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THE ϑ-TRANSFER TECHNIQUE: ON NOETHERIAN INVOLUTION RINGS AND SYMMETRY OF PRIMITIVITY

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Abstract. Let $\vartheta : \mathcal{R} \to \mathcal{S}$ be a ring anti-isomorphism. We study ϑ-homomorphisms between left $\mathcal{R}$-modules E and right $\mathcal{S}$-modules M, that is, homomorphisms of the additive groups $\theta : E \to M$ satisfying $\theta(r.x) = \theta(x).\vartheta(r)$ for $r \in \mathcal{R}$ and $x \in E$. We also study the class of Noetherian involution rings and the problem of symmetry of primitivity. In particular, suppose that for every semiprimitive Noetherian involution ring which has exactly two minimal prime ideals both of these satisfy (SP). Then every prime ideal of an arbitrary Noetherian ring satisfies (SP); we say that a prime ideal $\mathcal{P}$ of some ring satisfies (SP), the symmetry of primitivity, if it holds that $\mathcal{P}$ is left primitive if and only if it is right primitive. Besides, as an interesting fact, we note that any factor ring of the enveloping algebra of the Lie algebra $\mathfrak{sl}(2)$ over a field of characteristic zero is an involution algebra, and so it satisfies the Krull symmetry.

Introduction

The problem of validity of various symmetry properties for some class of rings or a category of bimodules ${}_{\mathcal{R}}X_{\mathcal{S}}$, for given rings $\mathcal{R}$ and $\mathcal{S}$, is very interesting. In particular, the following two: (KS) Krull symmetry, and (SP) symmetry of primitivity, are well known and yet unsolved problems for the class $\mathfrak{Noe}$ of Noetherian rings (see, e.g., [GW, Appendix]). Nevertheless, some results have been obtained for certain subclasses of $\mathfrak{Noe}$. For example, (KS) holds for FBN rings due to Gabriel [G] (see also [MR, Thm. 6.4.8], and [K, GR]) who showed that in this case the classical Krull dimension is equal to

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the (left, right) Krull dimension; and for factor rings of the enveloping algebra of a finite-dimensional solvable Lie algebra over a field of characteristic zero due to Heinicke [H]. Also, (SP) holds for the enveloping algebra of a finite dimensional Lie algebra over an algebraically closed field of characteristic zero due to Dixmier [Di] and Moeglin [Mg].

Suppose now that $(\mathcal{R}, \mathcal{S})$ is a pair of anti-isomorphic (resp. isomorphic) rings, and fix some anti-isomorphism (resp. isomorphism) $\vartheta : \mathcal{R} \to \mathcal{S}$. Let E be a left $\mathcal{R}$-module and M a right (resp. left) $\mathcal{S}$-module. The following generalizes the standard notion of a homomorphism of modules: A homomorphism of the additive groups $\theta : E \to M$ will be called a *ϑ-homomorphism* between E and M if $\theta(r.x) = \theta(x).\vartheta(r)$ (resp. $\theta(r.x) = \vartheta(r).\theta(x)$) for all $r \in \mathcal{R}$ and $x \in E$; if θ is moreover bijective it will be called a *ϑ-isomorphism.* The purpose of this paper is to study ϑ-homomorphisms between left $\mathcal{R}$-modules E and right $\mathcal{S}$-modules M. Namely, choosing convenient ϑ, many left-hand properties of E are "transferable via ϑ" to the corresponding right-hand properties of M. This transfering we call the *ϑ-transfer technique*; as we will see, it can be helpful while considering various symmetry properties. Next, we focus our attention at the class of antiautomorphic rings, and in particular at its subclass $\mathfrak{NoeInv}$ of Noetherian involution rings. Although the class $\mathfrak{NoeInv}$ is for many problems much easier to study than the class $\mathfrak{Noe}$ (for example, for $\mathfrak{NoeInv}$ the problem (KS) is trivial; see Corollary 2.2), it turns out that these also have some close connections. The following theorem supports the last claim; it will be proved in Section 3. (By this, we see that in order to deal with the problem (SP) it is sufficient to consider only the (finitely many) minimal primes of $\mathcal{R}$, when $\mathcal{R}$ is running through a rather restricted class of rings. Moreover, we may assume that every such $\mathcal{R}$ has exactly two minimal primes.)

THEOREM. *The following are equivalent.*

(i) (SP) *holds for the class* $\mathfrak{Noe}$.

(ii) (SP) *holds for the class* $\mathfrak{NoeInv}$.

(iii) *For every semiprimitive Noetherian involution ring $\mathcal{R}$, every minimal prime ideal of $\mathcal{R}$ satisfies* (SP).

(iv) *For every semiprimitive Noetherian involution ring $\mathcal{R}$ such that it has exactly two minimal prime ideals, say $\mathcal{P}_1$ and $\mathcal{P}_2$, it holds that $\mathcal{P}_i$ is left primitive if and only if it is right primitive, for $i = 1, 2$.*

REMARK. (1) Suppose that $(\mathcal{R}, \mathcal{S})$ and an anti-isomorphism ϑ are as above. Let $\mathcal{R}$-**mod** and **mod**-$\mathcal{S}$ be the categories of left $\mathcal{R}$-modules and right $\mathcal{S}$-modules, respectively. Consider $\Omega : \mathcal{R}\text{-}\mathbf{mod} \to \mathbf{mod}\text{-}\mathcal{S}$ defined on objects such that $\Omega(E)$ is E as an additive group, and $x.s := \vartheta^{-1}(s).x$ for $s \in \mathcal{S}$ and $x \in E$. Also, if ε is a homomorphism between left $\mathcal{R}$-modules E and F, define $\Omega(\varepsilon)(x) := \varepsilon(x)$ for $x \in E$; then $\Omega(\varepsilon)$ is a homomorphism between the right $\mathcal{S}$-modules $\Omega(E)$ and $\Omega(F)$. Thus, as is very well known, Ω is an equivalence of categories. But let us emphasize that the approach via the ϑ-transfer presented here, and the obtained conclusions as well, cannot be simply deduced just using the fact that now $\mathcal{R}$-**mod** and **mod**-$\mathcal{S}$ are equivalent categories.

(2) The class $\mathfrak{Noe}$ and the class of antiautomorphic rings are mutually incomparable. However, a number of important rings which appear in practice are antiautomorphic or even involution. For example, the ring $T_n(\mathcal{A})$ of all upper/lower triangular $n \times n$ matrices over an antiautomorphic (resp. involution) ring $\mathcal{A}$ is antiautomorphic (resp. involution) as well; see Claim 2.4. Next, let $\mathcal{U}$ be the enveloping algebra of the Lie algebra $\mathfrak{sl}(2)$ over a field of characteristic zero. Then, for any ideal I of $\mathcal{U}$, $\mathcal{U}/I$ is an involution algebra, and thus in particular it satisfies the Krull symmetry; see Proposition 2.5.

Notation, terminology and conventions

Let us briefly introduce the notation, terminology and conventions that will be freely used throughout the paper; we assume that the reader is familiar with the standard material, as can be found in [GW] and [MR].

Every ring will have an identity and every module will be unital. If not otherwise specified, the term "ideal" always means a two-sided ideal.

Let $\mathcal{R}$ be a ring. We denote by (0) the zero-ideal of $\mathcal{R}$, by $J(\mathcal{R})$ the Jacobson radical of $\mathcal{R}$, by $\operatorname{Id}\mathcal{R}$ the set of all ideals of $\mathcal{R}$ and by $\operatorname{Aut}\mathcal{R}$ the group of all automorphisms of $\mathcal{R}$. For a subset $A \subseteq \mathcal{R}$, $\langle A \rangle$ denotes the ideal of $\mathcal{R}$ generated by A. The prime spectrum of $\mathcal{R}$, the maximal spectrum of $\mathcal{R}$, the left primitive spectrum of $\mathcal{R}$ and the right primitive spectrum of $\mathcal{R}$ will be denoted by $\operatorname{Spec}\mathcal{R}$, $\operatorname{Max}\mathcal{R}$, $\operatorname{lPrim}\mathcal{R}$ and $\operatorname{rPrim}\mathcal{R}$, respectively.

Let now X be a (left, right) $\mathcal{R}$-module. We denote by $\mathcal{L}(X)$ the lattice of all submodules of X. For a subset $S \subseteq X$, we write $\operatorname{ann}(S)$ for the annihilator of S. If X is moreover a bimodule, for the sake of preciseness we will usually distinguish the left annihilator $\operatorname{lann}(S)$ of S from the right annihilator $\operatorname{rann}(S)$. The Krull dimension (see [RG], and also [MR, Sect. 6]) is denoted by Kdim.

Let $\mathcal{R}$ and $\mathcal{S}$ be two rings. An antihomomorphism from $\mathcal{R}$ to $\mathcal{S}$ is a homomorphism ϑ of the additive groups which moreover satisfies $\vartheta(xy) = \vartheta(y)\vartheta(x)$ for every $x, y \in \mathcal{R}$. A bijective (resp. injective, surjective) antihomomorphism is called an anti-isomorphism (resp. antimonomorphism, antiepimorphism). We say that two rings are anti-isomorphic if there exists an anti-isomorphism between them. An anti-isomorphism (resp. antihomomorphism) from $\mathcal{R}$ to $\mathcal{R}$ is called an antiautomorphism (resp. antiendomorphism) of $\mathcal{R}$, and then we say that $\mathcal{R}$ is a ring with antiautomorphism or antiautomorphic ring. By $\operatorname{aAut}\mathcal{R}$ we denote the set of all antiautomorphisms of $\mathcal{R}$. If not otherwise said, an involution of a ring $\mathcal{R}$ will always be denoted by *; in that case $\mathcal{R}$ is called a ring with involution or involution ring. By $\operatorname{Inv}\mathcal{R}$ we denote the set of all involutions of $\mathcal{R}$.

1. ϑ-homomorphisms of modules

Note. In the sequel, when we say, for example, that "$\vartheta : \mathcal{R} \to \mathcal{S}$ is an antihomomorphism" it always means that $\mathcal{R}$ and $\mathcal{S}$ are rings and that ϑ is some antihomomorphism between them.

The notions of a homomorphism of rings and homomorphism of modules are standard. In this preparatory section we focus our attention on the "dual" notions: ring antihomomorphisms, and ϑ-homomorphisms of modules for a fixed ring antihomomorphism ϑ (cf. [B, Ch. I, §§4,8 and Ch. II, §1]). We also give some preliminary results, and introduce some other notions closely related to the ϑ-transfer. We begin with the following clear lemma.

Lemma 1.1. *Suppose that $\vartheta : \mathcal{R} \to \mathcal{S}$ is an antiepimorphism.*

(i) *If I is a left (resp. right, two-sided) ideal of $\mathcal{R}$, then $\vartheta(I)$ is a right (resp. left, two-sided) ideal of $\mathcal{S}$. If I is moreover a maximal (left, right or two-sided) ideal and ϑ is an anti-isomorphism, then $\vartheta(I)$ is maximal too.*

(ii) *If J is a left (resp. right, two-sided) ideal of $\mathcal{S}$, then $\vartheta^{-1}(J)$ is a right (resp. left, two-sided) ideal of $\mathcal{R}$.*

By the previous lemma it follows that the following definition makes sense only for (two-sided) ideals.

Definition 1.2. An ideal I of a ring $\mathcal{R}$ with antiautomorphism α is called an *α-stable* ideal if $\alpha(I) \subseteq I$. If moreover $\alpha(I) = I$, we say that I is an *α-ideal.* (Note that every α-stable ideal is an α-ideal when $\alpha \in \operatorname{Inv}\mathcal{R}$.) An ideal of $\mathcal{R}$ which is an α-ideal for some (resp. every) $\alpha \in \operatorname{aAut}\mathcal{R}$ will be called *proper* (resp. *strongly proper*).

Remark 1.3. (1) For an antihomomorphism $\vartheta : \mathcal{R} \to \mathcal{S}$ and ideals I of $\mathcal{R}$ and J of $\mathcal{S}$ satisfying $\vartheta(I) \subseteq J$, the map $\overline{\vartheta} : \mathcal{R}/I \to \mathcal{S}/J$ defines an antihomomorphism of the quotient rings in a natural way. If ϑ is an anti-isomorphism and $\vartheta(I) = J$, then $\overline{\vartheta}$ is an anti-isomorphism too. In particular, for $\alpha \in \operatorname{aAut}\mathcal{R}$ and I α-stable (resp. an α-ideal), the map $\overline{\alpha}$ defines an antiendomorphism (resp. antiautomorphism) of the factor ring $\mathcal{R}/I$.

(2) In general, an α-stable ideal need not be an α-ideal. To see this one can consider the ring $\mathcal{R} := \mathbb{F}[X_i \mid i \in \mathbb{Z}]$, the algebra of polynomials in infinitely many variables indexed by the integers and with coefficients from a field $\mathbb{F}$. Let α be a shift defined on the generators of $\mathcal{R}$ as $\alpha(X_i) := X_{i+1}$ for every $i \in \mathbb{Z}$. Clearly, $\alpha \in \operatorname{Aut}\mathcal{R}$ ($= \operatorname{aAut}\mathcal{R}$; since $\mathcal{R}$ is commutative). Now, the ideal $I := \langle X_i \mid i \geq 0\rangle$ is α-stable but it is not an α-ideal.

For convenience of the reader and further needs we recall the definition of a skew group ring. (We prefer the "left-hand version"; for the right-hand one see, e.g., [MR, 1.5.4].) Let $\mathcal{A}$ be a ring, G a group and $\varphi : G \to \operatorname{Aut}\mathcal{A}$

a group homomorphism. Write a^g instead of $\varphi(g)(a)$. Then the *skew group ring* $\mathcal{A}\#G$ is a free left $\mathcal{A}$-module with elements of G as a basis, and with multiplication defined by $(a.g)(b.h) := ab^g.gh$ for $a, b \in \mathcal{A}$ and $g, h \in G$. When φ is trivial, $\mathcal{A}\#G$ becomes the ordinary group ring which we denote by $\mathcal{A}[G]$. The following lemma is a basic result; cf. Remark 1.5.

LEMMA 1.4. *Let $\mathcal{A}$ be a commutative ring and G any group. Then the skew group ring $\mathcal{A}\#G$ is an involution ring with the natural involution*

$$(a.g)^* := a^{g^{-1}}.g^{-1} \qquad \text{for } a \in \mathcal{A}, g \in G. \tag{1.1}$$

REMARK 1.5. The previous lemma can be slightly improved. For that, suppose $\mathcal{A}$ is a ring with an antiautomorphism (resp. involution) α. (Of course, for $\mathcal{A}$ commutative we can take $\alpha = 1_{\mathcal{A}}$.) Then the group ring $\mathcal{A}[G]$ is antiautomorphic (resp. involution) too. Moreover, suppose that every $\varphi(g)$, $g \in G$, commutes with α. Then it is easy to check that the map

$$\vartheta(a.g) := \alpha(a)^{g^{-1}}.g^{-1}$$

is an antiautomorphism (resp. involution) of $\mathcal{A}\#G$.

In the example below we will see that, having a concrete antiautomorphism α of an even relatively simple ring $\mathcal{R}$, it can be quite complicated to check whether a concrete ideal of $\mathcal{R}$ is α-stable or not. (Concerning this, note how one can expect that the problem of understanding of the set of all proper ideals in an arbitrary $\mathcal{R}$ could be very hard.)

EXAMPLE 1.6. Let $\mathcal{A}$ be a commutative ring, and G a finite cyclic group of order n with generator ϖ. Consider $\mathcal{R} := \mathcal{A}[G]$ with the natural involution as in (1.1). Let $r = \sum_{i<n} a_i.\varpi^i \in \mathcal{R}$ be arbitrary and then define the principal ideal $P := \langle r \rangle$ of $\mathcal{R}$. We would like to know whether $P^* = P$; or equivalently asked, whether

$$r^* = \rho r \qquad \text{for some } \rho \in \mathcal{R}. \tag{$\star$}$$

Putting $\rho = \sum_{i<n} x_i.\varpi^i$, $(\star)$ is further equivalent to the solvability of the system

$$\sum_{i<n} x_i a_{j-i} = a_{n-j} \qquad \text{for } j = 0, 1, \ldots, n-1, \tag{$\star\star$}$$

in the indeterminates x_i; here $a_k = a_l$ provided $k \equiv l \pmod n$.

Let now $\mathcal{A} = \mathbb{F}$ be an algebraically closed field and take $n = 3$. By elementary calculations it can be shown that the system $(\star\star)$ is non-solvable if and only if $\mathrm{char}(\mathbb{F}) \neq 3$ and simultaneously we have:

(a) $3a_0a_1a_2 = a_0^3 + a_1^3 + a_2^3$; and
(b) $(a_0 + a_1 + a_2)(a_0 - a_1)(a_0 - a_2)(a_1 - a_2) \neq 0$.

Consider the case $\mathrm{char}(\mathbb{F}) \neq 3$. Then define the principal ideals (e denotes the identity of G)

$$I := \langle e + \varpi + \varpi^2 \rangle, \quad K := \langle e - \varpi \rangle, \quad K_1 := \langle \varpi + \gamma\varpi^2 \rangle,$$

where $\gamma \in \mathbb{F}$ satisfies $\gamma^2 + 1 = \gamma$. Using the above criterion we easily conclude that $I^* = I$, $K^* = K$ and $K_1^* \neq K_1$. We also have $\mathcal{R} = I \oplus K$, and for the ideal $J := K \cap K_1$ it holds $J = \langle -\gamma e + \varpi + \gamma^2\varpi^2 \rangle$. Hence we deduce that $\mathcal{R} = I \oplus J \oplus J^*$. Let us observe that the maximal ideal $K = J \oplus J^*$ is a *-ideal, while the natural involution permutes the maximal ideals $K_1 = I \oplus J$ and $K_1^* = I \oplus J^*$. (Note that by Maschke's theorem (see, e.g., [P, §3.6]) we know that $\mathcal{R}$ is a semisimple ring, and hence that $\operatorname{Spec}\mathcal{R} = \{K, K_1, K_1^*\}$ and $\operatorname{Id}\mathcal{R} = \{(0)\} \cup \{I, J, J^*\} \cup \operatorname{Spec}\mathcal{R}$.)

The following definition introduces the ϑ-homomorphisms of modules.

Definition 1.7. Let $\vartheta : \mathcal{R} \to \mathcal{S}$ be an anti-isomorphism, E a left (resp. right) $\mathcal{R}$-module, and M a right (resp. left) $\mathcal{S}$-module. A homomorphism $\theta : E \to M$ of the additive groups, which moreover satisfies

$$\theta(r.x) = \theta(x).\vartheta(r) \qquad (\text{resp. } \theta(x.r) = \vartheta(r).\theta(x)) \tag{1.2}$$

for $r \in \mathcal{R}$ and $x \in E$, will be called a *ϑ-homomorphism* of E and M. Define

$$\vartheta\text{-}\operatorname{Hom}(E, M) := \text{the set of all } \vartheta\text{-homomorphisms between } E \text{ and } M.$$

The notions of *ϑ-monomorphism*, *ϑ-epimorphism*, *ϑ-isomorphism*, the kernel $\ker\theta$ and the image $\operatorname{im}\theta$ are defined in the usual way.

Note. Let $\vartheta : \mathcal{R} \to \mathcal{S}$ be an anti-isomorphism. The fact that E is a left (resp. right) $\mathcal{R}$-module, M a right (resp. left) $\mathcal{S}$-module and $\theta : E \to M$ some ϑ-homomorphism between these modules will be in the sequel often shortly expressed as $\theta \in \vartheta\text{-}\operatorname{Hom}(E, M)$.

The next result might be called the first ϑ-isomorphism theorem; a straightforward proof is omitted.

Lemma 1.8. *Let $\vartheta : \mathcal{R} \to \mathcal{S}$ be an anti-isomorphism and $\theta \in \vartheta$-$\operatorname{Hom}(E, M)$. Then* $\ker\theta$ *is a left (resp. right) $\mathcal{R}$-module,* $\operatorname{im}\theta$ *is a right (resp. left) $\mathcal{S}$-module, and the induced map $\overline{\theta} : E/\ker\theta \to \operatorname{im}\theta$ is a ϑ-isomorphism.*

2. Antiautomorphic rings

In this section several remarks and basic results concerning antiautomorphic rings are given; for example, these rings satisfy some intrinsic symmetry properties such as Noetherian, Artinian and Goldie. We also point out at certain rings within this class.

The proposition below uses the ϑ-transfer on computing of the Krull dimension; it can actually be considered as a variation of a well known Lenagan's theorem [Ln] (see also [GW, Ch. 7]).

PROPOSITION 2.1. *Let* $\vartheta : \mathcal{R} \to \mathcal{S}$ *be an anti-isomorphism and* $\theta \in \vartheta$-$\mathrm{Hom}(E, M)$.

(i) *If* θ *is a* ϑ*-monomorphism, then* $\mathrm{Kdim}(E) \leq \mathrm{Kdim}(M)$.
(ii) *If* θ *is a* ϑ*-epimorphism, then* $\mathrm{Kdim}(M) \leq \mathrm{Kdim}(E)$.

PROOF. For later use let us state the following claim.

CLAIM. The map $p_\theta : \mathcal{L}(E) \to \mathcal{L}(M)$, $F \mapsto \theta(F)$, is monotone. Moreover, if θ is a ϑ-isomorphism, then p_θ is strictly monotone.

PROOF OF THE CLAIM. Using the definition of a ϑ-homomorphism it immediately follows that $\theta(F) \in \mathcal{L}(M)$ for every $F \in \mathcal{L}(E)$; that is, p_θ is well defined. Obviously, p_θ is monotone. For the second claim consider the map $p_{\theta^{-1}}$ and use the fact that p_θ and $p_{\theta^{-1}}$ are inverses of each other. □

(i) Use the above Claim and [GW, Ex. 13Q].

(ii) By Lemma 1.8, it follows that the map $\overline{\theta} : E/\ker\theta \to M$ is a ϑ-isomorphism. Hence, by (i), we have $\mathrm{Kdim}(E/\ker\theta) = \mathrm{Kdim}(M)$, which together with the fact $\mathrm{Kdim}(E/\ker\theta) \leq \mathrm{Kdim}(E)$ proves the claim. □

The following well known corollary is now a direct consequence of Remark 1.3(1) and the above proposition. (Of course, we could formulate this corollary in a simpler equivalent form by stating that every antiautomorphic ring satisfies the Krull symmetry, but we wanted to emphasize again the role of the set of all proper ideals in a ring; cf. Proposition 2.5 and its proof.)

COROLLARY 2.2. *Let* $\mathcal{R}$ *be an antiautomorphic ring, and suppose that* I *is a* ϑ*-ideal for some* $\vartheta \in \mathrm{aAut}\,\mathcal{R}$*. Denote by* $\mathcal{Q}$ *the factor ring* $\mathcal{R}/I$*. Then* $\mathrm{Kdim}({}_{\mathcal{Q}}\mathcal{Q}) = \mathrm{Kdim}(\mathcal{Q}_{\mathcal{Q}})$.

REMARK 2.3. (1) As is well known, a lot of symmetry properties fail when only one-sided Noetherian rings are considered. On the other hand, antiautomorphic rings behave well with respect to various symmetry properties. It is worthy to note that this is somehow in accordance with the following simple claim.

CLAIM. An antiautomorphic ring is left Noetherian (resp. Artinian, Goldie) if and only if it is right Noetherian (resp. Artinian, Goldie).

(2) Concerning the above corollary and Lemma 1.4, let us say that the precise value of the Krull dimension of $\mathcal{A}[G]$, for a Noetherian ring $\mathcal{A}$ and a polycyclic by finite group G, has been computed by Smith [S].

For completeness and later use we include the next interesting, and easy fact which is mostly well-known; at least to those working with involutions.

CLAIM 2.4. Let $\mathcal{A}$ be a ring with antiautomorphism (resp. involution) α. Then the ring of all upper (or lower) triangular matrices $T_n(\mathcal{A})$ is antiautomorphic (resp. involution), for every $n \in \mathbb{N}$.

PROOF. First note that the full matrix ring $M_n(\mathcal{A})$ is antiautomorphic. More precisely, the map $\vartheta(A) := \alpha(A)^t$, for $A \in M_n(\mathcal{A})$, where "t" denotes the ordinary transpose, defines an antiautomorphism. Here for a matrix $A = [a_{ij}]$ the new matrix $[\alpha(a_{ij})]$ is written in short as $\alpha(A)$. Next, let $E_{p,q}$, for $1 \le p, q \le n$, denote the standard matrix having 1 in the place (p,q) and 0 elsewhere. Define the matrix $S := \sum_{i=1}^{n} E_{i,n+1-i}$, and then define the map on matrices

$$\vartheta(A) := \alpha(SAS)^t \qquad \text{for } A \in M_n(\mathcal{A}).$$

Now it is straightforward to check that $\vartheta(aE_{i,j}) = \alpha(a)E_{n+1-j,n+1-i}$ for $a \in \mathcal{A}$. In particular, this means that ϑ maps $T_n(\mathcal{A})$ onto itself. Further, using the obtained equality one can see at once that ϑ is an antihomomorphism. Hence the claim follows. □

Any factor of the enveloping algebra of a finite-dimensional solvable Lie algebra over a field of characteristic zero satisfies the Krull symmetry (see [H]). The following interesting fact is the first step related to the question whether the same holds for semisimple Lie algebras as well (cf. [Lv]).

PROPOSITION 2.5. *Any factor of the enveloping algebra of the Lie algebra* $\mathfrak{sl}(2)$ *over a field of characteristic zero is an involution algebra, and so in particular it satisfies the Krull symmetry.*

PROOF. Let $\mathbb{F}$ be an algebraically closed field of characteristic zero, and $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{F}$. By a famous Duflo's result we know that there exists an involution τ of the enveloping algebra $U(\mathfrak{g})$ such that every semiprime ideal of $U(\mathfrak{g})$ is a τ-ideal (see [Du, Cor. 2]); for example, let τ be the Chevalley involution defined for some Chevalley basis of $\mathfrak{g}$. Hence, given a semiprime ideal I of $U(\mathfrak{g})$, the map τ_I, $\tau_I(u+I) := \tau(u) + I$ for $u \in U(\mathfrak{g})$, defines an involution of the factor algebra $U(\mathfrak{g})/I$ (see Remark 1.3(1)); thus, in particular, this algebra satisfies the Krull symmetry.

Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{F})$. Then, to prove the proposition, one just needs that now moreover every ideal of $U(\mathfrak{g})$ is a τ-ideal. The latter is an easy consequence of well understanding of the set of all ideals in $U(\mathfrak{g})$ (see, e.g., [C]; cf. also [Š2, Sect. 3]); note that here $\mathbb{F}$ is not necessarily algebraically closed. □

3. ON THE SYMMETRY OF PRIMITIVITY

In this section we prove the theorem stated in the Introduction. We also present two instructive examples of involution algebras, and study the action of their involutions on the corresponding prime spectra. (This will suggest some further hints how to deal with the problem of symmetry of primitivity within the class of semiprimitive Noetherian involution rings; see Remark 3.13.)

Recall the following terminology. We say that a prime ideal $\mathcal{P}$ of a ring $\mathcal{R}$ satisfies the symmetry of primitivity, or (SP) in short, if it holds that $\mathcal{P}$ is left primitive if and only if it is right primitive; also, we will say that $\mathcal{R}$ satisfies (SP) if each of its primes satisfies this property. A ring $\mathcal{R}$ is semiprimitive if $J(\mathcal{R}) = 0$.

We begin with proposition which is a straightforward generalization of [B, Ch. I, §8, Thm. 5]; cf. [Š1, Sect. 2]. For completeness we provide an argument; for that we first need the following two obvious lemmas.

LEMMA 3.1. *Let G, H be groups and $f : G \to H$ an epimorphism of groups. Suppose that G_1 is a subgroup of G containing the kernel* $\ker f$*. Then* $f^{-1}(f(G_1)) = G_1$.

LEMMA 3.2. *Let $\vartheta : \mathcal{R} \to \mathcal{S}$ be an epimorphism (resp. antiepimorphism) of rings.*

(i) *If I_1, I_2 are ideals of $\mathcal{R}$, then $\vartheta(I_1)\vartheta(I_2)$ is equal to $\vartheta(I_1 I_2)$ (resp. $\vartheta(I_2 I_1)$).*
(ii) *If J_1, J_2 are ideals of $\mathcal{S}$, then $\vartheta^{-1}(J_1)\vartheta^{-1}(J_2)$ is included in $\vartheta^{-1}(J_1 J_2)$ (resp. $\vartheta^{-1}(J_2 J_1)$).*

PROPOSITION 3.3. *Let $\vartheta : \mathcal{R} \to \mathcal{S}$ be either an antiepimorphism or an epimorphism of rings.*

(i) *If I is a prime (resp. semiprime) ideal of $\mathcal{R}$ containing the kernel* $\ker\vartheta$*, then $\vartheta(I)$ is a prime (resp. semiprime) ideal of $\mathcal{S}$.*
(ii) *If J is a prime (resp. semiprime) ideal of $\mathcal{S}$, then $\vartheta^{-1}(J)$ is a prime (resp. semiprime) ideal of $\mathcal{R}$.*
(iii) *The maps $\psi_1 : I \mapsto \vartheta(I)$ and $\psi_2 : J \mapsto \vartheta^{-1}(J)$ from the set of ideals $\{I \in \operatorname{Id}\mathcal{R} \mid \ker\vartheta \subseteq I\}$ (resp. $\{I \in \operatorname{Spec}\mathcal{R} \mid \ker\vartheta \subseteq I\}$) to the set of ideals $\operatorname{Id}\mathcal{S}$ (resp. $\operatorname{Spec}\mathcal{S}$) and conversely, respectively, are inverses of each other.*

PROOF. We will assume that ϑ is an epimorphism.

(ii) Define $I := \vartheta^{-1}(J)$, and let I_1, I_2 be ideals of $\mathcal{R}$ such that $I_1 I_2 \subseteq I$ (see Lemma 1.1). Consider the ideals $I_i' := I_i + \ker\vartheta$ for $i = 1, 2$. Then $I_1' I_2' \subseteq I$, and hence $\vartheta(I_1')\vartheta(I_2') \subseteq J$ (by Lemma 3.2(i)). Since J is prime we have $\vartheta(I_{i_0}') \subseteq J$ for $i_0 = 1$ or 2, and thus $I_{i_0}' \subseteq I$ (use Lemma 3.1), what we had to see.

(i) Similar to (ii) (use Lemma 3.2(ii)).

(iii) By (i) and (ii), we see that ψ_1 and ψ_2 are well defined. It remains to note that, by Lemma 3.1 and a well known fact that for any sets X, Y and a surjective map $f : X \to Y$ we have $f(f^{-1}(S)) = S$ for every $S \subseteq Y$, we have $(\psi_2 \circ \psi_1)(I) = I$ and $(\psi_1 \circ \psi_2)(J) = J$, respectively. ∎

For what follows it will be convenient to have the following result which relates the ϑ-transfer and localization; a straightforward proof is omitted. When $\mathcal{R}$ is a ring and $\Sigma \subseteq \mathcal{R}$ a left (resp. right) denominator set, we denote the left (resp. right) localization of $\mathcal{R}$ at Σ by ${}_{\Sigma}\mathcal{R}$ (resp. $\mathcal{R}_{\Sigma}$). The elements of ${}_{\Sigma}\mathcal{R}$ (resp. $\mathcal{R}_{\Sigma}$) are written as $\sigma^{-1}r$ (resp. $r\sigma^{-1}$), for $\sigma \in \Sigma$ and $r \in \mathcal{R}$.

Lemma 3.4. *Let $\vartheta : \mathcal{R} \to \mathcal{S}$ be an anti-isomorphism and Σ a multiplicatively closed subset of $\mathcal{R}$.*

(i) *The set $\vartheta(\Sigma)$ is a multiplicatively closed subset of $\mathcal{S}$. If Σ is a left (resp. right) Ore set, then $\vartheta(\Sigma)$ is a right (resp. left) Ore set. If Σ is a left (resp. right) denominator set, then $\vartheta(\Sigma)$ is a right (resp. left) denominator set. If Σ is equal to the set of regular elements of $\mathcal{R}$, then $\vartheta(\Sigma)$ is equal to the set of regular elements of $\mathcal{S}$.*

(ii) *Let Σ be a left denominator set. Then the map*

$$\begin{aligned} &{}_\Sigma\vartheta : {}_\Sigma\mathcal{R} \longrightarrow \mathcal{S}_{\vartheta(\Sigma)}, \\ &{}_\Sigma\vartheta(\sigma^{-1}r) := \vartheta(r)\vartheta(\sigma)^{-1} \qquad \textit{for } \sigma \in \Sigma, r \in \mathcal{R}, \end{aligned}$$

is an anti-isomorphism of the corresponding localized rings. (For a right denominator set Σ, the map ϑ_Σ is defined analogously.)

To prove our theorem we will use the next two easy lemmas. The first one deals with the ϑ-transfer of primitive ideals. The second lemma will provide the base of induction in the proof of the theorem (cf. Corollary 3.10).

Lemma 3.5. *Let $\vartheta : \mathcal{R} \to \mathcal{S}$ be an anti-isomorphism. If I is a left (resp. right) primitive ideal of $\mathcal{R}$, then $\vartheta(I)$ is a right (resp. left) primitive ideal of $\mathcal{S}$. Moreover, it holds $\vartheta(\mathrm{lPrim}\,\mathcal{R}) = \mathrm{rPrim}\,\mathcal{S}$ (resp. $\vartheta(\mathrm{rPrim}\,\mathcal{R}) = \mathrm{lPrim}\,\mathcal{S}$).*

Proof. Take a maximal left ideal L of $\mathcal{R}$ satisfying $I = \mathrm{lann}(\mathcal{R}/L)$. It immediately follows that $\vartheta(I) = \mathrm{rann}(\mathcal{S}/\vartheta(L))$. But we know, by Lemma 1.1(i), that $\vartheta(L)$ is a maximal right ideal of $\mathcal{S}$. Hence the lemma follows. ∎

Lemma 3.6. *Let $\mathcal{R}$ be a left (or right) Artinian ring. Then*

$$\mathrm{Spec}\,\mathcal{R} = \mathrm{lPrim}\,\mathcal{R} = \mathrm{rPrim}\,\mathcal{R} = \mathrm{Max}\,\mathcal{R}. \tag{3.1}$$

Proof. It is well known that $\mathrm{Max}\,\mathcal{R} \subseteq \mathrm{lPrim}\,\mathcal{R}, \mathrm{rPrim}\,\mathcal{R} \subseteq \mathrm{Spec}\,\mathcal{R}$; of course, this is true for any ring. Now the lemma follows by [GW, Prop. 3.19]. ∎

Remark 3.7. Recall that an ideal I of a ring $\mathcal{R}$ is semiprimitive if the ring $\mathcal{R}/I$ is semiprimitive; or equivalently said, if I is an intersection of primitive ideals. (Note that there is no ambiguity in the last definition of a semiprimitive ideal (cf. [GW, p.37]). More precisely, an ideal I is semiprimitive if and only if $I = \bigcap_{\mathcal{P}\in\Phi} \mathcal{P}$ for some subset $\Phi \subseteq \mathrm{lPrim}\,\mathcal{R} \cup \mathrm{rPrim}\,\mathcal{R}$. To see this set $J := J(\mathcal{R}/I)$ and decompose $\Phi = \Phi_l \cup \Phi_r$, where $\Phi_l \subseteq \mathrm{lPrim}\,\mathcal{R}$ and $\Phi_r \subseteq \mathrm{rPrim}\,\mathcal{R}$. Using Claim B given below (in the proof of our theorem) and [GW, Prop. 2.16], we have $J \subseteq \bigcap_{\mathcal{P}\in\Phi_l} \mathcal{P}/I$ and $J \subseteq \bigcap_{\mathcal{P}\in\Phi_r} \mathcal{P}/I$. Thus $J \subseteq \bigcap_{\mathcal{P}\in\Phi} \mathcal{P}/I$. It remains to note that $\bigcap_{\mathcal{P}\in\Phi} \mathcal{P}/I = (0)$.)

Proof of the Theorem. Let $\boldsymbol{\mathcal{C}} = \mathfrak{NoeInv}$, the class of Noetherian involution rings.

(iii)⇒(ii) To prove this implication we will proceed by induction on the Krull dimension $\mathrm{Kdim}(\mathcal{R})$, $\mathcal{R} \in \boldsymbol{\mathcal{C}}$. (Note that by Corollary 2.2 we have $\mathrm{Kdim}({}_{\mathcal{R}}\mathcal{R}) = \mathrm{Kdim}(\mathcal{R}_{\mathcal{R}})$ for $\mathcal{R} \in \boldsymbol{\mathcal{C}}$; the latter number is written in short as $\mathrm{Kdim}(\mathcal{R})$.)

Take some $\mathcal{R} \in \boldsymbol{\mathcal{C}}$ such that $\mathrm{Kdim}(\mathcal{R}) = 0$. This in fact means that $\mathcal{R}$ is Artinian and so we have (3.1); thus the base of the induction is proved.

For later use we state the following two auxiliary claims. (The second one presents a well known result in a form suitable for our purposes; for the reader's convenience we include an argument. It will be used below for reducing of the general ring case to the semiprimitive ring case, and for providing the induction step.)

Claim A. The Jacobson radical of an antiautomorphic ring $\mathcal{R}$ is a strongly proper ideal.

Proof of the Claim A. We have to show that $\vartheta(J(\mathcal{R})) = J(\mathcal{R})$ for every $\vartheta \in \mathrm{aAut}\,\mathcal{R}$; see Definition 1.2. For that, first note the following more general fact: If $\vartheta : \mathcal{R} \to \mathcal{S}$ is an anti-isomorphism and $\theta \in \vartheta$-$\mathrm{Hom}(E, M)$, then $\theta(\mathrm{rad}\,E) \subseteq \mathrm{rad}\,M$ with equality if θ is a ϑ-isomorphism. (One just has to adapt the proof of [P, §4.1, Lemma], by replacing the notions "homomorphism"↔"ϑ-homomorphism".) Now the claim follows by putting above that $\theta = \vartheta$; recall that $J(\mathcal{R}) := \mathrm{rad}_{\mathcal{R}}\,\mathcal{R} = \mathrm{rad}\,\mathcal{R}_{\mathcal{R}}$. □

Claim B. Let $\mathcal{P}$ be a prime ideal of a ring $\mathcal{R}$ and I an ideal of $\mathcal{R}$ contained in $\mathcal{P}$. Suppose that $\mathcal{P}$ is a left (resp. right) primitive, but not right (resp. left) primitive, ideal. Then $\mathcal{P}/I$ is a left (resp. right) primitive, but not right (resp. left) primitive, ideal of $\mathcal{R}/I$.

Proof of the Claim B. Let L be a maximal left ideal of $\mathcal{R}$ such that $\mathcal{P} = \mathrm{lann}(\mathcal{R}/L)$. Define the factor ring $\overline{\mathcal{R}} := \mathcal{R}/I$, and observe that $\overline{L}$ is a maximal left ideal of $\overline{\mathcal{R}}$. It is easy to check that $\overline{\mathcal{P}} = \mathrm{lann}(\overline{\mathcal{R}}/\overline{L})$, which proves that $\overline{\mathcal{P}}$ is a left primitive ideal of $\overline{\mathcal{R}}$.

Let us now show that $\overline{\mathcal{P}}$ is not right primitive. In order to see this assume to the contrary; that is, that there exists a maximal right ideal Δ of $\overline{\mathcal{R}}$ satisfying $\overline{\mathcal{P}} = \mathrm{rann}(\overline{\mathcal{R}}/\Delta)$. Then consider the maximal right ideal D of $\mathcal{R}$ such that $\overline{D} = \Delta$, and the right primitive ideal $\mathcal{Q} := \mathrm{rann}(\mathcal{R}/D)$ of $\mathcal{R}$. It immediately follows that $\mathcal{P} = \mathcal{Q}$; a contradiction. □

Suppose now that (ii) holds for every ring from $\boldsymbol{\mathcal{C}}$ with Krull dimension less than n, $n \in \mathbb{N}$. Then consider some $\mathcal{R} \in \boldsymbol{\mathcal{C}}$, $\mathrm{Kdim}(\mathcal{R}) = n$. Also suppose that there exists a prime $\mathcal{P}$ of $\mathcal{R}$ which is left primitive, but not right primitive (or vice versa). By the above claims we can assume with no loss of generality that $\mathcal{R}$ is semiprimitive. Hence, by (iii), it follows that $\mathcal{P}$ is not a minimal prime ideal. Now define the ideal

$$I := \mathcal{P} \cap \mathcal{P}^*.$$

Note that $I \neq \mathcal{P}$. [Indeed, otherwise it would follow that $\mathcal{P} = \mathcal{P}^*$. But then Lemma 3.5 implies that $\mathcal{P}$ is a right primitive ideal; a contradiction.]

Let the set of regular elements of $\mathcal{R}$ be denoted by $\Sigma(\mathcal{R})$. We claim that

$$I \cap \Sigma(\mathcal{R}) \neq \emptyset. \tag{3.2}$$

To show this first observe that $\mathcal{P} \cap \Sigma(\mathcal{R}) \neq \emptyset$, since $\mathcal{P}$ is not a minimal prime; see, e.g., [GW, Prop. 6.3]. Then take some regular $\sigma \in \mathcal{P}$. Obviously, the element σ^* is also regular (cf. Lemma 3.4(i)), and thus $\sigma\sigma^*$ is a regular element from I. This shows (3.2).

Finally, define the factor ring $\mathcal{R}/I$, and consider in it the left, but not right, primitive ideal $\mathcal{P}/I$. By (3.2) we know that $\mathrm{Kdim}(\mathcal{R}/I) < \mathrm{Kdim}(\mathcal{R})$ and so the induction hypothesis applies; see, e.g., [MR, 6.3.9 Lemma]. (Note that I is a *-ideal and so $\mathcal{R}/I \in \boldsymbol{\mathcal{C}}$.)

(iv)⇒(iii) Let $\mathcal{R} \in \boldsymbol{\mathcal{C}}$ be a semiprimitive ring with the set of its minimal primes $\{\mathcal{P}_1, \ldots, \mathcal{P}_n\}$, and suppose that some $\mathcal{P}_i$ is for example left primitive but not right primitive. Then $\mathcal{P}_i^* = \mathcal{P}_j$ for some $j \neq i$. Similarly as above, define $I := \mathcal{P}_i \cap \mathcal{P}_j$. Then I is a semiprimitive ideal (cf. Remark 3.7; note that $\mathcal{P}_j$ is right, but not left, primitive). Thus, $\mathcal{R}/I$ is a semiprimitive ring from $\boldsymbol{\mathcal{C}}$ with the set of minimal primes $\{\mathcal{P}_i/I, \mathcal{P}_j/I\}$, where both of these primes fail to satisfy (SP).

(i)⇔(ii) Take any $\mathcal{R} \in \mathfrak{Noe}$ and set $\Lambda := \mathcal{R} \oplus \mathcal{R}^{opp}$; $\mathcal{R}^{opp}$ is the opposite ring. Then $\Lambda \in \boldsymbol{\mathcal{C}}$; see Remark 3.9 below. Now, if (ii) would hold, by Lemma 3.8 below we would have that $\mathcal{R}$ satisfies (SP); therefore the stated equivalence follows.

Since (i)⇒(iv) is obvious, the theorem is proved. ∎

Lemma 3.8. *Let $\mathcal{R}, \mathcal{S}$ be rings, and Λ be the direct sum $\mathcal{R} \oplus \mathcal{S}$. Suppose that Λ satisfies* (SP). *Then both $\mathcal{R}$ and $\mathcal{S}$ satisfy* (SP).

Proof. We will show that, for example, $\mathcal{R}$ satisfies (SP).

So, let $\mathcal{P} \in \mathrm{lPrim}\,\mathcal{R}$ and L a maximal left ideal of $\mathcal{R}$ such that $\mathcal{P} = \mathrm{lann}(\mathcal{R}/L)$. Set $\hat{L} := L \oplus \mathcal{S}$; then $\hat{L}$ is a maximal left ideal of Λ. Further, define $\hat{\mathcal{P}} := \mathrm{lann}(\Lambda/\hat{L})$; $\hat{\mathcal{P}}$ is a left primitive ideal of Λ. It is easy to see that

$$\hat{\mathcal{P}} = \mathcal{P} \oplus \mathcal{S}. \tag{3.3}$$

By the supposition of the lemma we have $\hat{\mathcal{P}} \in \mathrm{rPrim}\,\Lambda$, and therefore there exists a maximal right ideal $\hat{D}$ of Λ such that $\hat{\mathcal{P}} = \mathrm{rann}(\Lambda/\hat{D})$. Obviously, we can write $\hat{D} = D \oplus \mathcal{S}$ for some subset D of $\mathcal{R}$. But it is clear that D is a right ideal of $\mathcal{R}$ and moreover it is maximal. Then consider the right primitive ideal $\mathcal{Q} := \mathrm{rann}(\mathcal{R}/D)$. Proceeding analogously as in proving (3.3) (one only has to replace the words "left" and "right"), we obtain $\hat{\mathcal{P}} = \mathcal{Q} \oplus \mathcal{S}$. Hence we conclude that $\mathcal{P} = \mathcal{Q}$. This finishes the proof that $\mathcal{R}$ satisfies (SP). ∎

Remark 3.9. Note that if $\mathcal{R}, \mathcal{S}$ and Λ are as in the previous lemma, and moreover $\mathcal{R}$ and $\mathcal{S}$ are anti-isomorphic rings, then Λ is an involution ring. More precisely, if $\vartheta : \mathcal{R} \to \mathcal{S}$ is an anti-isomorphism, then the map α, $\alpha(r,s) := (\vartheta^{-1}(s), \vartheta(r))$ for $r \in \mathcal{R}$ and $s \in \mathcal{S}$, defines an involution of Λ.

Corollary 3.10. *Let $\mathcal{R}$ be a prime Noetherian antiautomorphic ring such that* $\mathrm{Kdim}(\mathcal{R}) \leq 1$ *(see the beginning of the proof of our theorem). Then $\mathcal{R}$ satisfies* (SP).

Proof. Suppose that $\mathcal{P}$ is a left, but not right, primitive ideal of $\mathcal{R}$; note that $\mathcal{P} \neq (0)$. Then take a maximal right ideal D of $\mathcal{R}$ containing $\mathcal{P}$. The ideal $\mathcal{Q} := \mathrm{rann}(\mathcal{R}/D)$ is right primitive and so it strictly contains $\mathcal{P}$. Hence the classical Krull dimension of $\mathcal{R}$ is greater than 1, which further implies that $\mathrm{Kdim}(\mathcal{R}) > 1$; a contradiction. ∎

Now we present the two announced examples in which we investigate the action of involutions of certain (finite-dimensional) matrix algebras on their prime spectra.

Example 3.11. The following notation will be used in this example. Given an arbitrary ring $\mathcal{X}$, by $M_n(\mathcal{X})$ we denote the matrix ring of $n \times n$ matrices with coefficients from $\mathcal{X}$. The identity of $M_n(\mathcal{X})$ will be denoted by $\imath$; it will be clear from the context which $\mathcal{X}$ and n we mean. The transpose of $M_n(\mathcal{X})$ will be denoted by $\tau_{\mathcal{X}}$, or simply by the superscript "t". For a subset $S \subseteq M_n(\mathcal{X})$ we write $S^t := \{m^t \mid m \in S\}$.

Let $\mathbb{F}$ be a field and $\mathcal{U}$ (resp. $\mathcal{L}$) the subring of $M_2(\mathbb{F})$ consisting of all upper (resp. lower) triangular matrices. Consider the $\mathbb{F}$-algebra

$$\mathcal{R} := M_2(\mathcal{U}) \oplus M_2(\mathcal{L}).$$

First we will describe the set of all ideals of $\mathcal{R}$. For this it is sufficient to determine the sets $\mathrm{Id}\, M_2(\mathcal{X})$ for $\mathcal{X} = \mathcal{U}, \mathcal{L}$; since one can immediately see that

$$I = I \cap M_2(\mathcal{U}) \oplus I \cap M_2(\mathcal{L}) \tag{3.4}$$

for every ideal I of $\mathcal{R}$. We will consider only the subring $M_2(\mathcal{U})$; then the conclusions for $M_2(\mathcal{L})$ are simply obtained by "transposing". Put $\boldsymbol{p}_1 := \left[\begin{smallmatrix} \mathbb{F} & \mathbb{F} \\ 0 & 0 \end{smallmatrix}\right]$, $\boldsymbol{p}_2 := \left[\begin{smallmatrix} 0 & \mathbb{F} \\ 0 & \mathbb{F} \end{smallmatrix}\right]$, and $\boldsymbol{q} := \boldsymbol{p}_1 \cap \boldsymbol{p}_2$. Then

$$\mathrm{Id}\,\mathcal{U} = \{(0), \boldsymbol{q}, \boldsymbol{p}_1, \boldsymbol{p}_2, \mathcal{U}\},$$

wherefrom it is easy to see that

$$\mathrm{Id}\, M_2(\mathcal{U}) = \{M_2(A) \mid A \in \mathrm{Id}\,\mathcal{U}\}.$$

Hence, using (3.4), we have the description of the set $\mathrm{Id}\,\mathcal{R}$. In particular, by denoting $\mathcal{P}_i := M_2(\boldsymbol{p}_i) \oplus M_2(\mathcal{L})$ and $\mathcal{Q}_i := M_2(\mathcal{U}) \oplus M_2(\boldsymbol{p}_i^t)$ for $i = 1, 2$, it follows that (cf. Lemma 3.6)

$$\mathrm{Spec}\,\mathcal{R} = \{\mathcal{P}_1, \mathcal{P}_2, \mathcal{Q}_1, \mathcal{Q}_2\}. \tag{3.5}$$

Now we will deal with involutions of $\mathcal{R}$.

Claim. If $\vartheta \in \operatorname{aAut}\mathcal{R}$, then either ϑ leaves invariant both $M_2(\mathcal{U})$ and $M_2(\mathcal{L})$, or ϑ interchanges these subrings of $\mathcal{R}$.

Proof of the Claim. To prove this claim suppose to the contrary. Also observe that $M_2(\mathcal{U})$ is a 12-dimensional ideal of $\mathcal{R}$. Therefore, using Lemma 1.1, $\vartheta(M_2(\mathcal{U}))$ must be equal to some of the following four ideals (use the above description of $\operatorname{Id}\mathcal{R}$):

$$M_2(\boldsymbol{p}_i) \oplus M_2(\boldsymbol{q}^t), \quad M_2(\boldsymbol{q}) \oplus M_2(\boldsymbol{p}_i^t), \qquad \text{for } i = 1, 2.$$

Assume the first possibility; the other three can be considered analogously. Let $\vartheta(\imath) = (a, b)$ for some $a \in M_2(\boldsymbol{p}_1)$, $b \in M_2(\boldsymbol{q}^t)$. Since $\vartheta(\imath) = \vartheta(\imath^2) = (a^2, 0)$, we conclude that $b = 0$. Hence

$$\vartheta(m) = \vartheta(m\imath) = \vartheta(\imath)\vartheta(m) \in M_2(\boldsymbol{p}_1) \qquad \text{for } m \in M_2(\mathcal{U}).$$

This is a contradiction, which proves the claim. ∎

Let us emphasize that the construction which follows was influenced by the result of Montgomery [Mo, Thm. 1] which describes $\mathbb{F}$-involutions of $M_2(\mathbb{F})$, for $\operatorname{char}(\mathbb{F}) \neq 2$.

Given an arbitrary ring $\mathcal{X}$, we consider the automorphisms $\tau_{\mathcal{X}}$ and $\sigma_{\mathcal{X}}$ of the $\mathcal{X}$-module $M_2(\mathcal{X})$, where $\sigma_{\mathcal{X}}$ is the "symplectic" map (cf. [Mo], and also [Š3, Sect. 3])

$$\sigma_{\mathcal{X}}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) := \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

In general, $\tau_{\mathcal{X}}$ and $\sigma_{\mathcal{X}}$ need not be involutions (cf. Claim 2.4). But if $\alpha \in \operatorname{Inv}\mathcal{X}$, then the map $\gamma \times \alpha$, which acts as

$$\gamma \times \alpha\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) := \gamma\left(\begin{bmatrix} \alpha(a) & \alpha(b) \\ \alpha(c) & \alpha(d) \end{bmatrix}\right),$$

defines an involution of $M_2(\mathcal{X})$ for $\gamma = \tau_{\mathcal{X}}, \sigma_{\mathcal{X}}$. Using this observation and the above Claim, we will construct several distinguished involutions of $\mathcal{R}$. Let it be said that for easier notation we write τ both for the anti-isomorphism $\tau_{\mathbb{F}\,|\mathcal{U}} : \mathcal{U} \to \mathcal{L}$ and for its inverse. Also, we write σ both for the involutions $\sigma_{\mathbb{F}\,|\mathcal{U}}$ and $\sigma_{\mathbb{F}\,|\mathcal{L}}$ of $\mathcal{U}$ and $\mathcal{L}$, respectively. Now we consider the following two pairs of mutually nonequal involutions:

$$\sigma_{\mathcal{U}} \times \sigma, \tau_{\mathcal{U}} \times \sigma \in M_2(\mathcal{U})$$

and

$$\sigma_{\mathcal{L}} \times \sigma, \tau_{\mathcal{L}} \times \sigma \in M_2(\mathcal{L}).$$

Also, we consider two pairs of anti-isomorphisms which are inverses of each other:

$$\sigma_{\mathcal{L}} \times \tau : M_2(\mathcal{U}) \rightleftharpoons M_2(\mathcal{L}) : \sigma_{\mathcal{U}} \times \tau$$

and

$$\tau_{\mathcal{L}} \times \tau : M_2(\mathcal{U}) \rightleftharpoons M_2(\mathcal{L}) : \tau_{\mathcal{U}} \times \tau.$$

Note that $\tau_{\mathcal{L}} \times \tau$ and $\tau_{\mathcal{U}} \times \tau$ act as the transpose in $M_4(\mathbb{F})$. (The involution δ_2 defined below acts as $\delta_2(m,n) = (n^t, m^t)$.)

NOTATION. In general, for antiautomorphisms α of $M_2(\mathcal{U})$ and β of $M_2(\mathcal{L})$ we define the antiautomorphism (α,β) of $\mathcal{R}$ by $(\alpha,\beta)(m,n) := (\alpha(m),\beta(n))$, for $(m,n) \in \mathcal{R}$. Also, for anti-isomorphisms $\alpha : M_2(\mathcal{U}) \rightleftharpoons M_2(\mathcal{L}) : \beta$ we define the antiautomorphism $\overline{(\alpha,\beta)}$ of $\mathcal{R}$ by $\overline{(\alpha,\beta)}(m,n) := (\beta(n),\alpha(m))$, for $(m,n) \in \mathcal{R}$.

Taking into account all that we said, we list the following involutions of $\mathcal{R}$:

$$\begin{aligned}
\vartheta_1 &:= (\sigma_{\mathcal{U}} \times \sigma, \sigma_{\mathcal{L}} \times \sigma),\\
\vartheta_2 &:= (\sigma_{\mathcal{U}} \times \sigma, \tau_{\mathcal{L}} \times \sigma),\\
\vartheta_3 &:= (\tau_{\mathcal{U}} \times \sigma, \sigma_{\mathcal{L}} \times \sigma),\\
\vartheta_4 &:= (\tau_{\mathcal{U}} \times \sigma, \tau_{\mathcal{L}} \times \sigma),\\
\delta_1 &:= \overline{(\sigma_{\mathcal{L}} \times \tau, \sigma_{\mathcal{U}} \times \tau)},\\
\delta_2 &:= \overline{(\tau_{\mathcal{L}} \times \tau, \tau_{\mathcal{U}} \times \tau)}.
\end{aligned}$$

The reader can now easily verify that none of the above six involutions leave the prime ideal $\mathcal{P}_1$ (see (3.5)) stable. More precisely, it holds $\vartheta_i(\mathcal{P}_1) = \mathcal{P}_2$ and $\delta_j(\mathcal{P}_1) = \mathcal{Q}_1$ for any i and j.

It is also interesting to note the following. For the ideals $\overline{\mathcal{P}_i}$ and $\overline{\mathcal{Q}_j}$ of $\overline{\mathcal{R}} := \mathcal{R}/J(\mathcal{R})$ (obviously, the Jacobson radical $J(\mathcal{R})$ equals $M_2(\boldsymbol{q}) \oplus M_2(\boldsymbol{q}^t)$), and involutions $\overline{\vartheta_i}$ and $\overline{\delta_j}$ of $\overline{\mathcal{R}}$, we have that $\overline{\mathcal{P}_1}$ is not stable for any of $\overline{\vartheta_i}$ and $\overline{\delta_j}$. But at the same time, since $\overline{\mathcal{R}} \simeq \bigoplus 4M_2(\mathbb{F})$ (four copies of $M_2(\mathbb{F})$), it is clear that there exist a lot of involutions of $\overline{\mathcal{R}}$ which leave stable every prime of $\overline{\mathcal{R}}$ (e.g., consider the pull-back of the "four times transpose" of $\bigoplus 4M_2(\mathbb{F})$).

Let it be said that when $\mathbb{F}$ is for example equal to the field $\mathbb{Q}$ of rational numbers, then none of the involutions of $\overline{\mathcal{R}}$ leaving the prime ideals of $\overline{\mathcal{R}}$ stable does arise as the $\overline{\vartheta}$ for some $\vartheta \in \operatorname{Inv}\mathcal{R}$. A more easier example concerning this phenomenon is the following one.

EXAMPLE 3.12. Let $\mathbb{F} = \mathbb{Q}$ and $\mathcal{U}$, $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, $\boldsymbol{q}$ as in the previous example. Then $J(\mathcal{U}) = \boldsymbol{q}$ and so the ring $\overline{\mathcal{U}} := \mathcal{U}/J(\mathcal{U})$ is isomorphic to $\mathbb{F} \oplus \mathbb{F}$; we identify these rings in the sequel. Further, it is immediate that any $\alpha \in \operatorname{Inv}\mathcal{U}$ has the form

$$\alpha\left(\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}\right) := \begin{bmatrix} z & b(x-z)+cy \\ 0 & x \end{bmatrix}$$

for some $b, c \in \mathbb{F}$, $c \neq 0$. Hence the involution $\overline{\alpha}$ of $\overline{\mathcal{U}}$ acts as $\overline{\alpha}(x,z) = (z,x)$, and so $\overline{\alpha}(\overline{\boldsymbol{p}_1}) = \overline{\boldsymbol{p}_2}$; that is, $\overline{\alpha}$ interchanges $\overline{\boldsymbol{p}_1}$ and $\overline{\boldsymbol{p}_2}$. At the same time the identity map $1_{\overline{\mathcal{U}}}$ leaves stable both $\overline{\boldsymbol{p}_1}$ and $\overline{\boldsymbol{p}_2}$. (Note that $\overline{\boldsymbol{p}_i} \simeq \mathbb{F}$ for $i = 1, 2$.)

We will finish the paper with the following remark.

REMARK 3.13. Let $\mathcal{R}$ be a semiprimitive Noetherian involution ring. By Goldie's theorem it follows the existence of the quotient ring $Q(\mathcal{R}) := \mathcal{R}_{\Sigma(\mathcal{R})}$ ($\Sigma(\mathcal{R})$ denotes the set of all regular elements of $\mathcal{R}$), and by Lemma 3.4 we know that this ring is also an involution ring. Further, $\mathcal{R}$ is embedded in $Q(\mathcal{R})$. Now we can state the following simple claim which is motivated by the above considerations. In particular, it suggests a possibility to deal with the problem of symmetry of primitivity via passing to the quotient ring of a given ring.

CLAIM. Let $\mathcal{R}$ be a semiprimitive Noetherian involution ring. Consider the following three statements.

(I) $\mathcal{R}$ satisfies (SP).

(II) For every minimal prime $\mathcal{P}$ of $\mathcal{R}$, there exists an involution $\alpha_{\mathcal{P}}$ of $\mathcal{R}$ (in general, $\alpha_{\mathcal{P}}$ depends of $\mathcal{P}$) such that $\mathcal{P}$ is an $\alpha_{\mathcal{P}}$-ideal.

(III) There exists an involution α of $Q(\mathcal{R})$ which leaves stable both the ring $\mathcal{R}$ and every prime ideal of $Q(\mathcal{R})$.

Then (III)$\Rightarrow$(II)$\Rightarrow$(I).

PROOF OF THE CLAIM. By Proposition 3.3 and our theorem, it is clear that (II)$\Rightarrow$(I).

Given a minimal prime $\mathcal{P}$ of $\mathcal{R}$, the localization $\mathcal{P}_{\Sigma(\mathcal{R})}$ is a prime of $Q(\mathcal{R})$ and it holds $\mathcal{P}_{\Sigma(\mathcal{R})} \cap \mathcal{R} = \mathcal{P}$. Hence the restriction $\alpha_{|\mathcal{R}}$ leaves stable every such $\mathcal{P}$. Thus (III)$\Rightarrow$(II). ∎

Note that if $\mathcal{R}$ is moreover Artinian, then by the Artin-Wedderburn structure theorem it follows that $Q(\mathcal{R}) = \mathcal{R}$. So, in particular, we have $Q(\overline{\mathcal{R}}) = \overline{\mathcal{R}}$ and $Q(\overline{\mathcal{U}}) = \overline{\mathcal{U}}$ for the rings $\mathcal{R}$ and $\mathcal{U}$ from Examples 3.11 and 3.12, respectively. Thus, as it was in fact already noted, now we have the existence of a "good" involution α of $Q(\overline{\mathcal{R}})$ (resp. $Q(\overline{\mathcal{U}})$) such that (III) is fulfilled. (Also, there is no such α for $Q(\mathcal{R})$ (resp. $Q(\mathcal{U})$); therefore we cannot drop the assumption, in the claim, that the ring $\mathcal{R}$ is semiprimitive.)

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