MATHEMATICAL COMMUNICATIONS 555
Math. Commun., Vol. **15**, No. 2, pp. 555-572 (2010)

Optimality criteria method for optimal design in hyperbolic problems[*,†]

Marko Vrdoljak[1,‡]

[1] *Department of Mathematics, University of Zagreb, Bijenička 30, HR-10 000 Zagreb, Croatia*

Received November 30, 2009; accepted May 26, 2010

Abstract. We consider an optimal design problem with a hyperbolic initial boundary value problem as the state equation. As a possible application consider a body made from a mixture of different materials which is vibrating under the given external force, subject to a prescribed boundary and initial values. The control function (the distribution of given materials in the given domain) uniquely determines the response (the state function) of the vibrating material. Our goal is to find a distribution of materials minimising given integral functional depending on state and control functions.
We derive the necessary condition of optimality, which enables us to formulate an optimality criteria method for a numerical solution. Two numerical examples are presented. The same procedure can be applied in the case of multiple state equations as well.

AMS subject classifications: 35B27, 35L05, 49K20, 65K10

Key words: optimal design, wave equation, homogenisation, optimality criteria method

1. Introduction

The homogenisation method proved to be well suited for treatment of optimal design for elliptic problems (in modelling both conductivity and elasticity), first for the theoretical questions on proper relaxation, but also as a starting point for application of classical methods of calculus of variations leading to numerical solutions [19, 23, 25, 1].

As long as the coefficients of the wave equation depend only on spatial variables, a similar approach is possible: the same H-topology on coefficients multiplying spatial derivatives can be used to obtain a proper relaxation [24, 2, 3]. We shall restrict our attention to the wave equation with a three-dimensional spatial variable.

Unfortunately, due to different regularity properties for the wave operator, necessary conditions of optimality cannot be obtained as easily as in the elliptic case. In the literature, this problem was resolved by assuming more regularity on the right-hand side [16, 17]. In these papers the authors used gradient Young measures as a

*Presented at the Sixth Conference on Applied Mathematics and Scientific Computing, Zadar, Croatia, September 2009. This work is supported in part by Croatian MZOS projects 037-0372787-2795 and 037-1193086-3226.
†For interpretation of color in all figures, the reader is referred to the web version of this article available at www.mathos.hr/mc.
‡Corresponding author. *Email address:* `marko@math.hr` (M. Vrdoljak)

relaxation tool, allowing them to consider space and time dependent coefficients (in one spatial variable) and quadratic cost functional depending on first derivatives of the state function.

For a numerical solution of our optimal design problem via the homogenisation method, a characterisation of G-closure of original materials is important, so we need to restrict ourselves to the case of two-phase isotropic mixtures (for the space and time dependent coefficients the G-closure problem was studied in one spatial variable in [14], see also [15]).

Similarly to elliptic optimal design problems, based on the necessary conditions of optimality, we can conclude that the optimal design can be found among sequential laminates, which is important for the implementation of the classical gradient method for a numerical solution [4, 5]. An analogous situation also occurs for multiple state optimal design problems.

However, the optimality criteria method, derived directly from necessary conditions of optimality, appears to be the most popular numerical method for optimal design problems. The key role of this paper is to present how this method can be applied to our problem. For this purpose we use some explicit calculations (Lemma 2 and 3, below) performed in [27].

Let $\Omega \subseteq \mathbf{R}^d$ be a bounded domain, $V = \mathrm{H}^1_0(\Omega)$, $H = \mathrm{L}^2(\Omega)$, and for $T > 0$ let $\Omega_T = \langle 0, T\rangle \times \Omega$. By $\mathcal{M}_S(\alpha, \beta; \Omega)$ we shall denote the set of all symmetric matrix functions on Ω with eigenvalues between α and β ($0 < \alpha < \beta$). For $\mathbf{A} \in \mathcal{M}_S(\alpha, \beta; \Omega)$ and $u^0 \in V$, $u^1 \in H$ and $f \in \mathrm{L}^2(0, T; H) = \mathrm{L}^2(\Omega_T)$, we consider the initial boundary value problem

$$\text{Find } u \in \mathrm{L}^2(0,T;V) \cap \mathrm{H}^1(0,T;H) \text{ such that}$$
$$\begin{cases} u'' - \mathsf{div}\,(\mathbf{A}\nabla u) = f \\ u(0,\cdot) = u^0 \\ u'(0,\cdot) = u^1 \,. \end{cases} \tag{1}$$

After introducing bilinear forms

$$a(u,v) = \int_\Omega \mathbf{A}(\mathbf{x})\nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x})\, d\mathbf{x}\,,$$
$$c(u,v) = \int_\Omega u(\mathbf{x}) v(\mathbf{x})\, d\mathbf{x}$$

the weak formulation of (1) reads

$$\begin{cases} (\forall\, v \in V) \quad \dfrac{d}{dt} c(u'(t,\cdot), v) + a(u(t,\cdot), v) = \displaystyle\int_\Omega f(t,\cdot) v \, d\mathbf{x} \quad \text{in } \mathcal{D}' \\ u(0,\cdot) = u^0 \\ u'(0,\cdot) = u^1 \,. \end{cases} \tag{2}$$

This problem has a unique solution satisfying $u'' \in \mathrm{L}^2(0,T;V')$ as well, together with the estimate

$$(\forall\, t \in [0,T]) \quad \|u(t)\|_V + \|u'(t)\|_H \le C \left(\|u^0\|_V + \|u^1\|_H + \|f\|_{\mathrm{L}^2(0,T;H)} \right) , \tag{3}$$

where the constant C depends only on numbers α and β [11, 10]. Moreover, the solution u belongs to $\mathrm{C}([0,T];V)$ with $u' \in \mathrm{C}([0,T];H)$. The same is true if $f \in \mathrm{L}^1(0,T;H)$; the only difference is that $u'' \in \mathrm{L}^1(0,T;V')$. Looking for the Gâteaux derivative of the functional that we shall consider in the optimal design problem, we come to a less regular right-hand side: $f \in \mathrm{L}^2(0,T;V')$. Although the weak formulation given above still makes sense, one gets only a much weaker a priori estimate

$$(\forall t \in [0,T]) \quad \|u(t)\|_H + \|u'(t)\|_{V'} \leq C\left(\|u^0\|_H + \|u^1\|_{V'} + \|f\|_{\mathrm{L}^2(0,T;V')}\right) . \tag{4}$$

This implies that for $u^0 \in H$ and $u^1 \in V'$ there exists a unique solution $u \in \mathrm{L}^2(0,T;H)$ with $u' \in \mathrm{L}^2(0,T;V')$. Actually, it belongs to $\mathrm{C}([0,T];H)$ with $u' \in \mathrm{C}([0,T];V')$.

2. Optimal design problem

Assume that we want to fill the set Ω with m given materials. Each material is characterised by a symmetric positive definite matrix $\mathbf{A}_i$ $(i = 1,\ldots,m)$ (mechanical properties of a particular material). Furthermore, for simplicity, suppose that all the materials are isotropic, i.e. $\mathbf{A}_i = \gamma_i\mathbf{I}$, where $\gamma_i \in [\alpha,\beta]$. In this situation, after denoting by χ_i the characteristic function of the i-th material, the coefficients in (1) can be expressed as

$$\mathbf{A} = \sum_{i=1}^{m} \chi_i\gamma_i\mathbf{I}\,. \tag{5}$$

The optimal design problem consists of minimisation of the functional

$$I(\boldsymbol{\chi}) = \int_{\Omega_T} F(t,\mathbf{x},\mathbf{A}(\mathbf{x}),u(t,\mathbf{x}))\,d\mathbf{x}\,dt\,,$$

where $\boldsymbol{\chi} = (\chi_1,\ldots,\chi_m) \in \mathrm{L}^\infty(\Omega;\{0,1\}^m), \sum_{i=1}^m \chi_i = 1$ a.e. on Ω, corresponds to the distribution of given materials and u is the state function determined by (1), with $\mathbf{A}$ as in (5). Due to a special type of admissible controls $\boldsymbol{\chi}$ used, the above cost functional can be written in a slightly different form. For $(t,\mathbf{x}) \in \Omega_T$ and $\lambda \in \mathbf{R}$ we define

$$g_i(t,\mathbf{x},\lambda) = F(t,\mathbf{x},\gamma_i\mathbf{I},\lambda)\;,$$

so that the cost functional reads

$$I(\boldsymbol{\chi}) = \int_{\Omega_T} \sum_{i=1}^{m} \chi_i(\mathbf{x})g_i(t,\mathbf{x},u(t,\mathbf{x}))\,d\mathbf{x}\,dt\,.$$

As ubiquitous in optimal design problems, the non-existence of a solution can occur. The main problem is the question of continuity of the mapping $\boldsymbol{\chi} \mapsto u$. In [19] Murat and Tartar, using the notion of $H-$convergence, introduced relaxation of the optimal design problem for the case where the state equation is the stationary diffusion equation. An analogous approach for the hyperbolic problems is presented in [2, 3].

Let us first write down the original problem as

$$\begin{cases} J(\boldsymbol{\chi}, \mathbf{A}) = \displaystyle\int_{\Omega_T} \sum_{i=1}^{m} \chi_i(\mathbf{x}) g_i(t, \mathbf{x}, u(t, \mathbf{x}))\, d\mathbf{x}\, dt \longrightarrow \min \\ (\boldsymbol{\chi}, \mathbf{A}) \in \mathcal{A}\,, \end{cases} \tag{6}$$

where function u is determined by (1), while

$$\mathcal{A} = \left\{ (\boldsymbol{\chi}, \mathbf{A}) : \boldsymbol{\chi} \in \mathrm{L}^\infty(\Omega; \{0,1\}^m),\ \sum_{i=1}^{m} \chi_i = 1,\ \mathbf{A} = \sum_{i=1}^{m} \chi_i \gamma_i \mathbf{I} \text{ a.e. on } \Omega \right\}.$$

As showed in [2], although J depends only on $\boldsymbol{\chi}$, we are forced to consider $\mathbf{A}$ as an additional argument in order to get continuity of the extension of functional J, defined by the same formula (6), to the set

$$\mathcal{T} = \left\{ (\boldsymbol{\theta}, \mathbf{A}) \in \mathrm{L}^\infty(\Omega; [0,1]^m) \times \mathcal{M}_S(\alpha, \beta; \Omega) : \sum_{i=1}^{m} \theta_i = 1 \text{ a.e. on } \Omega \right\}.$$

The following product topology on $\mathcal{T}$: L^∞ weak $*$ for $\boldsymbol{\theta}$ and H-topology for $\mathbf{A}$ is reasonable, according to the homogenisation result for the wave equation [24]: if the sequence $(\mathbf{A}_n)$ in $\mathcal{M}_S(\alpha, \beta; \Omega)$ $H-$converges to $\mathbf{A}$, then the corresponding sequence of solutions u_n for (1) (with coefficients $\mathbf{A}_n$ instead of $\mathbf{A}$) converges to u, determined by (1), weakly in $\mathrm{L}^2(0,T;V)$, with derivatives u_n' converging weakly to u' in $\mathrm{L}^2(0,T;H)$.

To get the continuity of J on $\mathcal{T}$ we consider the appropriate growth condition on g_i, $i = 1, \ldots, m$. In order to obtain necessary conditions of optimality under weaker assumptions, one needs an improvement of the result presented in [2].

Lemma 1. *Let g_i, $i = 1, \ldots, m$, be Carathéodory's functions (i.e. measurable in $t, \mathbf{x}$ and continuous in λ) satisfying*

$$|g_i(t, \mathbf{x}, \lambda)| \le \varphi_i(t, \mathbf{x}) + \psi_i(t)|\lambda|^q \qquad \text{for } \lambda \in \mathbf{R}, \quad a.e. \quad (t, \mathbf{x}) \in \Omega_T,$$

with $\varphi_i, \psi_i \in L^1$ and some $q \in [1, q^*\rangle$, where

$$q^* = \begin{cases} \infty & , \ d \le 2 \\ \frac{2d}{d-2} & , \ d > 2. \end{cases}$$

Then $J : \mathcal{T} \to \mathbf{R}$, given by the formula in (6), is continuous.

Proof. If we take a sequence $(\boldsymbol{\theta}_n, \mathbf{A}_n)$ in $\mathcal{T}$ such that $\boldsymbol{\theta}_n \xrightharpoonup{*} \boldsymbol{\theta}$ and $\mathbf{A}_n \xrightharpoonup{H} \mathbf{A}$, then the corresponding sequence (u_n) converges to u in $\mathrm{H}^1(\Omega_T)$. But $u, u_n \in \mathrm{C}([0,T]; V)$ and by the a priori estimate (3), for any $t \in [0,T]$ the sequence $(u_n(t, \cdot))$ is bounded in V. Therefore, we have

$$(\forall\, t \in [0,T]) \quad u_n(t, \cdot) \rightharpoonup u(t, \cdot) \text{ in } V\,.$$

By the Sobolev embedding theorem, $u_n(t, \cdot) \longrightarrow u(t, \cdot)$ in $\mathrm{L}^q(\Omega)$, for $q \in [1, q^*\rangle$. On the subsequence, we have the convergence almost everywhere on Ω, so that

$$g_i(t, x, u_n(t, x)) \longrightarrow g_i(t, x, u(t, x)) \text{ a.e. } (t, x) \in \Omega_T\,,\ i = 1, \ldots, m\,.$$

Now, using the growth condition on g_i, the Fubini theorem and the Lebesgue dominated convergence theorem, $g_i(t,\cdot,u_n(t,\cdot))$ converges to $g_i(t,\cdot,u(t,\cdot))$ strongly in $\mathrm{L}^1(\Omega)$ for almost every $t \in [0,T]$ implying the convergence

$$\int_\Omega \theta_n^i g_i(t,\mathbf{x},u_n(t,\mathbf{x}))\,d\mathbf{x} \longrightarrow \int_\Omega \theta^i g_i(t,\mathbf{x},u(t,\mathbf{x}))\,d\mathbf{x}$$

almost everywhere on $[0,T]$. By a second application of the Lebesgue dominated convergence theorem, we conclude the proof. □

Since $\mathcal{T}$ is compact, the proper relaxation of the original problem (6) consists only of replacing $\mathcal{A}$ by its closure in $\mathcal{T}$:

$$\begin{cases} J(\boldsymbol{\theta},\mathbf{A}) \longrightarrow \min\,, \quad (\boldsymbol{\theta},\mathbf{A}) \in \overline{\mathcal{A}}\,, \quad \text{where} \\ \overline{\mathcal{A}} = \{(\boldsymbol{\theta},\mathbf{A}) : \boldsymbol{\theta} \in [0,1]^m\,,\ \sum_{i=1}^m \theta_i = 1\,,\ \mathbf{A} \in \mathcal{K}(\boldsymbol{\theta}) \text{ a.e. on } \Omega\}\,. \end{cases}$$

The set $\mathcal{K}(\boldsymbol{\theta})$ denotes all effective matrix coefficients obtained by mixing original phases with local proportion $\boldsymbol{\theta}$, in literature known under the name G-closure of the set of original conductivities with prescribed ratio (originally called Gm-closure [12]). For the case $m = 2$ the complete characterisation of $\mathcal{K}(\boldsymbol{\theta})$ is described in [12, 13, 19, 22]. The presented concept for the relaxation follows [19] and relies on the local representation of G-closure [8, 21, 25].

The same approach holds if one adds the following constraints on the amount of given materials:

$$\int_\Omega \chi_i\,d\mathbf{x} \le \kappa_i\,,\ i = 1,\dots,m\,.$$

3. The Gâteaux derivative

In this section we shall calculate the Gâteaux derivative of the cost functional J, at the admissible point $(\boldsymbol{\theta},\mathbf{A}) \in \overline{\mathcal{A}}$. The corresponding state function, defined by (1), is denoted by u.

We need additional assumptions on functions g_i: Their partial derivatives $\frac{\partial g_i}{\partial\lambda}$ should be Carathéodory's functions satisfying the growth condition

$$\left|\frac{\partial g_i}{\partial\lambda}(t,\mathbf{x},\lambda)\right| \le \widetilde{\varphi}_i(t,\mathbf{x}) + \widetilde{\psi}_i(t)|\lambda|^r \qquad \text{for } \lambda \in \mathbf{R}, \quad \text{a.e.} \quad (t,\mathbf{x}) \in \Omega_T,$$

with $\widetilde{\varphi}_i \in \mathrm{L}^1(0,T;H)$, $\widetilde{\psi}_i \in \mathrm{L}^1(0,T)$ and some $r \in [1,r^*\rangle$, where

$$r^* = \begin{cases} \infty & ,\ d \le 2 \\ \frac{d}{d-2} & ,\ d > 2. \end{cases}$$

Let us consider the smooth path $\varepsilon \mapsto (\boldsymbol{\theta}_\varepsilon,\mathbf{A}_\varepsilon) \in \overline{\mathcal{A}}$, passing through $(\boldsymbol{\theta},\mathbf{A})$ for $\varepsilon = 0$; $u_\varepsilon \in \mathrm{L}^2(0,T;V) \cap \mathrm{H}^1(0,T;H)$ is the corresponding state function:

$$\begin{cases} u_\varepsilon'' - \mathsf{div}\,(\mathbf{A}_\varepsilon \nabla u_\varepsilon) = f \\ u_\varepsilon(0,\cdot) = u^0 \\ u_\varepsilon'(0,\cdot) = u^1\,. \end{cases}$$

Denoting by $\delta\boldsymbol{\theta} = \frac{d}{d\varepsilon}\boldsymbol{\theta}_\varepsilon\big|_{\varepsilon=0}$ and $\delta\mathbf{A} = \frac{d}{d\varepsilon}\mathbf{A}_\varepsilon\big|_{\varepsilon=0}$, we proceed to calculate the variation of J, more precisely, we look for $\delta J := \frac{d}{d\varepsilon}J(\boldsymbol{\theta}_\varepsilon, \mathbf{A}_\varepsilon)\big|_{\varepsilon=0}$:

$$\begin{aligned}\delta J &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\Omega_T}\left[\sum_{i=1}^m \theta_i^\varepsilon(\mathbf{x})g_i(t,\mathbf{x},u_\varepsilon(t,\mathbf{x})) - \theta_i(\mathbf{x})g_i(t,\mathbf{x},u(t,\mathbf{x}))\right]d\mathbf{x}\,dt\\ &= \lim_{\varepsilon\to 0}\bigg[\int_{\Omega_T}\sum_{i=1}^m \frac{\theta_i^\varepsilon-\theta_i}{\varepsilon}g_i(\cdot,\cdot,u_\varepsilon)\,d\mathbf{x}\,dt\\ &\quad + \int_{\Omega_T}\sum_{i=1}^m \theta_i\frac{u_\varepsilon-u}{\varepsilon}\int_0^1\frac{\partial g_i}{\partial\lambda}(\cdot,\cdot,u+s(u_\varepsilon-u))\,ds\;d\mathbf{x}\,dt\bigg]. \end{aligned}\tag{7}$$

As $\mathbf{A}_\varepsilon \longrightarrow \mathbf{A}$ in L^∞ for $\varepsilon\to 0$, which implies H-convergence, in an analogous way to the proof of Lemma 1 we have

$$(\forall\, t\in[0,T])\quad u_\varepsilon(t,\cdot) \rightharpoonup u(t,\cdot) \text{ in } V\,,\quad \text{or strongly in } \mathrm{L}^q(\Omega)\,,$$

for any $q\in[1,q^*\rangle$, with q^* given in Lemma 1. Consequently, by an application of the Lebesgue dominated convergence, one gets the convergence

$$\int_0^1\frac{\partial g_i}{\partial\lambda}(\cdot,\cdot,u+s(u_\varepsilon-u))\,ds \longrightarrow \frac{\partial g_i}{\partial\lambda}(\cdot,\cdot,u)\ \text{ in } \mathrm{L}^1(0,T;H)\ \text{ as } \varepsilon\to 0\,.$$

We introduce $w_\varepsilon := \dfrac{u_\varepsilon - u}{\varepsilon}$, which solves

$$\begin{cases} w_\varepsilon'' - \mathsf{div}\,(\mathbf{A}\nabla w_\varepsilon) = \mathsf{div}\,(\delta\mathbf{A}\nabla u_\varepsilon)\\ w_\varepsilon(0,\cdot) = 0\\ w_\varepsilon'(0,\cdot) = 0\,.\end{cases}\tag{8}$$

One should notice that the right-hand side $\mathsf{div}\,(\delta\mathbf{A}\nabla u_\varepsilon)$ belongs to $\mathrm{C}([0,T];V')$, so the solution w_ε belongs to $\mathrm{L}^2(0,T;H)$ with time derivative in $\mathrm{L}^2(0,T;V')$.

Moreover, as $\varepsilon\to 0$, it converges weakly in $\mathrm{L}^2(0,T;V')$, so the corresponding sequence of solutions (w_ε) converges in $\mathrm{L}^\infty(0,T;H)$ weakly $*$ to w, the solution of the initial boundary value problem

$$\text{Find } w\in\mathrm{L}^2(0,T;H)\cap\mathrm{H}^1(0,T;V') \text{ such that}\quad \begin{cases} w'' - \mathsf{div}\,(\mathbf{A}\nabla w) = \mathsf{div}\,(\delta\mathbf{A}\nabla u)\\ w(0,\cdot) = 0\\ w'(0,\cdot) = 0\,.\end{cases}\tag{9}$$

Thus, we can pass to the limit in (7) and obtain

$$\delta J = \int_{\Omega_T}\sum_{i=1}^m\left(\delta\theta_i(\mathbf{x})g_i(t,\mathbf{x},u) + \theta_i(\mathbf{x})\frac{\partial g_i}{\partial\lambda}(t,\mathbf{x},u)\,w(t,\mathbf{x})\right)d\mathbf{x}\,dt\,.$$

After introducing the adjoint state function p, we make the dependence of δJ on $\delta\mathbf{A}$ explicit:

$$\text{Find } p\in\mathrm{L}^2(0,T;V)\cap\mathrm{H}^1(0,T;H) \text{ such that}\quad \begin{cases} p'' - \mathsf{div}\,(\mathbf{A}\nabla p) = \sum_{i=1}^m\theta_i\frac{\partial g_i}{\partial\lambda}(\cdot,\cdot,u)\\ p(T,\cdot) = 0\\ p'(T,\cdot) = 0\,.\end{cases}\tag{10}$$

We would like to test (10) on w, which cannot be done directly. One should approximate the right-hand side in (9) by a sequence $(f_n) \subseteq \mathrm{L}^2(0,T;H)$:

$$f_n \longrightarrow \mathsf{div}\,(\delta\mathbf{A}\nabla u) \ \text{ in } \mathrm{L}^2(0,T;V')\,.$$

With this approximation f_n instead of $\mathsf{div}\,(\delta\mathbf{A}\nabla u)$ we obtain a solution w_n that converges to w in $\mathrm{L}^\infty(0,T;H)$. After testing (10) on w_n we have

$$-\int_{\Omega_T} p' w_n' \,d\mathbf{x}\,dt + \int_{\Omega_T} \mathbf{A}\nabla p\cdot\nabla w_n \,d\mathbf{x}\,dt = \int_{\Omega_T}\sum_{i=1}^m \theta_i \frac{\partial g_i}{\partial\lambda}(\cdot,\cdot,u)\, w_n \,d\mathbf{x}\,dt\,.$$

On the other hand, by testing the equation for w_n on p we get ($\langle\cdot,\cdot\rangle$ is the duality pairing between V' and V)

$$-\int_{\Omega_T} p' w_n' \,d\mathbf{x}\,dt + \int_{\Omega_T} \mathbf{A}\nabla p\cdot\nabla w_n \,d\mathbf{x}\,dt = -\int_0^T \langle f_n, p\rangle\,dt\,,$$

implying that

$$\int_{\Omega_T}\sum_{i=1}^m \theta_i \frac{\partial g_i}{\partial\lambda}(\cdot,\cdot,u)\, w_n \,d\mathbf{x}\,dt = -\int_0^T \langle f_n, p\rangle\,dt\,.$$

At the limit we have the equality

$$\int_{\Omega_T}\sum_{i=1}^m \theta_i \frac{\partial g_i}{\partial\lambda}(\cdot,\cdot,u)\, w \,d\mathbf{x}\,dt = -\int_{\Omega_T} \delta\mathbf{A}\nabla u\cdot\nabla p\,d\mathbf{x}\,dt\,,$$

which simplifies the expression for δJ:

$$\delta J = \int_{\Omega_T}\left[\sum_{i=1}^m \delta\theta_i(\mathbf{x}) g_i(t,\mathbf{x},u(t,\mathbf{x})) - \delta\mathbf{A}(\mathbf{x})\nabla u(t,\mathbf{x})\cdot\nabla p(t,\mathbf{x})\right] d\mathbf{x}\,dt\,.$$

4. Necessary condition of optimality

The necessary condition of optimality is obtained by following the same idea as for the optimal design problems for the stationary diffusion equation [19, 23, 25, 1]: One first considers variations only in $\mathbf{A}$, while keeping $\boldsymbol{\theta}$ fixed. As we shall see, the dependence on t does not introduce significant difficulties.

Let us denote the relaxed optimal design by $(\boldsymbol{\theta}^*, \mathbf{A}^*)$, and the corresponding state function $u^* \in \mathrm{L}^2(0,T;V)\cap\mathrm{H}^1(0,T;H)$, the solution of:

$$\begin{cases} u^{*\prime\prime} - \mathsf{div}\,(\mathbf{A}^*\nabla u^*) = f \\ \qquad\qquad u^*(0,\cdot) = u^0 \\ \qquad\qquad u^{*\prime}(0,\cdot) = u^1\,, \end{cases}$$

and the adjoint state function $p^* \in \mathrm{L}^2(0,T;V)\cap\mathrm{H}^1(0,T;H)$ solving

$$\begin{cases} p^{*\prime\prime} - \mathsf{div}\,(\mathbf{A}^*\nabla p^*) = \sum_{i=1}^m \theta_i^* \frac{\partial g_i}{\partial\lambda}(\cdot,\cdot,u^*) \\ \qquad\qquad p^*(T,\cdot) = 0 \\ \qquad\qquad p^{*\prime}(T,\cdot) = 0\,. \end{cases}$$

The necessary condition of optimality states that the variation δJ in an optimal point $(\boldsymbol{\theta}^*, \mathbf{A}^*)$ must be nonnegative, for any admissible variation $(\delta\boldsymbol{\theta}, \delta\mathbf{A})$. If we take $\delta\boldsymbol{\theta} = \mathbf{0}$, since any variation is given by $\delta\mathbf{A} = \mathbf{A} - \mathbf{A}^*$ for some $\mathbf{A} \in \mathcal{K}(\boldsymbol{\theta}^*)$, we arrive to the following necessary condition of optimality

$$(\forall\, \mathbf{A} \in \mathcal{K}(\boldsymbol{\theta}^*)) \qquad \int_{\Omega_T} \mathbf{A}\nabla u^* \cdot \nabla p^* \, d\mathbf{x}\, dt \leq \int_{\Omega_T} \mathbf{A}^*\nabla u^* \cdot \nabla p^* \, d\mathbf{x}\, dt\,.$$

Since $\mathbf{A}$'s are independent of t, we rewrite this maximisation problem in the following form (here $\cdot$ denotes the Euclidean scalar product on $\mathbf{R}^{d\times d}$)

$$\begin{cases} \displaystyle\int_{\Omega} \mathbf{A}(\mathbf{x}) \cdot \mathbf{M}^*(\mathbf{x}) \to \max \\ \mathbf{A} \in \mathcal{K}(\boldsymbol{\theta}^*)\,. \end{cases} \tag{11}$$

Here matrix function $\mathbf{M}^*$ stands for the symmetric part $\mathrm{Sym} \int_0^T \nabla u^* \otimes \nabla p^* \, dt$, where $\otimes$ denotes the tensor product on $\mathbf{R}^d$. It is easy to see (by contradiction) that (11) is equivalent to pointwise maximisation: For almost every $\mathbf{x} \in \Omega$, the optimal $\mathbf{A}^*(\mathbf{x})$ is the maximum point of

$$\begin{cases} \mathbf{A} \cdot \mathbf{M}^*(\mathbf{x}) \to \max \\ \mathbf{A} \in \mathcal{K}(\boldsymbol{\theta}^*(\mathbf{x}))\,. \end{cases} \tag{12}$$

It is important that we have finally obtained a finite-dimensional optimisation problem, similarly to the case in stationary diffusion [23], where it was proved that $\mathbf{A}^*$ can be found among simple laminates. A surprising fact is that this was obtained while mixing an arbitrary number of anisotropic materials (where optimal bounds for set $\mathcal{K}(\boldsymbol{\theta}^*)$ are not known), using only Wiener bounds (for example, in the case of mixing two isotropic phases these correspond to inequalities (14)). In our situation, this happens only if matrix function $\mathbf{M}^*$ is a rank-one matrix (almost everywhere on Ω) which fails in general, because of the integration over $[0, T]$. This is the reason why in the last section we consider only the case $m = 2$, where the optimisation problem (12) can be solved explicitly (under an additional assumption that $d \leq 3$).

Therefore, in the rest of the paper, we assume that $m = 2$, and for simplicity we write $\gamma_1 = \alpha, \gamma_2 = \beta$. Moreover, instead of $\boldsymbol{\theta}$ we use only a scalar function θ, the local fraction of the first material, with $\boldsymbol{\theta} = (\theta, 1 - \theta)$.

According to (12), we introduce function $f : [0,1] \times \mathbf{R}^{d\times d} \to \mathbf{R}$ given by

$$f(\theta, \mathbf{M}) = \max_{\mathbf{A}\in\mathcal{K}(\theta)} \mathbf{A} \cdot \mathbf{M}\,.$$

Now we take into account variations in θ: We consider a smooth path $\varepsilon \mapsto \theta_\varepsilon$ in $\mathrm{L}^\infty(\Omega; [0,1])$, passing through θ^* for $\varepsilon = 0$, and for any ε and $\mathbf{x} \in \Omega$ we take $\mathbf{A}_\varepsilon(\mathbf{x})$ to be the maximiser for $f(\theta_\varepsilon(\mathbf{x}), \mathbf{M}^*(\mathbf{x}))$.

Theorem 1. *The optimal design $(\theta^*, \mathbf{A}^*)$ satisfies $\mathbf{A}^*(\mathbf{x}) \cdot \mathbf{M}^*(\mathbf{x}) = f(\theta^*(\mathbf{x}), \mathbf{M}^*(\mathbf{x}))$, for almost every $\mathbf{x} \in \Omega$. Defining the quantity*

$$Q(\mathbf{x}) = -\frac{\partial f}{\partial \theta}(\theta^*(\mathbf{x}), \mathbf{M}^*(\mathbf{x})) + \int_0^T g_1(t, \mathbf{x}, u^*(t,\mathbf{x})) - g_2(t, \mathbf{x}, u^*(t,\mathbf{x}))\, dt\,,$$

the optimal θ^* satisfies

$$\begin{array}{rl} \theta^*(\mathbf{x}) = 0 & \text{if } Q(\mathbf{x}) > 0 \\ \theta^*(\mathbf{x}) = 1 & \text{if } Q(\mathbf{x}) < 0 \\ 0 \le \theta^*(\mathbf{x}) \le 1 & \text{if } Q(\mathbf{x}) = 0 \,. \end{array} \tag{13}$$

Remark 1. *The presence of a constraint on the amount of materials $\int_\Omega \theta \, d\mathbf{x} = c$ is treated as usual by adding $l \int_\Omega \theta \, d\mathbf{x}$ to the cost functional J, where l is a Lagrange multiplier, which leads to the same result for the function*

$$Q(\mathbf{x}) = -\frac{\partial f}{\partial \theta}(\theta^*(\mathbf{x}), \mathbf{M}^*(\mathbf{x})) + l + \int_0^T g_1(t, \mathbf{x}, u^*(t, \mathbf{x})) - g_2(t, \mathbf{x}, u^*(t, \mathbf{x})) \, dt \,.$$

5. Optimality criteria method

If a local proportion of the first material is θ, then the set $\mathcal{K}(\theta)$ of effective conductivities consists of all symmetric matrices with eigenvalues belonging to the set $\boldsymbol{\Lambda}(\theta) \subseteq \mathbf{R}^d$ [12, 13, 19, 22]:

$$\lambda_\theta^- \le \lambda_j \le \lambda_\theta^+ \,, \quad j = 1, \ldots, d \,, \tag{14}$$

$$\sum_{j=1}^d \frac{1}{\lambda_j - \alpha} \le \frac{1}{\lambda_\theta^- - \alpha} + \frac{d-1}{\lambda_\theta^+ - \alpha} \,, \tag{15}$$

$$\sum_{j=1}^d \frac{1}{\beta - \lambda_j} \le \frac{1}{\beta - \lambda_\theta^-} + \frac{d-1}{\beta - \lambda_\theta^+} \quad , \tag{16}$$

where λ_θ^+ and λ_θ^- stand for the arithmetic and harmonic average: $\lambda_\theta^+ = \theta\alpha + (1-\theta)\beta$ and $\lambda_\theta^- = \left(\frac{\theta}{\alpha} + \frac{1-\theta}{\beta}\right)^{-1}$.

Let us now fix a point $\mathbf{x} \in \Omega$ (the common Lebesgue point of $\theta^*, \mathbf{A}^*, \mathbf{M}^*$) and consider the optimisation problem (12). The classical von Neumann result (v. [18]) says that a maximum in (12) is attained at a matrix $\mathbf{A}$ such that $\mathbf{A}$ and $\mathbf{M}^*$ are simultaneously diagonalisable. Therefore, denoting by $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d)$ the eigenvalues of $\mathbf{A}$, and by $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_d)$ those of $\mathbf{M}^*$, we have

$$\max_{\mathbf{A} \in \mathcal{K}(\boldsymbol{\theta}^*)} \mathbf{A} \cdot \mathbf{M}^* = \max_{\boldsymbol{\lambda} \in \boldsymbol{\Lambda}(\theta^*)} \sum_{n=1}^d \lambda_n \mu_n \,. \tag{17}$$

This maximal value, denoted by $f(\theta^*, \mathbf{M}^*)$, can be calculated explicitly, as well as the optimal microstructure, for the case $d = 2$ [1, Lemma 3.2.17] or $d = 3$ [26, 27]. For completeness, we state the result in Lemma 3.

In general, since in (17) a linear function in $\boldsymbol{\lambda}$ is maximised, for any $\boldsymbol{\mu}$ there exists a maximum point $\boldsymbol{\lambda}$ satisfying the equality in (15) or (16). These parts of the boundary correspond to sequential laminates, which are defined inductively starting from simple (or rank-one) laminates. Suppose that two original materials are put in layers periodically, such that their thickness ratio is $\frac{\theta}{1-\theta}$. If we pass to the zero limit of the layer thickness (with the ratio preserved), we obtain a homogenised material described by a constant matrix with eigenvalues λ_θ^- (in the direction of lamination) and λ_θ^+ (perpendicular to the direction of lamination). A sequential rank-r laminate is obtained by lamination of a rank-$(r-1)$ sequential laminate with a single pure phase (always the same one which is then called a matrix phase).

It is known [1, 7] that the conductivity of rank-m sequential laminates is a matrix with exactly $d-m$ eigenvalues equal to λ_θ^+ and such that the equality in (15) is satisfied if the matrix material is $\alpha\mathbf{I}$, or (16) if the matrix material is $\beta\mathbf{I}$. The equalities in (15) and (16) hold simultaneously only in the case of first order laminates, where $m-1$ eigenvalues are equal to λ_θ^+ and one is λ_θ^-.

One concludes that if $\mathbf{M}^*(\mathbf{x}) \neq \mathbf{0}$, then the optimal $\mathbf{A}^*(\mathbf{x})$ corresponds to a sequential laminate of rank at most d. In zero points of $\mathbf{M}^*$ we have no information on optimal $\boldsymbol{\lambda}^*$ for (17). However, in these points one can change θ^* and $\mathbf{A}^*$ such that the new design is still optimal and $\mathbf{A}^*$ is a sequential laminate of rank at most d, using the technique of Artstein [6] — for applications in optimal design problems see [20, 23, 5]. This information simplifies the application of a classical gradient method for a numerical solution, as presented in [4, 5] for the stationary diffusion problem.

Let us now describe the application of the optimality criteria method in our situation. Suppose that we have a design $(\theta_n, \mathbf{A}_n)$ at the n-th iteration (starting from some initial design $(\theta_0, \mathbf{A}_0)$). Using these coefficients we calculate the corresponding state function u_n by (1) and then the adjoint function p_n by (10) using $\theta_n, \mathbf{A}_n$ and u_n instead of $\theta, \mathbf{A}$ and u, respectively. Furthermore, we define $\mathbf{M}_n(\mathbf{x}) = \operatorname{Sym}\int_0^T \nabla u_n(t,\mathbf{x}) \otimes \nabla p_n(t,\mathbf{x})\,dt$ and a function $Q_n : \Omega \times [0,1] \to \mathbf{R}$ by

$$Q_n(\mathbf{x},\theta) = -\frac{\partial f}{\partial \theta}(\theta, \mathbf{M}_n(\mathbf{x})) + \int_0^T g_1(t,\mathbf{x},u_n(t,\mathbf{x})) - g_2(t,\mathbf{x},u_n(t,\mathbf{x}))\,dt\,.$$

For the next iteration $\theta_{n+1}(\mathbf{x})$ of the local fraction at $\mathbf{x}$ we take the zero point of $Q(\mathbf{x},\cdot)$ (function $\theta \mapsto Q(\mathbf{x},\theta)$ is monotone [1]), otherwise let $\theta_{n+1}(\mathbf{x})$ be 0 (1) if $Q(\mathbf{x},\cdot)$ is positive (negative) on $[0,1]$. Finally, solve (12) to obtain $\mathbf{A}_{n+1}(\mathbf{x})$ (with $\theta_{n+1}(\mathbf{x}), \mathbf{M}_n(\mathbf{x})$ instead of $\theta, \mathbf{M}$) — this can be done explicitly by Lemma 3 below.

Lemma 2. *For given $\theta \in [0,1]$ and a symmetric 3×3 matrix $\mathbf{M}$ with eigenvalues $\mu_1 \le \mu_2 \le \mu_3$ we have*

A. If $\mu_1 = 0$, then $\frac{\partial f}{\partial \theta}(\theta,\mathbf{M}) = -(\beta-\alpha)(\mu_2+\mu_3)$.

B. If $\mu_1 > 0$, then

$$\frac{\partial f}{\partial \theta}(\theta,\mathbf{M}) = \begin{cases} -\beta(\beta-\alpha)(\alpha+2\beta)\left(\dfrac{\sqrt{\mu_1}+\sqrt{\mu_2}+\sqrt{\mu_3}}{\theta(\beta-\alpha)+\alpha+2\beta}\right)^2, & \theta < \theta_1^B\,, \\ -\beta(\beta^2-\alpha^2)\left(\dfrac{\sqrt{\mu_1}+\sqrt{\mu_2}}{\theta(\beta-\alpha)+\alpha+\beta}\right)^2 - \mu_3(\beta-\alpha)\,, & \theta_1^B \le \theta < \theta_2^B\,, \\ -\dfrac{\alpha\beta(\beta-\alpha)\mu_1}{(\theta(\beta-\alpha)+\alpha)^2} - (\beta-\alpha)(\mu_2+\mu_3)\,, & \theta \ge \theta_2^B\,, \end{cases}$$

with $\theta_1^B = 1 - \dfrac{(2\sqrt{\mu_3}-\sqrt{\mu_2}-\sqrt{\mu_1})\,\beta}{\sqrt{\mu_3}\,(\beta-\alpha)}$, $\theta_2^B = 1 - \dfrac{(\sqrt{\mu_2}-\sqrt{\mu_1})\,\beta}{\sqrt{\mu_2}\,(\beta-\alpha)}$.

C. The case $\mu_1 < 0$. If μ_2 and μ_3 are negative as well, then

$$\frac{\partial f}{\partial \theta}(\theta, \mathbf{M}) = \begin{cases} \alpha(\beta-\alpha)(2\alpha+\beta)\left(\dfrac{\sqrt{-\mu_1}+\sqrt{-\mu_2}+\sqrt{-\mu_3}}{\theta(\beta-\alpha)+3\alpha}\right)^2 , \; \theta > \theta_2^C , \\ \alpha(\beta^2-\alpha^2)\left(\dfrac{\sqrt{-\mu_1}+\sqrt{-\mu_2}}{\theta(\beta-\alpha)+2\alpha}\right)^2 - \mu_3(\beta-\alpha) , \; \theta_1^C < \theta \le \theta_2^C , \\ -\dfrac{\alpha\beta(\beta-\alpha)\mu_1}{(\theta(\beta-\alpha)+\alpha)^2} - (\beta-\alpha)(\mu_2+\mu_3) , \; \theta \le \theta_1^C , \end{cases}$$

with $\theta_1^C = \dfrac{(\sqrt{-\mu_1}-\sqrt{-\mu_2}\,)\,\alpha}{\sqrt{-\mu_2}\,(\beta-\alpha)}$, $\theta_2^C = \dfrac{(\sqrt{-\mu_1}+\sqrt{-\mu_2}-2\sqrt{-\mu_3}\,)\,\alpha}{\sqrt{-\mu_3}\,(\beta-\alpha)}$.

If $\mu_2 < 0$ and $\mu_3 \ge 0$, then θ_2^C is not defined, and we can express $\dfrac{\partial f}{\partial \theta}(\theta, \mathbf{M})$ by the formulae given above, but omitting its first case and the assumption $\theta \le \theta_2^C$ in the second case.

If $\mu_2 \ge 0$, then both θ_1^C and θ_2^C are not defined, and $\dfrac{\partial f}{\partial \theta}(\theta, \mathbf{M})$ is given by the formula given in the third case above, for any $\theta \in [0, 1]$.

To make the above update procedure clear, suppose that $\mathbf{M}_n(\mathbf{x})$ fits the case B of Lemma 2. For simplicity, we shall use notation $\mathbf{M}$ for $\mathbf{M}_n(\mathbf{x})$ and L for

$$\int_0^T g_1(t, \mathbf{x}, u_n(t, \mathbf{x})) - g_2(t, \mathbf{x}, u_n(t, \mathbf{x}))\, dt.$$

As

$$\theta \mapsto \frac{\partial f}{\partial \theta}(\theta, \mathbf{M})$$

is strictly increasing (in case B), it is enough to calculate its values for $\theta \in \{0, 1, \theta_1^B, \theta_2^B\}$ (if they all belong to the segment [0,1]; otherwise, the situation is even more simple) to find out which of the intervals $\langle 0, \theta_1^B\rangle$, $\langle \theta_1^B, \theta_2^B\rangle$ or $\langle \theta_2^B, 1\rangle$ contains the solution $\theta_{n+1}(\mathbf{x})$ of the equation

$$\frac{\partial f}{\partial \theta}(\theta, \mathbf{M}) = L.$$

When we know which interval is right, it is easy to solve this equation explicitly.

Moreover, knowing the location of $\theta_{n+1}(\mathbf{x})$ we can easily calculate $\mathbf{A}_{n+1}(\mathbf{x})$ by the following Lemma equivalent to Corollary 1 in [27], but written in such a way that fits this application better.

Lemma 3. *For given $\theta \in [0, 1]$ and a symmetric 3×3 matrix $\mathbf{M}$ with eigenvalues $\mu_1 \le \mu_2 \le \mu_3$, the maximiser $\mathbf{A}$ in the definition of $f(\theta, \mathbf{M})$ (i.e. such that $\mathbf{A} \cdot \mathbf{M} = f(\theta, \mathbf{M})$) is simultaneously diagonalisable with $\mathbf{M}$, with eigenvalues $\lambda_1 \le \lambda_2 \le \lambda_3$ defined by the following procedure*

A. If $\mu_1 = 0$, then $\lambda_1 = \lambda_\theta^-$ and $\lambda_2 = \lambda_3 = \lambda_\theta^+$.

B. The case $\mu_1 > 0$.

1. If $\theta < \theta_1^B$, then

$$\lambda_i = \alpha + \frac{\sqrt{-\mu_1} + \sqrt{-\mu_2} + \sqrt{-\mu_3}}{c_2(\theta)\sqrt{-\mu_i}}\,,\ i = 1,2,3\,;$$

with $c_2(\theta) = \dfrac{1}{\lambda_\theta^- - \alpha} + \dfrac{2}{\lambda_\theta^+ - \alpha}$.

2. If $\theta_1^B \le \theta < \theta_2^B$, then

$$\lambda_i = \alpha + \frac{\sqrt{-\mu_1} + \sqrt{-\mu_2}}{c_1(\theta)\sqrt{-\mu_i}}\,,\ i = 1,2\,,\ \lambda_3 = \lambda_\theta^+\,;$$

with $c_1(\theta) = \dfrac{1}{\lambda_\theta^- - \alpha} + \dfrac{1}{\lambda_\theta^+ - \alpha}$.

3. If $\theta \ge \theta_2^B$, then $\lambda_1 = \lambda_\theta^-$ and $\lambda_2 = \lambda_3 = \lambda_\theta^+$.

C. The case $\mu_1 < 0$.

1. If $\theta > \theta_2^C$, then

$$\lambda_i = \beta - \frac{\sqrt{\mu_1} + \sqrt{\mu_2} + \sqrt{\mu_3}}{d_2(\theta)\sqrt{\mu_i}}\,,\ i = 1,2,3\,;$$

with $d_2(\theta) = \dfrac{1}{\beta - \lambda_\theta^-} + \dfrac{2}{\beta - \lambda_\theta^+}$.

2. If $\theta_1^C < \theta \le \theta_2^C$, then

$$\lambda_i = \beta - \frac{\sqrt{\mu_1} + \sqrt{\mu_2}}{d_1(\theta)\sqrt{\mu_i}}\,,\ i = 1,2\,,\ \lambda_3 = \lambda_\theta^+\,;$$

with $d_1(\theta) = \dfrac{1}{\beta - \lambda_\theta^-} + \dfrac{1}{\beta - \lambda_\theta^+}$.

3. If $\theta \le \theta_1^C$, then $\lambda_1 = \lambda_\theta^-$ and $\lambda_2 = \lambda_3 = \lambda_\theta^+$.

6. Numerical examples

We shall present numerical results for two optimal design problems, differing only in initial condition u^0.

In both examples $\Omega = \langle -1, 1\rangle^3$, $T = 1$, $\alpha = 1$ and $\beta = 2$, with a volume constraint of 40% of the first material: $\int_\Omega \theta = 0.4|\Omega|$.

The cost functional is given by

$$J(\theta, \mathbf{A}) = -\int_{\Omega_T} u\, d\mathbf{x}\, dt \longrightarrow \min,$$

the right-hand side is $f = 3\chi_A$, where χ_A is the characterististic function of the set $A = \{(t, \mathbf{x}) \in \Omega_T : x_1 \ge 0\}$ and the initial velocity $u^1 = 0$.

The initial position u^0 in the first example equals zero, and for the second one

$$u^0(x, y, z) = (1 - x^2)\cos\frac{\pi y}{2}\cos\frac{\pi z}{2}\,.$$

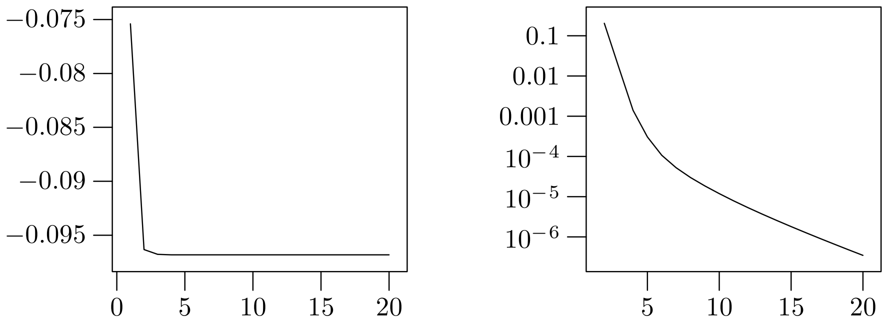

Figure 1. *Convergence history for the first example: a) cost functional* J *and b)* $\|\theta_i - \theta_{i-1}\|^2_{L^2}$*, in terms of the iteration number* i

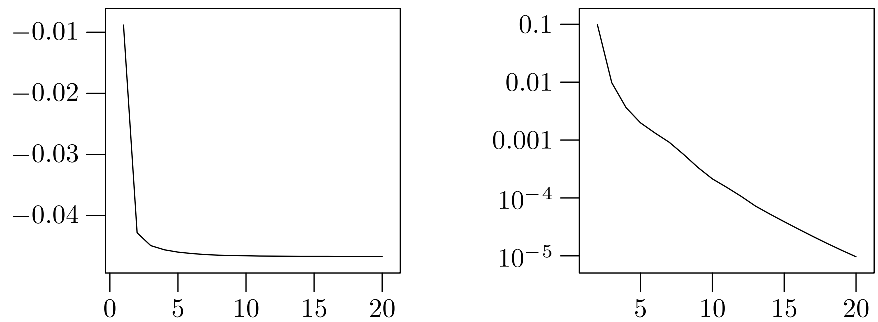

Figure 2. *Convergence history for the second example: a) cost functional* J *and b)* $\|\theta_i - \theta_{i-1}\|^2_{L^2}$*, in terms of the iteration number* i

We used deal.II C++ library [9] for the finite element solution of initial boundary value problems for wave equation. The space domain was divided into a $64 \times 64 \times 64$ cubic mesh, with bilinear elements and we used Gaussian quadratures of order two for numerical integration; time discretisation step was $\frac{1}{128}$. At each iteration of the algorithm (we made 20 iterations) two initial boundary value problems with the same coefficient matrix are solved: for the state function and the adjoint state function. Based on these solutions, the update of the design variables is calculated, as described in the previous Section. The initial design for the numerical solutions presented here is $\theta = 0.4$ on Ω, $\mathbf{A}$ being the simple laminate with layers orthogonal to $\mathbf{e}_1$, in both examples. However, many experiments show that a numerical solution practically does not change if we consider different initial design.

The convergence histories for these two examples are shown in Figures 1 and 2, respectively. The method shows a very quick convergence: the optimal design can be approximately reconstructed after just three iterations. In Figure 5 the local fraction θ is presented at the final iteration. As a reminder, the value $\theta = 1$ corresponds to the material with coefficient $\alpha\mathbf{I}$, and $\theta = 0$ to the other one. In Figure 3 we presented the level set of function θ at height 0.05, to see more closely what happens inside.

The second numerical example is presented in the same way in Figures 6 and 4.

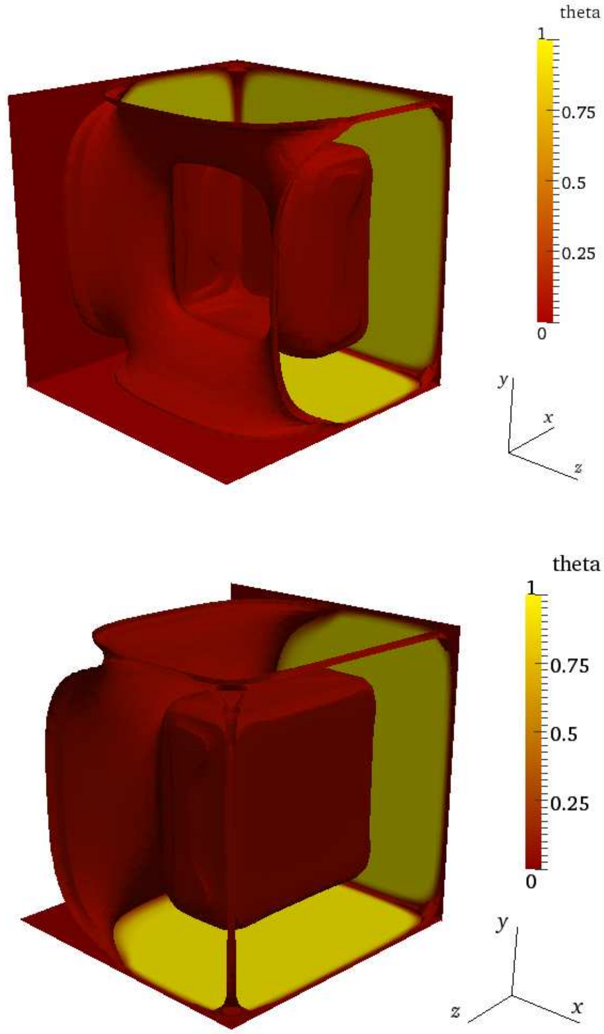

Figure 3. *Level set $\theta = 0.05$ for the first example, two different view angles*

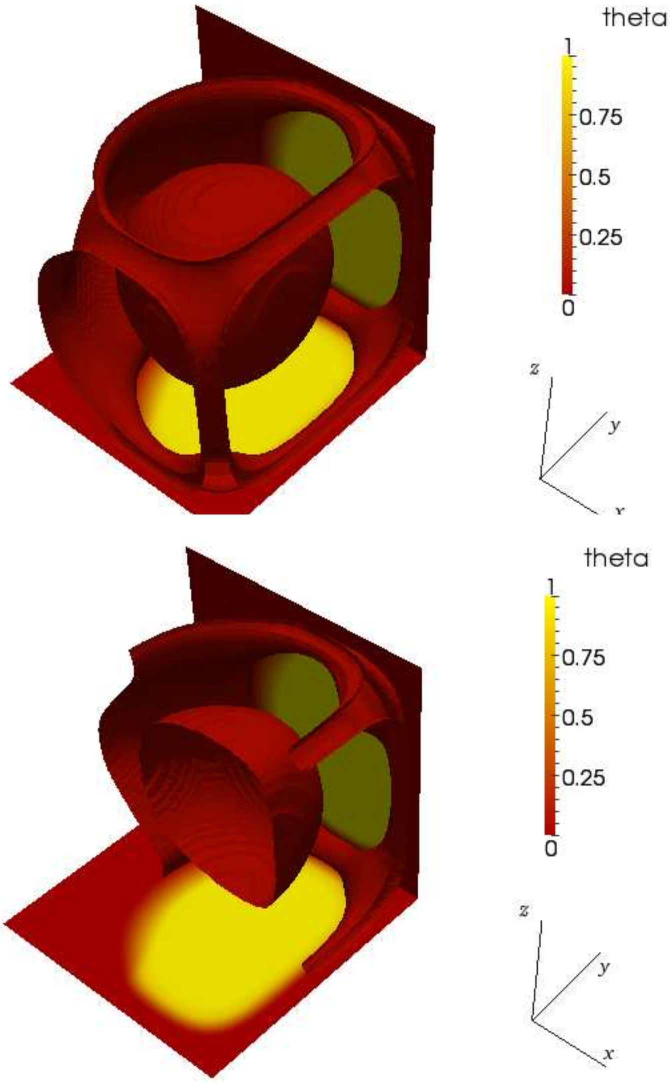

Figure 4. *Level set* $\theta = 0.05$ *for the second example: The whole cube* $[-1,1]^3$ *and its half* $[-1,1]\times[0,1]\times[-1,1]$

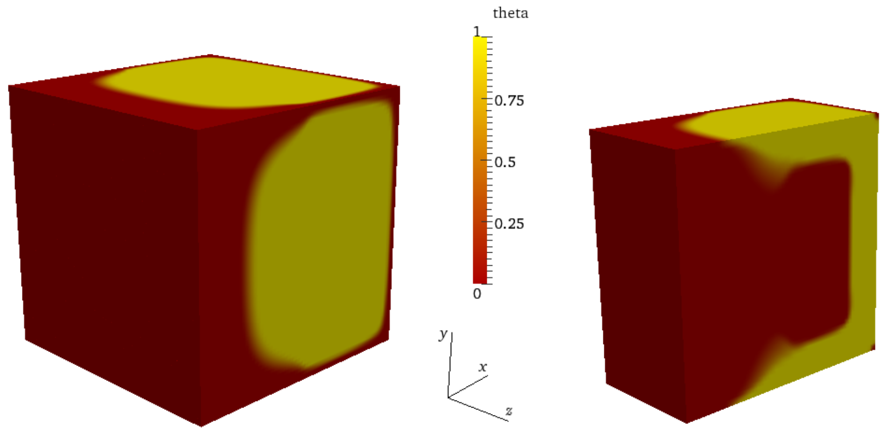

Figure 5. *Optimal distribution of materials for the first example: The whole cube* $[-1,1]^3$ *and its half* $[-1,1]^2 \times [-1,0]$

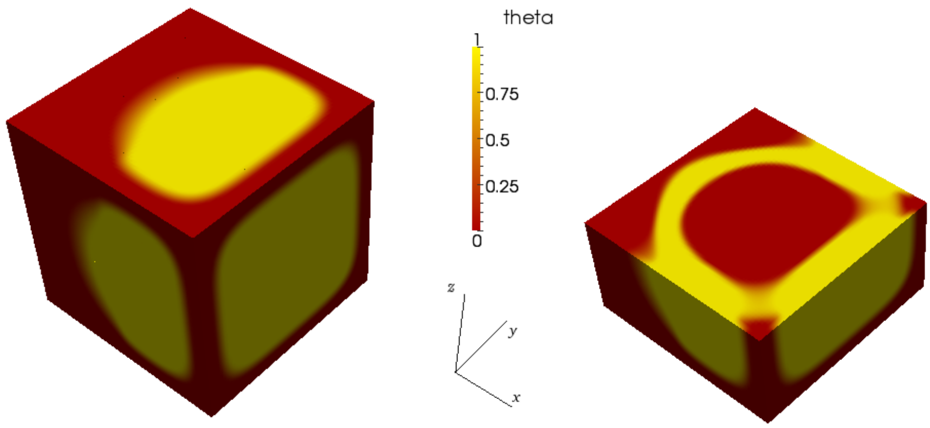

Figure 6. *Optimal distribution of materials for the second example: The whole cube* $[-1,1]^3$ *and its half* $[-1,1]^2 \times [-1,0]$

7. Multiple state equations

We consider the response of a body obtained by a mixture of two isotropic phases, under N different right-hand sides:

$$\begin{cases} u_j \in \mathrm{L}^2(0,T;V) \ \ , \ \ u_j' \in \mathrm{L}^2(0,T;H) \\ u_j'' - \mathsf{div}\,(\mathbf{A}\nabla u_j) = f_j \\ \qquad u_j(0,\cdot) = u_j^0 \\ \qquad u_j'(0,\cdot) = u_j^1\,. \end{cases} \qquad j = 1,\dots,N$$

Denoting the solution by $\mathsf{u} = (u_1, \dots, u_N)$ we state the optimal design problem

$$\begin{cases} J(\boldsymbol{\chi}, \mathbf{A}) = \int_{\Omega_T} \chi(\mathbf{x}) g_1(t, \mathbf{x}, \mathsf{u}(t,\mathbf{x})) + (1-\chi(\mathbf{x})) g_2(t, \mathbf{x}, \mathsf{u}(t,\mathbf{x}))\, d\mathbf{x}\, dt \longrightarrow \min \\ \boldsymbol{\chi} \in \mathrm{L}^\infty(\Omega; \{0,1\}),\ \mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I}\,. \end{cases} \tag{18}$$

The analysis presented in Section 2 is applicable to this problem as well, so the relaxation consists of considering the closure of the original admissible set of designs in L^∞ weak $*$ topology for $\boldsymbol{\chi}$ and H-topology for $\mathbf{A}$. A similar situation occurs for the case of stationary diffusion optimal design problems [1, 4, 5]. If the analogous conditions on functions g_i are set, by defining the adjoint state $\mathsf{p}^* = (p_1^*, \dots, p_N^*)$ as a solution to

$$\begin{cases} p_j^* \in \mathrm{L}^2(0,T;V)\ \ ,\ \ p_j^{*\prime} \in \mathrm{L}^2(0,T;H) \\ p_j^{*\prime\prime} - \mathsf{div}\,(\mathbf{A}^*\nabla p_j^*) = \sum_{i=1}^m \theta_i^* \frac{\partial g_i}{\partial \lambda_j}(\cdot,\cdot,\mathsf{u}^*) \\ p_j^*(T,\cdot) = 0 \\ p_j^{*\prime}(T,\cdot) = 0\,, \end{cases} \qquad j = 1, \dots, N$$

one obtains the following result.

Theorem 2. *Let* $(\theta^*, \mathbf{A}^*)$ *be the optimal relaxed design for (18),* u^* *the corresponding state function,* p^* *the adjoint state function and*

$$\mathbf{M}^* = \mathrm{Sym} \int_0^T \sum_{j=1}^N \nabla u_j^* \otimes \nabla p_j^*\, dt\,.$$

Then the optimal design satisfies $\mathbf{A}^*(\mathbf{x}) \cdot \mathbf{M}^*(\mathbf{x}) = f(\theta^*(\mathbf{x}), \mathbf{M}^*(\mathbf{x}))$*, for almost every* $\mathbf{x} \in \Omega$. *Defining the quantity*

$$Q(\mathbf{x}) = -\frac{\partial f}{\partial \theta}(\theta^*(\mathbf{x}), \mathbf{M}^*(\mathbf{x})) + \int_0^T g_1(t, \mathbf{x}, \mathsf{u}^*) - g_2(t, \mathbf{x}, \mathsf{u}^*)\, dt\,,$$

the optimal θ^* *satisfies*

$$\begin{array}{rl} \theta^*(\mathbf{x}) = 0 & \text{if } Q(\mathbf{x}) > 0 \\ \theta^*(\mathbf{x}) = 1 & \text{if } Q(\mathbf{x}) < 0 \\ 0 \le \theta^*(\mathbf{x}) \le 1 & \text{if } Q(\mathbf{x}) = 0\,. \end{array} \tag{19}$$

Based on this necessary condition of optimality, the optimality criteria method can be used in the same way as it was done in the case of one state equation in Section 5.

References

[1] G. Allaire, *Shape optimization by the homogenization method*, Springer, Berlin, 2002.

[2] N. Antonić, M. Vrdoljak, *Optimal design and hyperbolic problems*, Math. Commun. **4**(1999), 121–129.

[3] N. Antonić, N. Balenović, M. Vrdoljak, *Optimal design for vibrating plates*, Z. Angew. Math. Mech. **80**(2000), 783–784.

[4] N. ANTONIĆ, M. VRDOLJAK, *Sequential laminates in multiple state optimal design problems*, Math. Probl. Eng. **2006**(2006), Article ID 68695.

[5] N. ANTONIĆ, M. VRDOLJAK, *Gradient methods for multiple state optimal design problems*, Ann. Univ. Ferrara Sez. VII Sci. Mat. **53**(2007), 177–187.

[6] Z. ARTSTEIN, *Discrete and continuous bang-bang and facial spaces or: look for the extreme points*, SIAM Rev. **22**(1980), 172–185.

[7] A. CHERKAEV, *Variational methods for structural optimization*, Springer, New York, 2000.

[8] G. DAL MASO, R.V. KOHN, *The local character of G-closure*, to appear.

[9] W. BANGERTH, R. HARTMANN, G. KANSCHAT ET AL., *The deal.II library*, available at `http://www.dealii.org/`.

[10] R. DAUTRAY, J.-L. LIONS, *Mathematical analysis and numerical methods for science and technology*, Vol.V, Springer, Berlin, 1992.

[11] J.-L. LIONS, E. MAGENES, *Non-homogeneous boundary value problems and applications*, Vol.I, Springer, Berlin, 1972.

[12] K.A. LURIE, A.V. CHERKAEV, *Exact estimates of conductivity of composites formed by two isotropically conducting madia, taken in prescribed proportion*, Proc. Royal Soc. Edinburgh **99A**(1984), 71–87.

[13] K.A. LURIE, A.V. CHERKAEV, *Exact estimates of the conductivity of a binary mixture of isotropic materials*, Proc. Royal Soc. Edinburgh **104A**(1986), 21–38.

[14] K.A. LURIE, *A stable spatio-temporal G-closure and G_m-closure of a set of isotropic dielectrics with respect to one-dimensional wave propagation*, Wave Motion **40**(2004), 95–110.

[15] K.A. LURIE, *An introduction to the mathematical theory of dynamic materials*, Springer, New York, 2007.

[16] F. MAESTRE, A. MÜNCH, P. PEDREGAL, *Optimal design under the one-dimensional wave equation*, Interfaces Free Bound. **10**(2008), 87–117.

[17] F. MAESTRE, A. MÜNCH, P. PEDREGAL, *A spatio-temporal design problem for a damped wave equation*, SIAM J. Appl. Math. **68**(2007), 109–132.

[18] L. MIRSKY, *On the trace of matrix products*, Math. Nach. **20**(1959), 171–174.

[19] F. MURAT, L. TARTAR, *Calculus of variations and homogenization*, in: *Topics in the Mathematical Modelling of Composites Materials*, (A. Cherkaev and R. V. Kohn, Eds.), Birkhäuser, 1997, 139–173.

[20] U. RAITUMS, *Extension of extremal problems connected with linear elliptic equations*, Dokl. Akad. Nauk SSSR **243**(1978), 281–283.

[21] U. RAITUMS, *On the local representation of G-closure*, Arch. Rational Mech. Anal. **158**(2001), 213–234.

[22] L. TARTAR, *Estimations fines des coefficients homogénéisés*, in: *Ennio DeGiorgi colloquium*, (P. Krée, Ed.), Pitman, 1985, 168–187.

[23] L. TARTAR, *Remarks on homogenization method in optimal design problems, Homogenization and applications to material sciences*, GAKUTO Internat. Ser. Math. Sci. Appl. **9**(1995), 393–412.

[24] L. TARTAR, *Homogenization and hyperbolicity, Dedicated to Ennio De Giorgi*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **25**(1997), 785–805.

[25] L. TARTAR, *An introduction to the homogenization method in optimal design*, in: *Optimal shape design*, (A. Cellina and A. Ornelas, Eds.), Springer, 2000, 47–156.

[26] M. VRDOLJAK, *Multiple state optimal design problems and Hashin-Shtrikman bounds*, Int. J. Pure Appl. Math. **42**(2008), 319–324.

[27] M. VRDOLJAK, *On Hashin-Shtrikman bounds for mixtures of two isotropic materials*, Nonlinear Anal. Real World Appl. **11**(2010), 4597–4606.