

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Received October 10, 2009; accepted May 1, 2010

Abstract. In this paper, first we prove fixed point theorems for different variant of compatible maps, satisfying a contractive condition of integral type in fuzzy metric spaces, which improve the results of Branciari [2], Rhoades [33], Kumar et al. [23], Subramanyam [35] and results of various authors cited in the literature of "Fixed Point Theory and Applications". Secondly, we introduce the notion of any kind of weakly compatible maps and prove a fixed point theorem for weakly compatible maps along with the notion of any kind of weakly compatible. At the end, we prove a fixed point theorem using variants of R-Weakly commuting mappings in fuzzy metric spaces.

AMS subject classifications: 47H10, 54H25

Key words: Common fixed point, compatible map, weakly compatible maps, any kind of weakly compatible

1. Introduction

The notion of a probabilistic metric space corresponds to the situation when we do not know the distance between the points but one knows only the possible values of the distance. Since the 16th century, probability theory has been studying a kind of uncertainty known as the randomness of the occurrence of an event; in this case, the event itself is completely certain. The study of mathematics began to explore the restricted zone-fuzziness, which followed the study of uncertainty and randomness. Fuzziness is a kind of uncertainty. It is applied to those events, whose chances of occurrence are uncertain, i.e., they are in non-black or non-white state. Zadeh [38] introduced the concept of fuzzy set as a new way to represent vagueness in our everyday life. A fuzzy set A in X is a function with domain X and values in $[0, 1]$. Since then, many authors have developed a lot of literature regarding the theory of fuzzy sets and its applications. However, when the uncertainty is due to fuzziness rather than randomness, as in the measurement of an ordinary length, it seems that the concept of a fuzzy metric space is more suitable.

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There are many viewpoints of the notion of the metric space in fuzzy topology.We can divide them into following two groups:
The first group involves those results in which a fuzzy metric on a set X is treated as a map d: X x $X \to \mathbb{R}^+$, where X represents the totality of all fuzzy points of a set and satisfy some axioms which are analogous to the ordinary metric axioms. Thus, in such an approach numerical distances are set up between fuzzy objects. On the other hand, in the second group, we keep those results in which the distance between objects is fuzzy and the objects themselves may or may not be fuzzy.

Especially, Erceg [10], Kaleva and Seikkala [20], Kramosil and Michalek [21] have introduced the concept of fuzzy metric space in different ways. Grabiec [13] followed Kramosil and Michalek [21] and obtained a fuzzy version of Banach contraction principle. Grabiec [13] results were further generalized for a pair of commuting mappings by Subramanyam [35]. Moreover, George and Veermani [11] modified the concept of fuzzy metric spaces, introduced by Kramosil and Michalek [21]. Further, George and Veermani [12] introduced the concept of Hausdorff topology on fuzzy metric spaces and showed that every metric induces a fuzzy metric.

Definition 1. *A fuzzy set A in X is a function with domain X and values in* $[0, 1]$.

Definition 2. *A binary operation* $* : [0, 1]$ *x* $[0, 1] \to [0, 1]$ *is a continuous t-norm if* $\{[0, 1], .\}$ *is an abelian topological monoid with unit* 1 *such that* $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$, $a, b, c, d \in [0, 1]$.

Definition 3. *The* 3*-tuple* $(X, M, *)$ *is called a fuzzy metric space (shortly, FM-space) if* X *is an arbitrary set,* $*$ *is a continuous t-norm and* M *is a fuzzy set in* X^2 *x* $[0, \infty)$ *satisfying the following conditions:*
For all $x, y, z \in X$ *and* $s, t > 0$,

(FM-1) $M(x, y, 0) = 0$

(FM-2) $M(x, y, t) = 1$ *for all* $t > 0$ *if and only if* $x = y$,

(FM-3) $M(x, y, t) = M(y, x, t)$,

(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$

(FM-5) $M(x, y, .) : [0, \infty) \to [0, 1]$ *is left continuous for all* $x, y, z \in X$ *and* $s, t > 0$.

Note that $M(x, y, t)$ can be thought of as the degree of nearness between x and y with respect to t. We identify $x = y$ with $M(x, y, t) = 1$ for all $t > 0$ and $M(x, y, t) = 0$ with respect to t. Since $*$ is a continuous t-norm, it follows from (FM-4) that the limit of the sequence in FM-space is uniquely determined. In 1994, George and Veermani [11] introduced the concept of Hausdorff topology on fuzzy metric spaces and showed that every metric space induces a fuzzy metric space.
We can fuzzify examples of metric spaces into fuzzy metric spaces in a natural way:

Let (X, d) be a metric space. Define $a * b = ab$ for all x, y in X and $t > 0$.
Define

$$M(x, y, t) = \frac{t}{(t + d(x, y))}$$

for all x, y in X and $t > 0$. Then $(X, M, *)$ is a fuzzy metric space and this fuzzy metric induced by a metric d is called the standard fuzzy metric.

Definition 4. *Let $(X, M, *)$ be a fuzzy metric space . A sequence $\{x_n\}$ in X is said to*

(i) be convergent to a point $x \in X$ (denoted by $\lim_{n\to\infty} x_n = x$), if
$\lim_{n\to\infty} M(x_n, x, t) = 1$, *for all $t > 0$.*

(ii) be a Cauchy sequence if
$\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$, *for all $t > 0$ and $n, p \in \mathbb{N}$*

(iii) be a complete if every Cauchy sequence in X is convergent to some point in X.

A map $f : X \to X$ is called continuous at x_0 if $\{f(x_n)\}$ converges to $f(x_0)$ for each $\{x_n\}$ converging to x_0.
For more properties and examples of fuzzy metric spaces, see [4]– [13], [20]– [21], [27], [29], [35]– [33]. Moreover, we know that every metric induces a fuzzy metric.

Remark 1. *Since $*$ is continuous, it follows from (FM-4) that the limit of the sequence in FM-space is uniquely determined. In this paper, we consider $(X, M, *)$ a fuzzy metric space with the condition*

$$(FM-6) \quad \lim_{t\to\infty} M(x, y, t) = 1, \text{ for all } x, y \in X.$$

In 1994, Mishra et al. [27] introduced the concept of compatible mapping in FM-space akin to concept of compatible mapping in metric space as follows:

Definition 5. *Let f and g be mappings from a fuzzy metric space $(X, M, *)$ into itself. A pair of map $\{f, g\}$ is said to be compatible if $\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ for some $u \in X$ and for all $t > 0$.*

In 2001, Chugh and Kumar [6] defined the concept of compatible mapping of type (A) as follows:

Definition 6. *Let f and g be mappings from a fuzzy metric space $(X, M, *)$ into itself. A pair of map $\{f, g\}$ is said to be compatible of type (A) if $\lim_{n\to\infty} M(fgx_n, ggx_n, t) = 1$ and $\lim_{n\to\infty} M(gfx_n, ffx_n, t) = 1$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ for some $u \in X$ and for all $t > 0$.*

Chugh, Rathi and Kumar [7] introduced the concept of compatible mappings of Type (P) as follows:

Definition 7. *Let f and g be self mappings on a fuzzy metric space $(X, M, *)$. A pair of map $\{f, g\}$ is said to be compatible of type (P) if $\lim_{n\to\infty} M(ffx_n, ggx_n, t) = 1$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ for some $u \in X$ and for all $t > 0$.*

In 2003, Chugh, Rathi and Kumar [7] defined the concept of weak compatible mapping of type (A) as follows:

Definition 8. *Let f and g be mappings from a fuzzy metric space $(X, M, *)$ into itself. A pair of map $\{f, g\}$ is said to be weak compatible maps of type (A) if* $\lim_{n\to\infty} M(fgx_n, ggx_n, t) = 1$ *or* $\lim_{n\to\infty} M(gfx_n, ffx_n, t) = 1$ *whenever $\{x_n\}$ is a sequence in X such that* $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ *for some $u \in X$ and for all $t > 0$.*

Recently in 2008, Al-Thagafi and Shahzad [3] weakened the concept of weakly compatible maps by giving the new concept of occasionally weakly compatible maps.

Definition 9. *Two self-maps f and g of X are called occasionally weakly compatible maps (shortly owc) if there is a point x in X such that $fx = gx$ at which f and g commute.*

This notion was used in 2006 by Jungck and Rhoades [18] to prove some common fixed point theorems in symmetric spaces.

Definition 10 (see [1])**.** *Let f and g be mappings from a fuzzy metric space $(X, M, *)$ into itself. A pair of map $\{f, g\}$ is said to satisfy the property (E.A) if there is a sequence $\{x_n\}$ in X such that* $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ *for some $t \in X$.*

2. Compatible maps

In 1995, Subramanyam [35] proved the following theorem:

Theorem 1. *Let f and g be self maps of a complete fuzzy metric space $(X, M, *)$ satisfying the following conditions:*

(a) $f(X) \subset g(X)$,

(b) g is continuous,

(c) $M(fx, fy, \alpha t) \geq M(gx, gy, t)$ *for all x, y in X and* $0 < \alpha < 1$.

Then f and g have a unique common fixed point provided f and g commute.

Now, we generalize Theorem 1 for a pair of compatible of type (A) (compatible of type (P), weak compatible of type (A)), weakly compatible and any kind of weakly compatible map along with the notion of weakly compatible maps satisfying a contractive condition of integral type.

Lemma 1 (Lebesgue Dominated Convergence Theorem)**.** *If a sequence $\{f_n\}$ of Lebesgue measurable functions converges almost everywhere to f and if there exists an integrable function $g \geq 0$ such that $|f_n(x)| \leq g(x)$ for every n, then $|f(x)| \leq g(x)$ and* $\lim_{n\to\infty} \int f_n(x)d\mu = \int f(x)d\mu$.

Theorem 2. *Let f and g be compatible maps of type (A) of a complete fuzzy metric space $(X, M, *)$ satisfying the following conditions:*

(2.1) $f(X) \subset g(X)$,

(2.2) any one of the mapping f or g is continuous,

(2.3)
$$\int_0^{1-M(fx,fy,ct)}\phi(p)dp < \int_0^{1-M(gx,gy,t)}\phi(p)dp,$$

for each $x, y \in X, t > 0, c \in [0,1)$, where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue-integrable mapping which is a summable, non-negative, and such that

(2.4) $\int_0^{\epsilon}\phi(p)dp > 0$ *for each $\epsilon > 0$.*

Then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$. Since $f(X) \subset g(X)$, choose $x_1 \in X$ such that $gx_1 = fx_0$. choose x_{n+1} such that $y_n = gx_{n+1} = fx_n$. We now show that $\{y_n\}$ is a Cauchy sequence. For each integer $n \geq 1$, and from (2.3), we have

$$\begin{aligned}\int_0^{1-M(y_n,y_{n+1},t)}\phi(p)dp &= \int_0^{1-M(fx_n,fx_{n+1},t)}\phi(p)dp < \int_0^{1-M(gx_n,gx_{n+1},\frac{t}{c})}\phi(p)dp \\ &= \int_0^{1-M(fx_{n-1},fx_n,\frac{t}{c})}\phi(p)dp < \int_0^{1-M(gx_{n-1},gx_n,\frac{t}{c^2})}\phi(p)dp,\end{aligned}$$

In this fashion one obtains

$$\int_0^{1-M(y_n,y_{n+1},t)}\phi(p)dp < \int_0^{1-M(y_0,y_1,\frac{t}{c^n})}\phi(p)dp.$$

Now letting $n \to \infty$ and using Lebesgue dominated convergence theorem it follows in view of (2.4), $\lim_{n\to\infty}, M(y_n, y_{n+1}, t) = 1$. Similarly, $\lim_{n\to\infty}, M(y_{n+1}, y_{n+2}, t) = 1$.

Now for any positive integer p, we have

$$\begin{aligned}M(y_n, y_{n+p}, t) &\geq M(y_n, y_{n+1}, \frac{t}{p}) * \ldots p - times \ldots * M(y_{n+p-1}, y_{n+p}, \frac{t}{p}) \\ &\geq M(y_n, y_{n+1}, \frac{t}{p}) * \ldots p - times \ldots * M(y_n, y_{n+1}, \frac{t}{p})\end{aligned}$$

Since $\lim_{n\to\infty} M(y_n, y_{n+1}, t) = 1$, for $t > 0$, it follows that

$$\lim_{n\to\infty} M(y_n, y_{n+p}, t) \geq 1 * \ldots * 1 \geq 1.$$

Thus $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, so there exists a point $z \in g(X)$ such that $\lim_{n\to\infty} y_n = z$. Hence $\lim_{n\to\infty} gx_{n+1} = \lim_{n\to\infty} fx_n = z$. Since either f or g is continuous, for definiteness assume that g is continuous, therefore $\lim_{n\to\infty} gfx_n = gz$. But f and g are weak compatible mappingss of type (A), by Remark 1, $\lim_{n\to\infty} fgx_n = gz$. Now from (2.3), we have

$$\int_0^{1-M(fgx_n,fx_n,t)}\phi(t)dt < \int_0^{1-M(ggx_n,gx_n,\frac{t}{c})}\phi(t)dt,$$

where $c \in [0, 1)$.

Letting $\lim_{n\to\infty}$ and using Lebesgue dominated convergence theorem it follows in view of (2.4) that $gz = z$. Again from (2.3), we have

$$\int_0^{1-M(fx_n,fz,t)} \phi(t)dt < \int_0^{1-M(gx_n,gz,\frac{t}{c})} \phi(t)dt.$$

Taking $\lim_{n\to\infty}$ and using Lebesgue dominated convergence theorem it follows in view of (2.4) that $fz = z$. Hence , z is a common fixed point of f and g.
Uniqueness: Suppose that $w(\neq z)$ is also another fixed point of f and g. Then, we have

$$\begin{aligned}\int_0^{1-M(z,w,t)} \phi(t)dt &= \int_0^{1-M(fz,fw,t)} \phi(t)dt \\ &< \int_0^{1-M(gz,gw,\frac{t}{c})} \phi(t)dt \\ &= \int_0^{1-M(z,w,\frac{t}{c})} \phi(t)dt\end{aligned}$$

where $c \in [0, 1)$, which implies $z = w$ and so uniqueness follows. □

Next we prove a theorem for weakly compatible maps that satisfy a contractive condition of integral type.

Theorem 3. *Let f and g be weakly compatible self maps of a fuzzy metric space $(X, M, *)$ satisfying* (2.1), (2.3), (2.4) *and the following condition*
(2.6) *any one of $f(X)$ or $g(X)$ is closed subset of X .*
Then f and g have a coincidence point. Moreover, f and g have a unique common fixed point.

Proof. From the proof of Theorem 2, we conclude that $\{y_n\}$ is a Cauchy sequence in X and since either $f(X)$ or $g(X)$ is closed, for definiteness assume that $g(X)$ is a closed subset of X. Note that the sequence $\{y_{2n}\}$ is contained in $g(X)$ and it has a limit in $g(X)$, call it z. Let $u \in g^{-1}z$. Then $gu = z$. Now we show that $fu = z$. From (2.3), we have

$$\int_0^{1-M(fx_n,fu,t)} \phi(t)dt < \int_0^{1-M(gx_n,gu,\frac{t}{c})} \phi(t)dt,$$

where $c \in [0, 1)$. Letting $\lim_{n\to\infty}$ and using Lebesgue dominated convergence theorem and $c \in [0, 1)$, it follows in view of (2.4) , $fu = z$. Since f and g are weakly compatible, therefore, it follows that $fz = fgu = gfu = gz$. Now we show that z is a common fixed point of f and g. From (2.3), we have

$$\begin{aligned}\int_0^{1-M(fz,z,t)} \phi(p)dp &= \int_0^{1-M(fz,fu,t)} \phi(p)dp \\ &< \int_0^{1-M(gz,gu,\frac{t}{c})} \phi(p)dp \\ &= \int_0^{1-M(fz,z,\frac{t}{c})} \phi(p)dp,\end{aligned}$$

which is a contradiction, since $c \in [0,1)$, therefore, $fz = z = gz$ and hence, z is a common fixed point of f and g.
Uniqueness: Suppose that $w(\neq z)$ is also another fixed point of f and g. Then, we have

$$\begin{aligned}\int_0^{1-M(z,w,t)} \phi(p)dp &= \int_0^{1-M(fz,fw,t)} \phi(p)dp \\ &< \int_0^{1-M(gz,gw,\frac{t}{c})} \phi(p)dp \\ &= \int_0^{1-M(z,w,\frac{t}{c})} \phi(p)dp\end{aligned}$$

which implies $z = w$. Hence uniqueness follows. □

Remark 2. *On setting $\phi(t) = 1$ in (2.3), Theorem 1 can be significantly improved by employing compatible and weakly compatible maps instead of commutativity of maps.*

We now provide an example in support of our theorem.

Example 1. *Let $X = [0,1]$ be equipped with the usual metric space. Define*

$$M(x,y,t) = \frac{t}{t + d(x,y)}$$

for all $x, y \in X$ and for each $t > 0$. Define mappings $f, g : X \to X$ by $fx = \frac{x}{3}$ and $gx = \frac{x}{2}$ for all $x \in X$. Clearly

$$fX = [0, \frac{1}{3}] \subset gX = [0, \frac{1}{2}].$$

Moreover, ϕ defined by $\phi(t) = t$ for $t > 0$ is a Lebesgue-integral mapping which is summable (with finite integral) on each compact subset of $\mathbb{R}^+$, non-negative, and such that for each $\epsilon > 0$,

$$\int_0^{\epsilon} \phi(p)dp > 0.$$

Now

$$\int_0^{M(fx,fy,ct)} \phi(p)dp < \int_0^{M(gx,gy,t)} \phi(p)dp,$$

where

$$1 - M(fx, fy, ct) = \frac{d(x,y)}{(3ct + d(x,y))}$$

and

$$1 - M(gx, gy, ct) = \frac{d(x,y)}{(2ct + d(x,y))}.$$

Thus all hypotheses of Theorem 2 are satisfied with $\phi(t) = t$, for $t > 0$, $\phi(0) = 0$, and $c \in [\frac{2}{3}, 1)$ and 0 is the unique common fixed point of f and g.

Example 2. *Let $X = [3, 22]$ and d be a usual metric on X. Let $f, g : X \to X$ be defined by*

$$f(x) = \begin{cases} 3: & \text{if } x = 3 \text{ or } x > 7 \\ 8: & \text{if } 3 < x \leq 7, \end{cases} \quad , \quad g(x) = \begin{cases} 10 & : \text{if } 3 < x \leq 7 \\ \frac{x+2}{3} & : \text{if } x > 7 \\ 3 & : \text{if } x = 3 \end{cases} .$$

Define $\psi, \phi : [0, \infty) \to [0, \infty)$, where $\psi(t) = (t+1)^{(t+1)} - 1$ and $\phi(t) = \psi'(t)$. Further, define

$$M(x, y, t) = \frac{t}{t + d(x, y)}$$

for all $x, y \in X$ and for each $t > 0$. Moreover, $fX = \{3, 8\}$ and $gX = [3, 10]$. Hence $fX \subseteq gX$. To see that f and g are non-compatible maps, let us consider the sequence $\{x_n\}$, where x_n is defined by

$$x_n = \{(7 + \frac{1}{n}), n \geq 1\}.$$

Therefore, $\lim_{n\to\infty} fx_n = 3$, $\lim_{n\to\infty} gx_n = 3$, $\lim_{n\to\infty} fgx_n = 8$ and $\lim_{n\to\infty} gfx_n = 3$. Hence, f and g are non-compatible maps. But they are any kind of weakly compatible since they commute at the coincidence point at $x = 3$. Thus f and g satisfy all the conditions of the above theorem and have a unique common fixed point at $x = 3$.

Remark 3. *The contractive condition of integral type can be reduced to a contractive condition, not involving integrals, by setting $\phi(t) = 1$ over $\mathbb{R}^+$ on setting $\phi(t) = 1$ in* (2.3)*, Theorem* 3 *can be significantly improved by employing weakly compatible maps instead of commutativity of maps.*

3. Any kind of weakly compatible maps

Now we shall define any kind of weakly compatible maps in fuzzy metric spaces as follows:

Definition 11. *A pair of self-mappings (f, g) of a fuzzy metric space $(X, M, *)$ is said to be any kind of weakly compatible maps if and only if there is a sequence $\{x_n\}$ in X satisfying $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$, and $fgt = gft$ at this point.*

Example 3. *Let $(X, M, *)$ be a fuzzy metric spaces, where $X = [0, 2]$ with a t-norm defined by $a * b = \min\{a, b\}$ for all $a, b \in X$ and*

$$M(x, y, t) = \frac{t}{t + d(x, y)},$$

for all $x, y \in X$ and for each $t > 0$. $M(x, y, 0) = 0$ for all $x, y \in X$.

Define $f, g : [0, 2] \to [0, 2]$ by

$$f(x) = \begin{cases} 2 & : \text{if } x \in [0, 1] \\ \frac{x}{2} & : \text{if } x \in (1, 2] \end{cases} \quad , \quad g(x) = \begin{cases} 2 & : \text{if } x \in [0, 1] \\ \frac{x+3}{5} & : \text{if } x \in (1, 2] \end{cases} .$$

Consider the sequence $\{x_n\}$ *where* $x_n = (2 - \frac{1}{2n})$. *Clearly* $f(1) = g(1) = 2$ *and* $f(2) = g(2) = 1$. *Here* 1 *and* 2 *are two coincidence points for maps* f *and* g. *Note that* f *and* g *commute at* 1 *and* 2, *i.e.,* $fg(1) = gf(1) = 1$, *also* $fg(2) = gf(2) = 2$ *and so* f *and* g *are weakly compatible maps on* $\mathbb{R}$. *Now,*

$$fx_n = (1 - \frac{1}{4n}), \quad gx_n = (1 - \frac{1}{10n}).$$

Therefore,

$$fx_n \to 1, \quad gx_n \to 1, \quad fgx_n = 2, \quad gfx_n = (\frac{4}{5} - \frac{1}{20n})$$

and

$$\lim_{n\to\infty} M(fgx_n, gfx_n, t) = \frac{t}{t + \frac{6}{5}} < 1,$$

so, f *and* g *are not compatible maps on* $\mathbb{R}$ *but they are any kind of weakly compatible on* $\mathbb{R}$.

Example 4. *Let* $X = [\frac{2}{3}, +\infty)$. *Define* $f, g : X \to X$ *by* $gx = \frac{x+1}{3}$ *and* $fx = \frac{2x+1}{3}$, *for all* x *in* X. *Suppose that* f *and* g *are any kind of weakly compatible maps. Then, there exists a sequence* $\{x_n\}$ *in* X *satisfying* $fx_n = gx_n = z$ *for some* $z \in X$. *Therefore,* $\lim_{n\to\infty} x_n = 3z - 1$ *and* $\lim_{n\to\infty} x_n = \frac{3z-1}{2}$ *.Thus,* $z = \frac{1}{3}$ *, which is a contradiction, since* $\frac{1}{3}$ *is not contained in* X. *Hence* f *and* g *are not any kind of weakly compatible maps.*

Example 5. *Let* $X = [0, 1]$ *with the usual metric space* d *i.e.,* $d(x, y) = |x - y|$. *Define*

$$M(x, y, t) = \frac{t}{t + d(x, y)},$$

for all $x, y \in X$ *and for all* $t > 0$.

Hence $(X, M, *)$ *is a fuzzy metric space. Also define*

$$f(x) = \begin{cases} 1 - x : \text{if } x \in [0, \frac{1}{2}] \\ \frac{3}{4} \quad : \text{if } x \in (\frac{1}{2}, 1] \end{cases}, \quad g(x) = \begin{cases} \frac{1}{2} : \text{if } x \in [0, \frac{1}{2}] \\ 0 : \text{if } x \in (\frac{1}{2}, 1] \end{cases}.$$

Consider the sequence $\{x_n\} = \frac{1}{2} - \frac{1}{n}, n \geq 2$, *we have*

$$\lim_{n\to\infty} f(\frac{1}{2} - \frac{1}{n}) = \frac{1}{2} = \lim_{n\to\infty} g(\frac{1}{2} - \frac{1}{n})$$

and

$$fg(\frac{1}{2}) = gf(\frac{1}{2}) = \frac{1}{2}.$$

Thus the pair (f, g) *is any kind of weakly compatible maps.*

Further, f *and* g *are weakly compatible since* $x = \frac{1}{2}$ *is their unique coincidence point and* $fg(\frac{1}{2}) = f(\frac{1}{2}) = g(\frac{1}{2}) = gf(\frac{1}{2})$. *We further observe that*

$$\lim_{n\to\infty} d(fg(\frac{1}{2} - \frac{1}{n}), gf(\frac{1}{2} - \frac{1}{n})) \neq 0$$

showing that

$$\lim_{n\to\infty} M(fgx_n, gfx_n, t) \neq 1,$$

therefore, the pair (f, g) *is not subcompatible.*

Thus any kind of weakly compatible maps does not imply subcompatibility. Here, we notice that subcompatible and any kind of weakly maps are independent of each other.

Example 6. *Let* $X = \mathbb{R}^+$ *and* d *be the usual metric on* X*. Define*

$$M(x, y, t) = \frac{t}{t + d(x, y)},$$

for all $x, y \in X$ *and for all* $t > 0$*. Hence (X,M,*) is a fuzzy metric space. Also define* $f, g : X \to X$ *by* $fx = 0$*, if* $0 < x \leq 1$ *and* $fx = 1$*, if* $x > 1$*or* $x = 0$*; and* $gx = [|x|]$*, the greatest integer that is less than or equal to* x*, for all* $x \in X$*. Consider a sequence* $x_n = 1 + \frac{1}{n}, n \geq 2$ *which is in* $(1, 2)$*, then we have* $\lim_{n\to\infty} fx_n = 1 = \lim_{n\to\infty} gx_n$*. Thus the pair* (f, g) *is any kind of weakly compatible maps . However,* f *and* g *are not weakly compatible as each* $u_1 \in (0, 1)$ *and* $u_2 \in (1, 2)$ *are coincidence points of* f *and* g*, where they do not commute. Moreover, they commute at* $x = 0, 1, 2, \ldots,$ *but none of these points are coincidence points of* f *and* g*. Further, we note that a pair* (f, g) *is a non-compatible maps. Thus we can conclude that any kind of weakly compatible maps does not imply weak compatibility. Here, we notice that weakly compatible and any kind of weakly maps are independent of each other, see a paper on metric space by H. K. Pathak, Rosana Rodriguez-Lopez and R. K. Verma [32].*

Now we prove a theorem for a pair of weakly compatible maps along with the notion of any kind of weakly compatible maps.

Theorem 4. *Let* $(X, M, *)$ *be a fuzzy metric space. Suppose* f *and* g *are weak compatible self-maps of* X *satisfying* (2.1), (2.3), (2.4) *and the following:*

(3.1) f, g *are any kind of weakly compatible maps*

(3.2) $f(X)$ *or* $g(X)$ *is a closed subset* X*.*

Then f *and* g *have a coincidence point. Moreover,* f *and* g *have a unique common fixed point.*

Proof. Since f and g are any kind of weakly compatible maps, therefore, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u \in X$. Since either $f(X)$ or $g(X)$ is a closed subspace of X, for definiteness we assume that $g(X)$ is a closed subset of X. Further, note that the sequence $\{y_{2n}\}$ which is contained in $g(X)$, so there is a limit in $g(X)$. Call it be u such that $u = ga$. Therefore, $\lim_{n\to\infty} fx_n = u = ga = \lim_{n\to\infty} gx_n$ for some $a \in X$. This implies $u = ga \in gX$. Now we show that $u = fa = ga$. Now from (2.3), we have

$$\int_0^{F(fa, fx_n, ct)} \phi(p)dp < \int_0^{F(ga, gx_n, t)} \phi(p)dp.$$

Letting $\lim_{n\to\infty}$ and using the Lebesgue dominated convergence theorem and $c \in [0,1)$, it follows in view of (2.4) that $F(fa, fa, ct) \le F(ga, fa, t)$, this implies that $u = ga = fa$.

Thus a is the coincidence point of f and g. Since f and g are weakly compatible, therefore, $fu = fga = gfa = gu$. Now we show that $fu = u$. From (2.3), we have

$$\int_0^{F(fu,fa,ct)} \phi(p)dp < \int_0^{F(gu,ga,t)} \phi(p)dp,$$

which in turns implies that $fu = u$. Hence u is the common fixed point of f and g. *Uniqueness:* Suppose that $w(\neq z)$ is also another common fixed point of f and g. From (2.3), we have

$$\begin{aligned} \int_0^{F(z,w,t)} \phi(p)dp &= \int_0^{F(fz,fw,t)} \phi(p)dp \\ &< \int_0^{F(gz,gw,\frac{t}{c})} \phi(p)dp \\ &= \int_0^{F(z,w,\frac{t}{c})} \phi(p)dp, \end{aligned}$$

where $c \in [0,1)$. Thus $z = w$ and, therefore, uniqueness follows. □

4. Variants of weakly commuting mappings

In 1994, Pant [31] introduced the notion of R-weakly commuting maps in metric spaces. Later on, Vasuki [36] initiated the concept of non compatibility of mappings in fuzzy metric spaces, by introducing the notion of R-weakly commuting mappings in fuzzy metric spaces and proved some common fixed point theorems for these maps.

Definition 12. *A pair of self-mappings (f,g) of a fuzzy metric space $(X,M,*)$ is said to be R-weakly commuting if there exists some $R > 0$ such that*

$$M(fgx, gfx, t) \ge M(fx, gx, \frac{t}{R}),$$

for all $x \in X$ and $t > 0$.

Of course, R-weak commutativity implies weak commutativity only when $R \le 1$.

Example 7 (see [37]). *Let $X = \mathbb{R}$, the set of all real numbers. Define $a * b = ab$ and*

$$M(x,y,t) = \begin{cases} \frac{-|x-y|}{t} & : \text{if } t > 0 \\ 0 & : \text{if } t = 0 \end{cases}$$

for all $x, y \in X$. Then $(X,M,*)$ is a fuzzy metric space.

Let $f(x) = 2x - 1$ and $g(x) = x^2$. Then

$$M(fgx, gfx, t) = e^{\frac{-2(x-1)^2}{t}}$$

and

$$M(fx, gx, \frac{t}{R}) = e^{\frac{-R(x-1)^2}{t}}.$$

Therefore, for $R = 2$, f and g are R-weakly commuting. However, f and g are not weakly commuting mappings since the exponential function is strictly increasing.

Later on, Pathak et al. [30] improved the notion of R-weakly commuting mappings in metric spaces by introducing the notions of R-weakly commutativity of type (A_g) and R-weakly commutativity of type (A_f). In 2008, Imdad and Ali [19] embarked the notion of R-weakly commutativity of type (A_g) and R-weakly commutativity of type (A_f) in fuzzy metric with inspiration from Pathak et al. [30] and they further introduced the notion of R-weakly commuting mappings of type (P) in fuzzy metric spaces.

Definition 13. *A pair of self-mappings (f, g) of a fuzzy metric space $(X, M, *)$ is said to be*

- *(i) R-weakly commuting mappings of type (A_g) if there exists some $R > 0$ such that $M(gfx, ffx, t) \geq M(fx, gx, \frac{t}{R})$ for all $x \in X$ and $t > 0$.*
- *(ii) R-weakly commuting mappings of type (A_f) if there exists some R¿0 such that $M(fgx, ggx, t) \geq M(fx, gx, \frac{t}{R})$, for all $x \in X$ and $t > 0$.*
- *(iii) R-weakly commuting mappings of type (P) if there exists some $R > 0$ such that $M(ffx, ggx, t) \geq M(fx, gx, \frac{t}{R})$, for all $x \in X$ and $t > 0$.*

Remark 4. *We have some suitable examples to show that R- weakly commuting mappings, R-weakly commuting of type (A_f), R-weakly commuting of type (A_g) and R-weakly commuting of type (P) are distinct.*

Example 8. *Let $X = [-1, 1]$ be the set of all real numbers with usual metric d defined by $d(x, y) = |x - y|$ for all x, y in X. Define*

$$M(x, y, t) = \frac{t}{t + d(x, y)},$$

for all $x, y \in X$ and for all $t > 0$.Then $(X, M, *)$ is a fuzzy metric space. Define $fx = |x|$ and $gx = |x| - 1$.

Then by a straightforward calculation, one can find that

$$d(fx, gx) = 1, \quad d(fgx, gfx) = 2(1 - |x|), \quad d(fgx, ggx) = 1,$$
$$d(gfx, ffx) = 1, \quad d(ffx, ggx) = 2|x|,$$

for all $x \in X$.

Now we conclude the following:

- *(i) pair (f, g) is not weakly commuting,*
- *(ii) for $R = 2$, pair (f, g) is R-weakly commuting, R-weakly commuting of type (P), R-weakly commuting of type (A_g) and R-weakly commuting of type (A_f),*

(iii) For $R = \frac{3}{2}$, pair (f, g) is R-weakly commuting of type (A_f) but not R-weakly commuting of type (P) and R-weakly commuting.

Example 9. *Let $X = [0, 1]$ with usual metric d defined by $d(x, y) = |x - y|$ for all x, y in X. Define*

$$M(x, y, t) = \frac{t}{t + d(x, y)},$$

for all $x, y \in X$ and for all $t > 0$. Then $(X, M, *)$ is a fuzzy metric space. Further define $fx = x$ and $gx = x^2$.

Then by a straightforward calculation, one can show that

$$ffx = x, gfx = x^2, \quad fgx = x^2, ggx = x^4$$

and

$$d(fgx, gfx) = 0, \quad d(fgx, ggx) = |x^2(x-1)(x+1)|, \quad d(gfx, ffx) = |x(x-1)|,$$

and

$$d(ffx, ggx) = |x(x-1)(x^2+x+1)|, \quad d(fx, gx) = |x(x-1)|,$$

for all $x \in X$. Therefore, we conclude that

- *i) pair (f, g) is R- weakly commuting for all positive, real values of R,*
- *ii) for $R = 3$, pair (f, g) is R-weakly commuting of type (A_f), R-weakly commuting of type (A_g) and R-weakly commuting of type (P),*
- *iii) for $R = 2$, pair (f, g) is R-weakly commuting of type (A_f) and R-weakly commuting of type (A_g) and not R-weakly commuting of type (P) (for this take $x = \frac{3}{4}$).*

Example 10. *Consider $X = [\frac{1}{2}, 2]$ with usual metric d. Define*

$$M(x, y, t) = \frac{t}{t + d(x, y)},$$

for all $x, y \in X$ and for all $t > 0$. Then $(X, M, *)$ is a fuzzy metric space. Let us define self maps f and g by $fx = \frac{x+1}{3}, gx = \frac{x+2}{5}$. We calculate the following:

$$d(fx, gx) = \frac{2x-1}{15}, \quad d(fgx, gfx) = 0, \quad d(fgx, ggx) = \frac{2x-1}{75},$$

and

$$d(gfx, ffx) = \frac{2x-1}{45}, \quad d(ffx, ggx) = \frac{8(2x-1)}{225}.$$

Now we conclude the following, i.e.

the pair (f, g) is R-weakly commuting for all positive real numbers:

- *i) for $R \geq \frac{8}{15}$, it is R-weakly commuting of type (A_f), R-weakly commuting of type (A_g) and R-weakly commuting of type (P),* *ii) for $\frac{1}{3} \leq R < \frac{8}{15}$, it is R-weakly commuting of type (A_g) and R-weakly commuting of type (A_f), but not R-weakly commuting of type (P),*

iii) for $\frac{1}{5} \leq R < \frac{1}{3}$, it is R-weakly commuting of type (A_f), but neither R-weakly commuting of type (A_g) nor R-weakly commuting of type (P).

Moreover, such mappings commute at their coincidence points. It is also obvious that f and g can fail to be point-wise R-weakly commuting only if there exists some $x \in X$ such that $fx = gx$ but $fgx \neq gfx$, that is, only if they possess a coincidence point at which they do not commute. Therefore, the notion of a point-wise R-weak commutativity type mapping is equivalent to commutativity at coincidence points, as mentioned in [31].

Theorem 5. *Theorem 3 (Theorem 4) remains true if a weakly compatible property is replaced by any one of the following (retaining the rest of hypotheses):*

(a) R-weakly commuting property,

(b) R-weakly commuting property of type (A_g),

(c) R-weakly commuting property of type (A_f),

(d) R-weakly commuting property of type (P),

(e) weakly commuting property.

Proof. Since all the conditions of Theorem 3 (Theorem 4) are satisfied, then the existence of coincidence points for both pairs is insured. If x is an arbitrary point of coincidence for the pair (f, g), then using R-weak commutativity one gets

$$M(fgx, gfx, t) \geq M(fx, gx, \frac{t}{R}) = 1,$$

which amounts to say that $fgx = gfx$.

Thus the pair (f, g) is weakly compatible. Now applying Theorem 3 (Theorem 4) one concludes that f and g have a unique common fixed point. In case (f, g) is an R-weakly commuting pair of type (A_f), then

$$M(fgx, gfx, t) \geq M(fx, gx, \frac{t}{R}) = 1,$$

which amounts to say that $fgx = ggx$.

Now

$$M(fgx, gfx, t) \geq M(fgx, ggx, \frac{t}{2}) * M(ggx, gfx, \frac{t}{2}) = 1 * 1 = 1,$$

yielding thereby $fgx = gfx$.

Similarly, if a pair is an R-weakly commuting mapping of type (A_g) or type (P) or weakly commuting, then pair (f, g) also commutes at their points of coincidence. Now in view of Theorem 3 (Theorem 4), in all the cases f and g have a unique common fixed point. This completes the proof. □

As an application of Theorem 3 (Theorem 4) we prove a common fixed point theorem for two finite families of mappings which runs as follows:

Theorem 6. *Let* $\{f_1, f_2, ..., f_m\}$ *and* $\{g_1, g_2, ..., g_n\}$ *be two finite families of self-mappings of a fuzzy metric space* $(X, M, *)$ *such that* $f = f_1 f_2 ... f_m, g = g_1 g_2 ... g_n$, *satisfy conditions* (2.1), (2.3), (2.4),

iii) $g_m(X)$ *is a closed subspace of* X *for all* m.

Then f *and* g *have a point of coincidence. Moreover, if* $f_i f_j = f_j f_i$ *and* $g_k g_l = g_l g_k$ *for all* $i, j \in I_1 = \{1, 2, ..., m\}$, $k, l \in I_2 = \{1, 2, ..., n\}$, *then (for all* $i \in I_1, k \in I_2$*)* f_i *and* g_k *have a common fixed point.*

Proof. The conclusions are immediate, i.e., f and g have a point of coincidence as f and g satisfy all the conditions of Theorem 3. Now appealing to component-wise commutativity of various pairs, one can immediately prove that $fg = gf$; hence, obviously pair (f, g) is coincidentally commuting. Note that all the conditions of Theorem 3 are satisfied ensuring the existence of a unique common fixed point, say z. Now one needs to show that z remains the fixed point of all the component maps. For this consider

$$\begin{aligned} f(f_i z) = ((f_1 f_2 ... f_m) f_i) z &= (f_1 f_2 ... f_{m-1})((f_m f_i) z) = (f_1 ... f_{m-1})(f_i f_m z) \\ &= (f_1 ... f_{m-2})(f_{m-1} f_i (f_m z)) = (f_1 ... f_{m-2})(f_i f_{m-1}(f_m z)) \\ &\vdots \\ &= f_1 f_i (f_2 f_3 f_4 ... f_m z) = f_i f_1 (f_2 f_3 ... f_m z) = f_i(f_z) = f_i z. \end{aligned}$$

Similarly, one can show that $f(g_k z) = g_k(fz) = g_k z, g(g_k z) = g_k(gz) = g_k z$ and $g(f_i z) = f_i(gz) = f_i z$, which shows that (for all I and k) $f_i z$ and $g_k z$ are other fixed points of the pair (f, g). Now appealing to the uniqueness of common fixed points of both pairs separately, we get $z = f_i z = g_k z$, which shows that z is a common fixed point of f_i, g_k for all i and k.

By setting

$$f = f_1 = f_2 = ... = f_m, g = g_1 = g_2 = ... = g_n,$$

we deduce the following:

Corollary 1. *Let* f *and* g *be two self-mappings of a fuzzy metric space* $(X, M, *)$ *such that* f_m *and* g_n *satisfy conditions* (2.1), (2.3), (2.4). *If one of* $f_m(X)$ *or* $g_n(X)$ *is a complete subspace of* X, *then* f *and* g *have a unique common fixed point provided* (f, g) *commute.*

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