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THE SMOOTH IRREDUCIBLE REPRESENTATIONS OF $U\,(2)$

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Abstract. In this paper we parametrize all smooth irreducible representations of $U\,(2)$, the compact unitary group in two variables.

1. Introduction and Notation

Let E/F be a quadratic extension of local fields. If $(W_1, (,)_1)$ is a Hermitian space over E and $(W_2, (,)_2)$ is a skew-Hermitian space over E, the unitary groups $G_1 = G\,(W_1)$ and $G_2 = G\,(W_2)$ form a reductive dual pair in $Sp\,(W)$, where $W = W_1 \otimes_E W_2$ has the symplectic form $\langle , \rangle = \frac{1}{2} Tr_{E/F} \left(\overline{(,)_1} \otimes (,)_2 \right)$ over F and $x \to \bar{x}$ is the non trivial element of Galois group $\Gamma = \Gamma\,(E/F)$. We consider the special case in which $\dim_E W_1 = 1$, and $\dim_E W_2 = 2$. In this case, $G_1 = U\,(1)$, and $G_2 = U\,(2)$ are unitary groups in one and two variables, respectively. The structure of $U\,(1)$ is simple and its representations are all one dimensional and easy to find. The structure of $U\,(2)$ is more complicated and is the semidirect-product of two compact groups. Since our group is compact, all its smooth irreducible representations are finite dimensional. Although new methods and results for finding the representations of p-adic groups were published recently ([12]), in this paper we will be using the method used by Manderscheid ([5]) to construct the representations of SL_2, to parametrize the representations of $U\,(2)$. Our motivation for finding representations of $U\,(2)$, in addition to its own interest, is that they are needed to parametrize the theta correspondence for the reductive dual pair $(U\,(1)\,, U\,(2))$.

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This paper consists of four sections. The first section is devoted to introduction and notation. In the second section we describe the structure of $U(2)$. In the third section we find all representations of $U(2)$ whose dimensions are bigger than one. The final section consists of the description of all characters (one-dimensional representations) of $U(2)$. Besides characters, other finite-dimensional representations will be described as fully induced representations from characters of certain subgroups. This is given in Theorem 3.29. In Theorems 4.5 and 4.6, we will formalize all the results obtained in sections 3 and 4.

To begin, let F be a local p-$field$, where p is an odd prime integer number and let ϖ be the uniformizer of it. Let D be the unique (up to an isomorphism) quaternion division algebra over F. Let π be a uniformizer of D such that $\pi^2 = \varpi$. Also let ϵ be a unit in D such that ϵ^2 is a unit in F and $\epsilon\pi = -\pi\epsilon$. For any $x \in D$ we can write:

$$x = x_1 + x_2\epsilon + x_3\pi + x_4\epsilon\pi.$$

where x_1, x_2, x_3 and x_4 are in F. For any $x \in D$, $x = x_1 + x_2\epsilon + x_3\pi + x_4\epsilon\pi$, define $\bar{x}$ as follows:

$$\bar{x} = x_1 - x_2\epsilon - x_3\pi - x_4\epsilon\pi.$$

Then $x \to \bar{x}$ is an involution on D whose restriction to E is the non trivial element of Galois group $\Gamma = \Gamma(E/F)$. We denote by $\nu = \nu_{D/F}$ the reduced norm map and is defined as:

$$\nu(x) = x\bar{x},$$

and by $Tr = Tr_{D/F}$ the reduced trace map and is defined as:

$$Tr(x) = x + \bar{x}.$$

For $x \in E$, $Tr_{E/F}$ and $\nu_{E/F}$ are defined similarly:

$$Tr_{E/F}(x) = x + \bar{x},$$

and

$$\nu(x) = \nu_{E/F}(x) = x\bar{x}.$$

Also let $O = O_F$ be the ring of integers of F with maximal ideal $P = P_F$ generated by ϖ, and $k = k_F$ the residue class field O/P with cardinality q. Let $\upsilon_D(x)$ be the order of an element, x, in D. So for any x in D we can write $x = u\pi^{\upsilon_D(x)}$ for some unit u in D. Let O_D, P_D, and $\mathbf{k} = k_D = O_D/P_D$ be the ring of integers, maximal ideal generated by π, and residue class field of D, respectively. We denote by D° the set of all traceless elements in D and $O_{D^\circ} = O_D \cap D^\circ$. For any integer r, we define P_D^r as follows:

$$P_D^r = O_D\pi^r = \{a\pi^r \mid a \in O_D\}.$$

Also we define $P^r_{D^\circ}$ as $P^r_{D^\circ} = O_{D^\circ} \cap P^r_D$. The norm one elements group of D, denoted by D^1 is defined as follows:

$$D^1 = \{x \in D \mid \nu(x) = 1\}.$$

For any positive integer r, D^1_r is the following set:

$$D^1_r = \left\{x \in D^1 \mid x = 1 + a\pi^r, \text{ for some } a \in O_D\right\}.$$

Let E/F be a quadratic extension contained in D. Let γ be a generator of E/F , i.e., $E = F(\gamma)$. We can and will take $\gamma = \epsilon$ when E/F is unramified and $\gamma = \pi$ when E/F is ramified ([13]). Set $W_1 = E$ with the following Hermitian form:

$$(x, y)_1 = \frac{1}{2} x\bar{y},\ x, y \in E,$$

and set $W_2 = D$ with the following skew-Hermitian form:

$$(x, y)_2 = \frac{1}{2} Tr_{D/E}(\gamma x \bar{y}),\ x, y \in D,$$

where $Tr_{D/E}$ is the trace map from D to E and is equal to the first coordinate of x in E, i.e.,

$$Tr_{D/E}(x) = a, \text{ for any } x \in D, x = (a, b), \text{ and } a, b \in E.$$

One can show that:

$$Tr = Tr_{E/F} \circ Tr_{D/E}.$$

Now set $W = W_1 \otimes_E W_2$. Then W is isomorphic to $W_2 = D$ via $a \otimes x \longmapsto ax$ (its inverse is $x \longmapsto 1 \otimes x$) with the following skew-Hermitian form:

$$\langle x, y \rangle = \frac{1}{2} Tr(\gamma x \bar{y}),\ x, y \in D \cong W.$$

For any $x \in D$, let $x = (a, b)$, where a and b in E are coordinates of x in E.(note that D is a two dimensional vector space over E.) From here we get $\bar{x} = (\bar{a}, -b)$. We also denote by E^1 the norm one elements group of E, $E^1 = \{x \in E \mid \nu(x) = 1\}$.

2. Structure of $U(2)$

While the structure of $U(2)$ is well-known and is the semidirect-product of compact groups D^1 and E^1 ([2]), we are giving its detailed structure here in our notation.

Let all notations be as above and for simplicity for the rest of this paper we assume that E/F is unramified (for ramified case see Remarks 2.14 and 3.15). Thus we have $E = F(\epsilon)$ and $\mathcal{B} = \{1, \pi\}$ is a basis of D over E. Also note that the skew-Hermitian space $(W, \langle , \rangle)$ is anisotropic space, i.e., $\langle x, x \rangle = 0$ if and only if $x = 0$, and the matrix of the skew-Hermitian form $\langle , \rangle$ in this basis, $\mathcal{B}$, is:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -\pi^2 \end{pmatrix}.$$

THEOREM 2.1. *The group of isometries of* $(W, \langle , \rangle)$, G_2, *is given as follows:*

$$G_2 = \left\{ \begin{pmatrix} a & \lambda\pi^2\bar{c} \\ c & \lambda\bar{a} \end{pmatrix} \mid a \in E^\times, c \in E, \lambda \in E^1, \nu(a) - \nu(c)\pi^2 = 1 \right\}.$$

PROOF. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of G_2, where a, b, c and d are in E. Then we must prove that:

$$g^* A g = A. \tag{2.1}$$

where $g^* = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}$ and A is as in above. From equation (2.1) we also get:

$$A^{-1} = g A^{-1} g^*. \tag{2.2}$$

Now since $A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -\pi^{-2} \end{pmatrix}$, multiplying equation (2.2) by $-\pi^2$ we get

$$g \begin{pmatrix} -\pi^2 & 0 \\ 0 & 1 \end{pmatrix} g^* = \begin{pmatrix} -\pi^2 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.3}$$

From equations (2.1), (2.2) and(2.3) we get:

$$\begin{aligned} \nu(a) - \nu(c)\pi^2 &= 1, \\ \bar{a}b - \pi^2\bar{c}d &= 0, \\ \nu(b) - \nu(d)\pi^2 &= -\pi^2, \\ \pi^2\bar{c}a - \bar{d}b &= 0, \\ \nu(d) - \nu(c)\pi^2 &= 1, \\ \pi^2\nu(a) - \nu(b) &= \pi^2. \end{aligned}$$

The above conditions lead to $\nu(a) = \nu(d)$ and $\nu(b) = \nu(c)\pi^4$. From $\nu(a) = \nu(d)$ and equation $\nu(a) - \nu(c)\pi^2 = 1$ we deduce that $a \neq 0, d \neq 0$. Because if $a = d = 0$, then we must have $-\nu(c)\pi^2 = 1$, i.e $\nu(c\pi) = 1$, which is not true. See ([7]). From here and $\pi^2\bar{c}a - \bar{d}b = 0$ we get:

$$\frac{a}{\bar{d}} = \frac{d}{\bar{a}} = \lambda,$$

and

$$\frac{a}{\bar{d}} = \frac{b}{\bar{c}\pi^2} = \lambda, \text{ when } c \neq 0.$$

If $c = 0$, then $b = 0$ and $g = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ with $\nu(a) = \nu(d) = 1$. For $c \neq 0$ we have $d = \lambda\bar{a}$, $b = \lambda\pi^2\bar{c}$ and $\nu(\lambda) = 1$, $\nu(a) - \nu(c)\pi^2 = 1$. So

$$g = \begin{pmatrix} a & \lambda\pi^2\bar{c} \\ c & \lambda\bar{a} \end{pmatrix}.$$

i.e.,

$$G_2 = \left\{ \begin{pmatrix} a & \lambda\pi^2\bar{c} \\ c & \lambda\bar{a} \end{pmatrix} \mid a \in E^\times, c \in E, \lambda \in E^1, \nu(a) - \nu(c)\pi^2 = 1 \right\}.$$

□

PROPOSITION 2.2. *The group E^1 is isomorphic to a subgroup of G_2 and D^1 is isomorphic to a normal subgroup of G_2.*

PROOF. Define $f : E^1 \to G_2$ by

$$f(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, \lambda \in E^1.$$

Then one can check that f is a one to one homomorphism from E^1 to G_2. For the other part define $f : D^1 \to G_2$ by

$$f(x) = \begin{pmatrix} a & \pi^2 b \\ \bar{b} & \bar{a} \end{pmatrix},$$

where $x = a + b\pi$, $a, b \in E$ and $\nu(x) = 1$. Then for $x = a + b\pi$ and $y = c + d\pi$ in D^1 we have:

$$\begin{aligned} f(xy) &= f\left(ac + b\bar{d}\pi^2 + (ad + b\bar{c})\pi\right) \\ &= \begin{pmatrix} ac + b\bar{d}\pi^2 & (ad + b\bar{c})\pi^2 \\ \overline{(ad + b\bar{c})} & \overline{ac + b\bar{d}\pi^2} \end{pmatrix} \\ &= \begin{pmatrix} a & \pi^2 b \\ \bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} c & \pi^2 d \\ \bar{d} & \bar{c} \end{pmatrix} \\ &= f(x) f(y). \end{aligned}$$

So f is a homomorphism. One can check that f is one to one. So we can identify D^1 with its image, $f\left(D^1\right)$, in G_2. To show D^1 is normal in G_2, let $\delta = \begin{pmatrix} x & \pi^2 y \\ \bar{y} & \bar{x} \end{pmatrix} \in D^1$ and $g = \begin{pmatrix} a & \lambda\pi^2\bar{c} \\ c & \lambda\bar{a} \end{pmatrix} \in G_2$ with $a \in E^\times, c \in E, \lambda \in E^1, \nu(a) - \nu(c)\pi^2 = 1$, and $\nu(x) - \nu(y)\pi^2 = 1$. From here we get $\det g = \lambda\nu(a) - \lambda\nu(c)\pi^2 = \lambda$. Since $\nu(\lambda) = 1$, so $\lambda^{-1} = \bar{\lambda}$ and $g^{-1} = \begin{pmatrix} \bar{a} & -\pi^2\bar{c} \\ -c\bar{\lambda} & \bar{\lambda}a \end{pmatrix}$. This gives us:

$$\begin{aligned} g^{-1}\delta g &= \begin{pmatrix} a & \lambda\pi^2\bar{c} \\ c & \lambda\bar{a} \end{pmatrix} \begin{pmatrix} x & \pi^2 y \\ \bar{y} & \bar{x} \end{pmatrix} \begin{pmatrix} \bar{a} & -\pi^2\bar{c} \\ -c\bar{\lambda} & \bar{\lambda}a \end{pmatrix} \\ &= \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} x_{11} &= x\nu(a) - \pi^2\bar{c}\bar{y}a + \pi^2\bar{a}yc - \pi^2\nu(c)\bar{x}, \\ x_{12} &= \lambda x\pi^2\bar{a}\bar{c} - \lambda\pi^4\bar{c}\bar{c}\bar{y} + \lambda\pi^2\bar{a}\bar{a}y - \lambda\pi^2\bar{a}\bar{c}\bar{x}, \\ x_{21} &= -ax\bar{\lambda}c + a^2\bar{\lambda}\bar{y} - \pi^2\bar{\lambda}c^2y + ac\bar{\lambda}\bar{x}, \\ x_{22} &= -\pi^2x\nu(c) + a\pi^2\bar{y}\bar{c} - \pi^2cy\bar{a} + \nu(a)\bar{x}. \end{aligned}$$

Now one can show that $x_{22} = \overline{x_{11}}$ and $x_{21} = \frac{\overline{x_{12}}}{\pi^2}$, i.e.,

$$g^{-1}\delta g = \begin{pmatrix} x_{11} & \pi^2\overline{\left(\frac{\overline{x_{12}}}{\pi^2}\right)} \\ \frac{\overline{x_{12}}}{\pi^2} & \overline{x_{11}} \end{pmatrix} \in D^1.$$

□

THEOREM 2.3. *G_2 is semidirect product of D^1 and E^1, $G_2 = D^1 \rtimes E^1$.*

PROOF. This is well known, see e.g. [2], and Corollary 2.13 in this paper in our notation. □

LEMMA 2.4. *Let $\sigma : E^1 \to Aut(W)$, be defined as follows:*

$$\lambda \mapsto \sigma_\lambda, \text{ for any } \lambda \in E^1,$$

where

$$\sigma_\lambda : W \to W$$

is defined as:

$$\sigma_\lambda(w) = \sigma_\lambda(a + b\pi) = a + \lambda b\pi,$$

for any $w = a + b\pi \in W, a, b \in E$. Then $\lambda \mapsto \sigma_\lambda$ is an isomorphism of E^1 into the $Aut(W)$. Further for any $\lambda \in E^1$, σ_λ satisfies the following:

1. *$\overline{\sigma_\lambda(w)} = \sigma_\lambda(\overline{w})$ for any $w \in W$,*
2. *$\sigma_\lambda(e) = e$ for any $e \in E$,*
3. *σ_λ is in the $GL_E(W)$ as well as in $GL_F(W)$.*

PROOF. Let $w = a + b\pi$ and $w' = c + d\pi$ be two elements in W, where a, b, c and d are in E. Let λ, λ' be in E^1. Then:

$$ww' = ac + b\bar{d}\pi^2 + (ad + b\bar{c})\pi,$$

so by our definition we have:

$$\begin{aligned} \sigma_\lambda(ww') &= ac + b\bar{d}\pi^2 + \lambda(ad + b\bar{c})\pi \\ &= (a + \lambda b\pi)(c + \lambda d\pi) \\ &= \sigma_\lambda(w)\sigma_\lambda(w'), \end{aligned}$$

i.e., for any $\lambda \in E^1$, σ_λ is a homomorphism of W. One can check that $\ker\sigma_\lambda = \{1\}$. Also note that for any $w' = c + d\pi \in W$, we have:

$$\sigma_\lambda\left(c + \bar{\lambda}d\pi\right) = c + d\pi = w',$$

which shows that σ_λ is onto. Also we have:

$$\begin{aligned}\sigma_{\lambda\lambda'}(w) &= \sigma_{\lambda\lambda'}(a+b\pi) = a + \lambda\lambda' b\pi \\ &= \sigma_\lambda(a+\lambda' b\pi) = \sigma_\lambda\sigma_{\lambda'}(w),\end{aligned}$$

i.e., $\sigma_{\lambda\lambda'} = \sigma_\lambda\sigma_{\lambda'}$. This shows that σ is homomorphism. It is easy to check that σ is one to one. So σ is an imbedding. For the remaining statements we have:

1. For any $w = a + b\pi \in W$ and $\lambda \in E^1$, we have:

$$\sigma_\lambda(\bar{w}) = \sigma_\lambda(\bar{a} - b\pi) = \bar{a} - \lambda b\pi = \overline{\sigma_\lambda(w)}.$$

2. Any element e in E is in the form $e + 0\pi$, so $\sigma_\lambda(e) = e$.
3. Let $\mu \in E$. Then we have:

$$\begin{aligned}\sigma_\lambda(w+\mu w') &= \sigma_\lambda(a+\mu c + (b+\mu d)\pi) \\ &= a + \mu c + \lambda(b+\mu d)\pi \\ &= a + \lambda b\pi + \mu(c + \lambda d\pi) \\ &= \sigma_\lambda(w) + \mu\sigma_\lambda(w').\end{aligned}$$

So for any $\lambda \in E^1$, σ_λ is an *E-linear* map and hence an *F-linear* map as well. As a linear map, σ_λ, is one to one, too.

□

LEMMA 2.5. *For any $\lambda \in E^1$ and any $\delta \in D^1$. we have $\sigma_\lambda(\delta) \in D^1$.*

PROOF. By Lemma 2.4 one has:

$$\begin{aligned}\nu(\sigma_\lambda(\delta)) &= \sigma_\lambda(\delta)\overline{\sigma_\lambda(\delta)} = \sigma_\lambda(\delta)\sigma_\lambda\left(\bar{\delta}\right) \\ &= \sigma_\lambda\left(\delta\bar{\delta}\right) = \sigma_\lambda(1) = 1.\end{aligned}$$

□

COROLLARY 2.6. *Let $\sigma : E^1 \to Aut\left(D^1\right)$, be defined as follows:*

$$\lambda \mapsto \sigma_\lambda, \text{ for any } \lambda \in E^1,$$

where σ_λ is as in Lemma 2.4, restricted to $D^1 \subset D \cong W$. Then σ is an imbedding.

LEMMA 2.7. *Let $\lambda \in E^1$. Then for all $w \in W$ we have:*

$$Tr_{D/E}(\sigma_\lambda(w)) = Tr_{D/E}(w) = \sigma_\lambda\left(Tr_{D/E}(w)\right), \tag{2.4}$$

and

$$Tr(\sigma_\lambda(w)) = Tr(w) = \sigma_\lambda(Tr(w)). \tag{2.5}$$

PROOF. We prove (2.5). (2.4) is the same. Write $w = a + b\pi$. Then by definition of Tr and part 2 of Lemma 2.4 we get:

$$\begin{aligned} Tr\left(\sigma_\lambda\left(w\right)\right) &= \sigma_\lambda\left(w\right) + \overline{\sigma_\lambda\left(w\right)} = a + \lambda b\pi + \bar{a} - \lambda b\pi \\ &= a + \bar{a} = w + \bar{w} = Tr\left(w\right) = \sigma_\lambda\left(Tr\left(w\right)\right). \end{aligned}$$

$\square$

LEMMA 2.8. *For any $\lambda \in E^1$ we have $\sigma_\lambda \in G_2$ and $\sigma_\lambda \in Sp\left(W\right)$.*

PROOF. Let w and w' be two elements in W. Then by definition of $(,)_2$ and equation (2.4) in Lemma 2.7 we have:

$$\begin{aligned} \left(\sigma_\lambda\left(w\right), \sigma_\lambda\left(w'\right)\right)_2 &= \frac{1}{2} Tr_{D/E}\left(\epsilon\sigma_\lambda\left(w\right)\overline{\sigma_\lambda\left(w'\right)}\right) \\ &= \frac{1}{2} Tr_{D/E}\left(\epsilon\sigma_\lambda\left(w\right)\sigma_\lambda\left(\bar{w}'\right)\right) \\ &= \frac{1}{2} Tr_{D/E}\left(\epsilon\sigma_\lambda\left(w\bar{w}'\right)\right) = \frac{1}{2} Tr_{D/E}\left(\epsilon w\bar{w}'\right) = \left(w, w'\right)_2. \end{aligned}$$

So $\sigma_\lambda \in G_2$. The same computations and equation (2.5) in Lemma 2.7 give the result $\sigma_\lambda \in Sp\left(W\right)$. $\square$

LEMMA 2.9. *Let $\mathcal{G} = D^1 \times E^1$ be the set theoretic Cartesian product of D^1 and E^1, and define the following operation on it:*

$$(\delta, \lambda) * (\delta', \lambda') = \left(\sigma_\lambda\left(\delta'\right)\delta, \lambda\lambda'\right), \tag{2.6}$$

for (δ, λ) and $(\delta', \lambda') \in \mathcal{G}$. Then $\mathcal{G}$ is a group equal to the semidirect product of $D^1 \times \{1_{E^1}\} \cong D^1$ and $\{1_{D^1}\} \times E^1 \cong E^1$ via σ and we denote it by $D^1 \rtimes_\sigma E^1$.

PROOF. By Corollary 2.6 the action defined by equation (2.6) is well-defined and $(1, 1) \in \mathcal{G}$ is its unit element. Further for any $(\delta, \lambda) \in \mathcal{G}$, we have:

$$(\delta, \lambda)\left(\sigma_{\bar{\lambda}}\left(\bar{\delta}\right), \bar{\lambda}\right) = \left(\sigma_{\bar{\lambda}}\left(\bar{\delta}\right), \bar{\lambda}\right)(\delta, \lambda) = (1, 1).$$

So $(\delta, \lambda)^{-1} = \left(\sigma_{\bar{\lambda}}\left(\bar{\delta}\right), \bar{\lambda}\right)$. On the other hand for any $(\delta, \lambda) \in \mathcal{G}$ we have:

$$(\delta, \lambda) = (\delta, 1) * (1, \lambda),$$

and obviously we have:

$$D^1 \times \{1_{E^1}\} \cap \{1_{D^1}\} \times E^1 = \{(1, 1)\}.$$

$\square$

REMARK 2.10. From now on we will write $(\delta, \lambda)(\delta', \lambda')$ for $(\delta, \lambda) * (\delta', \lambda')$.

LEMMA 2.11. *There is a subgroup, say $\mathcal{H}$, of $Sp\left(W\right)$ such that $D^1 \rtimes_\sigma E^1 \cong \mathcal{H}$.*

Proof. Define $f : D^1 \rtimes_\sigma E^1 \to Sp(W)$, $(\delta, \lambda) \mapsto f_{(\delta,\lambda)}$, where $f_{(\delta,\lambda)}$ is given as follows:

$$f_{(\delta,\lambda)} : W \to W, \qquad f_{(\delta,\lambda)}(w) = \sigma_\lambda(w)\,\delta.$$

Then f is a homomorphism because for any (δ, λ) and $(\delta', \lambda') \in D^1 \rtimes_\sigma E^1$, and any $w \in W$ we have:

$$\begin{aligned} f_{(\delta,\lambda)(\delta',\lambda')}(w) &= f_{(\sigma_\lambda(\delta')\delta,\lambda\lambda')}(w) = \sigma_{\lambda\lambda'}(w)\,\sigma_\lambda(\delta')\,\delta \\ &= \sigma_\lambda(\sigma_{\lambda'}(w)\,\delta')\,\delta = f_{(\delta,\lambda)}(\sigma_{\lambda'}(w)\,\delta') \\ &= f_{(\delta,\lambda)} f_{(\delta',\lambda')}(w). \end{aligned}$$

This shows that $f(\delta, \lambda) f(\delta', \lambda') = f((\delta, \lambda)(\delta', \lambda'))$. Now let $(\delta, \lambda) \in \ker f$. Then, for any $w = a + b\pi \in W$, we must prove that:

$$f_{(\delta,\lambda)}(w) = (a + \lambda b\pi)\,\delta = a + b\pi.$$

This forces $(\delta, \lambda) = (1, 1)$. Let $(\delta, \lambda) \in D^1 \rtimes_\sigma E^1$. Then for any w, w' in W by Lemma 2.4 and relations (2.4), (2.5) in Lemma 2.7 we have:

$$\begin{aligned} \left\langle f_{(\delta,\lambda)}(w), f_{(\delta,\lambda)}(w') \right\rangle &= \langle \sigma_\lambda(w)\,\delta, \sigma_\lambda(w')\,\delta \rangle = \frac{1}{2} Tr\left(\epsilon \sigma_\lambda(w)\,\delta \overline{\sigma_\lambda(w')\,\delta}\right) \\ &= \frac{1}{2} Tr\left(\epsilon \sigma_\lambda(w)\,\delta\bar{\delta}\overline{\sigma_\lambda(w)}\right) = \frac{1}{2} Tr\left(\epsilon \sigma_\lambda(w)\,\sigma_\lambda(\bar{w})\right) \\ &= \frac{1}{2} Tr\left(\sigma_\lambda(\epsilon w\bar{w})\right) = \frac{1}{2} Tr(\epsilon w \bar{w}) = \langle w, \bar{w} \rangle. \end{aligned}$$

So $f_{(\delta,\lambda)} \in Sp(W)$. Now set $\mathcal{H} = \left\{ f_{(\delta,\lambda)} \in Sp(W) \mid (\delta, \lambda) \in D^1 \rtimes_\sigma E^1 \right\}$. ∎

Proposition 2.12. *Let all notations be as above. We have* $\mathcal{H} \cong G_2$.

Proof. First note that for any $f_{(\delta,\lambda)} \in \mathcal{H}$ we have $f_{(\delta,\lambda)} \in G_2$, because for any w, w' in W by Lemma 2.4, and equation (2.4) in Lemma 2.7 we have:

$$\begin{aligned} &\left\langle f_{(\delta,\lambda)}(w), f_{(\delta,\lambda)}(w') \right\rangle \\ &\quad = \langle \sigma_\lambda(w)\,\delta, \sigma_\lambda(w')\,\delta \rangle = \frac{1}{2} Tr_{D/E}\left(\epsilon \sigma_\lambda(w)\,\delta \overline{\sigma_\lambda(w')\,\delta}\right) \\ &\quad = \frac{1}{2} Tr_{D/E}\left(\epsilon \sigma_\lambda(w)\,\delta\bar{\delta}\overline{\sigma_\lambda(w')}\right) = \frac{1}{2} Tr_{D/E}\left(\sigma_\lambda(\epsilon w\bar{w}')\right) \\ &\quad = \frac{1}{2} Tr_{D/E}(\epsilon w \bar{w}') = \langle w, \bar{w} \rangle. \end{aligned}$$

Now define $\mathcal{F} : \mathcal{H} \to G_2$ as follows:

$$\mathcal{F}\left(f_{(\delta,\lambda)}\right) = \begin{pmatrix} a & \lambda\pi^2\bar{b} \\ b & \lambda\bar{a} \end{pmatrix},$$

where $\delta = a + b\pi \in D^1$ and $\lambda \in E^1$. $\mathcal{F}$ is a homomorphism because, for $\delta = a + b\pi \in D^1$ and $\delta' = c + d\pi \in D^1$ and λ, $\lambda' \in E^1$ we have:

$$\begin{aligned}
\mathcal{F}\left(f_{(\delta,\lambda)} f_{(\delta',\lambda')}\right) &= \mathcal{F}\left(f_{(\sigma_\lambda(\delta')\delta,\lambda\lambda')}\right) \\
&= \begin{pmatrix} ac + \lambda\bar{b}d\pi^2 & \lambda\lambda'\pi^2\left(\bar{\lambda}a\bar{d} + \bar{b}\bar{c}\right) \\ bc + \lambda\bar{a}d & \lambda\lambda'\left(\bar{a}\bar{c} + \bar{\lambda}b\bar{d}\pi^2\right) \end{pmatrix} \\
&= \begin{pmatrix} a & \lambda\pi^2\bar{b} \\ b & \lambda\bar{a} \end{pmatrix} \begin{pmatrix} c & \lambda'\pi^2\bar{d} \\ d & \lambda'\bar{c} \end{pmatrix} \\
&= \mathcal{F}\left(f_{(\delta,\lambda)}\right)\mathcal{F}\left(f_{(\delta',\lambda')}\right).
\end{aligned}$$

Now let $f_{(\delta,\lambda)} \in \ker\mathcal{F}$, where $\delta = a + b\pi \in D^1$ and $\lambda \in E^1$. Then we have:

$$\mathcal{F}\left(f_{(\delta,\lambda)}\right) = \begin{pmatrix} a & \lambda\pi^2\bar{b} \\ b & \lambda\bar{a} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This gives us $\delta = \lambda = 1$, i.e., $\mathcal{F}$ is one to one. On the other hand for any $Q = \begin{pmatrix} a & \lambda\pi^2\bar{b} \\ b & \lambda\bar{a} \end{pmatrix} \in G_2$, if we take $\delta = a + b\pi$, then $\mathcal{F}\left(f_{(\delta,\lambda)}\right) = Q$, which shows that $\mathcal{F}$ is onto. ∎

Corollary 2.13. *$G_2 = D^1 \rtimes E^1 \cong D^1 \rtimes_\sigma E^1 = U(2)$.*

Remark 2.14. If E/F is ramified then we may assume that $E = F(\pi)$. Then $\mathcal{B}=\{1, \epsilon\}$ will be a basis for D over E. So for any $x \in D$ we can write $x = a + \epsilon b$ for some $a, b \in E$. In this case the bilinear forms $(,)_1$ and $\langle,\rangle$ will be redefined as follows:

$$(x, y)_1 = \frac{1}{2}\overline{x}y, \text{ for } x, y \in E$$

and

$$\langle x, y\rangle = \frac{1}{2}Tr\left(\pi\overline{x}y\right), \quad \text{for } x, y \in D$$

Also for any $\lambda \in E, \sigma_\lambda$ will be given by

$$\sigma_\lambda(x) = \sigma_\lambda(a + \epsilon b) = a + \epsilon\lambda b$$

for $x = a + \epsilon b \in D$, where $a, b \in E$. The same argument as in unramified case gives us that

$$U(2) = D^1 \rtimes E^1 \cong D^1 \rtimes_\sigma E^1$$

3. Representations of $U(2)$

In this section we study certain subgroup because it will be used for the induced representations in the main Theorem.

Let E be the unramified quadratic extension of F contained in D specified in previous sections. Let O_E, P_E denote the ring of integers and maximal ideal of E, respectively. We may and will assume that ϖ (the uniformizer of F) is the generator of P_E. Also for any integer r, P_E^r is defined as $P_E^r =$

$O_E\varpi^r = O_E\pi^{2r}$ and $E_r^1 = \{x \in E^1 \mid x = 1 + a\varpi^r$, for some $a \in O_E\}$. The residue class field of E will be denoted by k_E.

Definition 3.1. *Let χ be a non trivial additive character of F. The conductor of χ is the smallest integer, n say, such that χ is trivial on P^n. Alternatively we may say P^n is the conductor of χ.*

Remark 3.2. Throughout this paper, χ is a non-trivial additive character of F with the conductor $0\,(O)$.

Lemma 3.3. *Let L_1 and L_2 be any two unramified quadratic extension of F contained in D. Then there exists a unit $d \in O_D$ such that $L_2 = dL_1d^{-1}$.*

Proof. See [9, page 104.]. □

Lemma 3.4. *For any non zero element $d \in D$, the map $\alpha \mapsto d\alpha d^{-1}$ is an isomorphism of O_{D° onto O_{D°.*

Proof. This is clear. □

Remark 3.5. In what follows $\alpha \in D^\circ$ is an element with $\upsilon_D(\alpha) = -n-1$, where n is a positive integer. Set $r = \left[\frac{n+1}{2}\right]$, where $[\ ]$ is the greatest integer part. We will identify $D_r^1 \rtimes_\sigma \{1_{E^1}\}$ with D_r^1 as a normal subgroup of $U(2)$.

Lemma 3.6. *Define $\chi_\alpha : D_r^1 \to \mathbb{C}^\times$ by :*

$$\chi_\alpha(h) = \chi(Tr(\alpha(h-1))),\ h \in D_r^1.$$

Then χ_α is a character of D_r^1 with conductor equal n.

Proof. See [7]. □

Definition 3.7. *Let φ be any homomorphism on a subgroup H of a group G. For any $g \in G$ set $H^{(g)} = gHg^{-1}$ and define φ^g on it as follows:*

$$\varphi^g(h) = \varphi\left(g^{-1}hg\right),\ h \in H^{(g)}.$$

φ^g is called the conjugation of φ by g.

Lemma 3.8. *Notation is as in Lemma 3.6. For any unit element $d \in O_D$, χ_α and $\chi^d_{d^{-1}\alpha d}$ are the same characters of D_r^1.*

Proof. Let $\delta \in D_r^1 \cap d\ D_r^1 d^{-1} = D_r^1$. Then we have:

$$\begin{aligned}\chi^d_{d^{-1}\alpha d}(\delta) &= \chi_{d^{-1}\alpha d}\left(d^{-1}\delta d\right) = \chi\left(Tr\left(d^{-1}\alpha d\right)\left(d^{-1}\delta d - 1\right)\right)\\ &= \chi\left(Tr\left(d^{-1}\alpha(\delta-1)d\right)\right) = \chi(Tr(\alpha(\delta-1))) = \chi_\alpha(\delta).\end{aligned}$$

□

Corollary 3.9. *Let L_1 and L_2 be any two unramified quadratic extension of F contained in D. Then the characters of L_1 and L_2 can be identified in a certain way.*

REMARK 3.10. On the basis of Lemmas 3.3 and 3.8, and for our purposes, any unramified quadratic extension of F contained in D might be taken equal E.

LEMMA 3.11. *Let α and r be as in Lemma 3.6. Let $L = F(\alpha)$ be a quadratic extension of F contained in D and set $L^1 = \{x \in L \mid \nu(x) = 1\}$. Then there exist $\varphi \in \left(L^1\right)^{\wedge}$ and $\psi \in \left(E^1\right)^{\wedge}$ such that $\varphi_{|L^1 \cap D_r^1} = \chi_{\alpha|L^1 \cap D_r^1}$ and $\psi_{|E^1 \cap D_r^1} = \chi_{\alpha|E^1 \cap D_r^1}$, where $(\ )^{\wedge}$ is Pontryagin's dual.*

PROOF. See [7]. □

DEFINITION 3.12. *Let α, r and L be as in Lemma 3.11. Set:*

$$\Phi(\alpha) = \left\{\varphi \in \left(L^1\right)^{\wedge} \mid \varphi_{|L^1 \cap D_r^1} = \chi_{\alpha|L^1 \cap D_r^1}\right\},$$

and

$$\Psi(\alpha) = \left\{\psi \in \left(E^1\right)^{\wedge} \mid \psi_{|E^1 \cap D_r^1} = \chi_{\alpha|E^1 \cap D_r^1}\right\}.$$

LEMMA 3.13. *Let $\lambda \in E^1$, and let $\alpha \in D^{\circ}$ be an element with $\upsilon_D(\alpha) = -n-1$, where n is a positive even integer and set $r = \left[\frac{n+1}{2}\right]$. Then for any $h \in D_r^1$, we have $\chi_\alpha(h) = \chi_\alpha(\sigma_\lambda(h))$ if and only if $\lambda \in E^1_{\left[\frac{n-r+1}{2}\right]}$.*

PROOF. Write $h = 1 + x$, for some $x \in P_D^r$. Then we must prove that:

$$\chi(Tr(\alpha\sigma_\lambda(x))) = \chi(Tr(\alpha x)).$$

This is the same as $(\alpha\sigma_\lambda(x) - \alpha x) \in P_D^{-1}$. Now write $x = a\pi^r = (a_1 + a_2\pi)\pi^r$, for some unit $a \in O_D$ where $a_1, a_2 \in O_E$. Then we have:

$$\sigma_\lambda(x) = \begin{cases} (a_1 + \lambda a_2\pi)\pi^r, & \text{if } r \text{ is even,} \\ \\ (\lambda a_1 + a_2\pi)\pi^r, & \text{if } r \text{ is odd.} \end{cases}$$

So from here when r is even we get:

$$(\alpha\sigma_\lambda(x) - \alpha x) = \alpha(\lambda - 1)a_2\pi^{r+1}.$$

Now note $(\alpha\sigma_\lambda(x) - \alpha x) \in P_D^{-1}$ if and only if $\alpha(\lambda-1)a_2\pi^{r+1} \in P_D^{-1}$, or $(\lambda - 1) \in P_D^{n-r-1}$ so

$$(\lambda - 1) \in P_D^{n-r-1} \cap O_E = P_E^{\left[\frac{n-r}{2}\right]} = P_E^{\left[\frac{n-r+1}{2}\right]}.$$

This implies that $\lambda \in E^1_{\left[\frac{n-r+1}{2}\right]}$. If r is odd then:

$$(\alpha\sigma_\lambda(x) - \alpha x) = \alpha(\lambda - 1)a_1\pi^r.$$

Now $(\alpha\sigma_\lambda(x) - \alpha x) \in P_D^{-1}$ if and only if $\alpha(\lambda - 1)a_1\pi^r \in P_D^{-1}$ or $(\lambda - 1) \in P_D^{n-r}$. Thus

$$(\lambda - 1) \in P_D^{n-r} \cap O_E = P_E^{\left[\frac{n-r+1}{2}\right]},$$

and this is the same as $\lambda \in E^1_{\left[\frac{n-r+1}{2}\right]}$. □

THEOREM 3.14. *Let $\alpha \in D^{\circ}$ be an element with $\upsilon_D(\alpha) = -n-1$, where n is a positive integer and $r = \left[\frac{n+1}{2}\right]$. Then the stabilizer of χ_α, $St(\chi_\alpha)$, in $U(2)$ is:*

$$St(\chi_\alpha) = \begin{cases} E^1 D^1_{r-1} \rtimes_\sigma E^1 & \text{if } n \text{ is odd,} \\ \\ L^1 D^1_r \rtimes_\sigma E^1_{\left[\frac{r+1}{2}\right]} & \text{if } n \text{ is even.} \end{cases}$$

PROOF. First suppose n is odd. Then $L = F(\alpha)$ is unramified. So by Remark 3.10 we may assume $L = E$. Now let $g = (\delta, \lambda) \in U(2)$ where $\delta \in D^1$ and $\lambda \in E^1$ be an element of $St(\chi_\alpha)$. Then $g^{-1} = \left(\sigma_{\bar\lambda}\left(\bar\delta\right), \bar\lambda\right)$. Let $h \in D^1_r$. Now we must prove that:

$$\chi_\alpha\left(g^{-1}hg\right) = \chi_\alpha(h). \tag{3.7}$$

On the other hand we have:

$$\begin{aligned} g^{-1}hg &= \left(\sigma_{\bar\lambda}\left(\bar\delta\right), \bar\lambda\right)(h,1)(\delta,\lambda) = \left(\sigma_{\bar\lambda}(h)\,\sigma_{\bar\lambda}\left(\bar\delta\right), \bar\lambda\right)(\delta,\lambda) \\ &= \left(\sigma_{\bar\lambda}(\delta)\,\sigma_{\bar\lambda}(h)\,\sigma_{\bar\lambda}\left(\bar\delta\right), 1\right) = \left(\sigma_{\bar\lambda}\left(\delta h \bar\delta\right), 1\right). \end{aligned} \tag{3.8}$$

By writing $h = 1 + x$, equation (3.7) becomes:

$$\chi\left(Tr\left(\alpha\sigma_{\bar\lambda}\left(\delta x\bar\delta\right)\right)\right) = \chi\left(Tr(\alpha x)\right). \tag{3.9}$$

Now since σ_λ is *E-linear* equation (3.9) becomes as follows:

$$\chi\left(Tr\left(\alpha\sigma_{\bar\lambda}\left(\delta x\bar\delta\right)\right)\right) = \chi\left(Tr\left(\sigma_{\bar\lambda}\left(\alpha\delta x\bar\delta\right)\right)\right). \tag{3.10}$$

Apply Lemma 2.7 to equation (3.10) to get;

$$\chi\left(Tr\left(\alpha\sigma_{\bar\lambda}\left(\delta x\bar\delta\right)\right)\right) = \chi\left(Tr\left(\alpha\delta x\bar\delta\right)\right) = \chi\left(Tr(\alpha x)\right),$$

or

$$\chi_\alpha\left(\delta h\bar\delta\right) = \chi_\alpha(h).$$

This last equality implies that $\delta \in E^1 D^1_{r-1}$. So $g = (\delta,\lambda) \in E^1 D^1_{r-1} \rtimes_\sigma E^1$.

Now suppose n is even. Then $L = F(\alpha)$ is ramified. Apply Lemma 3.13 to equation (3.9) to get:

$$\chi_\alpha\left(\sigma_{\bar\lambda}\left(\delta h\bar\delta\right)\right) = \chi_\alpha\left(\sigma_{\bar\lambda}(h)\right), \tag{3.11}$$

if and only if $\lambda \in E^1_{\left[\frac{r+1}{2}\right]}$. On the other hand equation (3.11) forces that $\delta \in L^1 D^1_r$. So $g = (\delta,\lambda) \in L^1 D^1_r \rtimes_\sigma E^1_{\left[\frac{r+1}{2}\right]}$. ∎

REMARK 3.15. In ramified case the stabilizer of $\chi_\alpha, St(\chi_\alpha)$, in $U(2)$ is:

$$St(\chi_\alpha) = \begin{cases} E^1 D^1_r \rtimes_\sigma E^1 & \text{if } n \text{ is even, and } \alpha \in E \\ L^1 D^1_r \rtimes_\sigma E^1_r & \text{if } n \text{ is even, and } \alpha \notin E \\ L^1 D^1_{r-1} \rtimes_\sigma E^1_{r-1} & \text{if } n \text{ is odd.} \end{cases}$$

LEMMA 3.16. *Let notation be as in Definition 3.12.*

1. *For any $\varphi \in \Phi(\alpha)$, define $\varphi_\alpha : L^1 D^1_r \to \mathbb{C}^\times$ by $\varphi_\alpha(l\delta) = \varphi(l)\,\chi_\alpha(\delta)$ for $l \in L^1$ and $\delta \in D^1_r$. Then φ_α is a well-defined character of $L^1 D^1_r$.*

2. *For any* $\psi \in \Psi(\alpha)$*, define* $\psi_\alpha : E^1 D_r^1 \to \mathbb{C}^\times$ *by* $\psi_\alpha(e\delta) = \psi(e)\chi_\alpha(\delta)$ *for* $e \in E^1$ *and* $\delta \in D_r^1$*. Then* ψ_α *is a well-defined character of* $E^1 D_r^1$*.*

PROOF. See [7]. □

LEMMA 3.17. *Let* $\alpha \in D^\circ$ *be an element with* $\upsilon_D(\alpha) = -n-1$*, where* n *is a positive integer and set* $r = \left[\frac{n+1}{2}\right]$*.*

1. *Let* n *be odd. For any* $\psi \in \Psi(\alpha)$ *and any character* ξ *of* E^1 *define:*

$$\psi_{(\alpha,\xi)} : E^1 D_r^1 \rtimes_\sigma E^1 \to \mathbb{C}^\times,$$

by

$$\psi_{(\alpha,\xi)}(x,\lambda) = \psi_\alpha(x)\,\xi(\lambda),$$

where ψ_α *is as in Lemma 3.16. Then* $\psi_{(\alpha,\xi)}$ *is a character of* $E^1 D_r^1 \rtimes_\sigma E^1$*.*

2. *Let* n *be even. For any* $\varphi \in \Phi(\alpha)$ *and any character* ξ *of* $E^1_{\left[\frac{r+1}{2}\right]}$ *define:*

$$\varphi_{(\alpha,\xi)} : L^1 D_r^1 \rtimes_\sigma E^1_{\left[\frac{r+1}{2}\right]} \to \mathbb{C}^\times,$$

by

$$\varphi_{(\alpha,\xi)}(x,\lambda) = \varphi_\alpha(x)\,\xi(\lambda),$$

where φ_α *is as in Lemma 3.16. Then* $\varphi_{(\alpha,\xi)}$ *is a character of* $L^1 D_r^1 \rtimes_\sigma E^1_{\left[\frac{r+1}{2}\right]}$*.*

PROOF. 1. Let $g = (x,\lambda)$ and $g' = (x',\lambda') \in E^1 D_r^1 \rtimes_\sigma E^1$. Then:

$$gg' = (\sigma_\lambda(x')\,x, \lambda\lambda').$$

So we have:

$$\psi_{(\alpha,\xi)}(gg') = \psi_\alpha(\sigma_\lambda(x')\,x)\,\xi(\lambda\lambda') = \psi_\alpha(\sigma_\lambda(x'))\,\psi_\alpha(x)\,\xi(\lambda)\,\xi(\lambda'). \tag{3.12}$$

Now note:

$$\sigma_\lambda(x') = (1,\lambda)(x',1)\left(1,\bar{\lambda}\right). \tag{3.13}$$

and since $\{1\} \rtimes_\sigma E^1$ is in the $St(\chi_\alpha)$, so by 3.13 we can rewrite 3.12 as follows:

$$\begin{aligned}\psi_{(\alpha,\xi)}(gg') &= \psi_\alpha(x')\,\psi_\alpha(x)\,\xi(\lambda)\,\xi(\lambda') = \psi_\alpha(x)\,\xi(\lambda)\,\psi_\alpha(x')\,\xi(\lambda') \\ &= \psi_{(\alpha,\xi)}(g)\,\psi_{(\alpha,\xi)}(g').\end{aligned}$$

2. Same argument as in part 1 works in this part.

□

Lemma 3.18. *Let n be a positive odd integer such that $r=\frac{n+1}{2}$ is even. Set:*

$$H_{r-1}=\left\{h\in D^1_{r-1}/D^1_n \mid h=\frac{1-a\pi^{r-1}}{1+a\pi^{r-1}}D^1_n,\ a\in O\right\}.$$

Then H_{r-1} is a subgroup of D^1_{r-1}/D^1_n with order $|H_{r-1}|=q^{\frac{r}{2}}$.

Proof. See [7]. □

Lemma 3.19. *Let notation be as in Lemma 3.18. Set*

$$\mathfrak{D}_{r-1}=D^1_r/D^1_nH_{r-1}.$$

Then $\mathfrak{D}_{r-1}$ is a subgroup of D^1_{r-1}/D^1_n and we have $|\mathfrak{D}_{r-1}|=q\left|D^1_r/D^1_n\right|$.

Proof. See [7]. □

Lemma 3.20. *Let $E^1_1=F(1+P_E)\cap E^1$. All other notations are as in Lemma 3.18. For any $\lambda\in E^1_1$ and $h\in H_{r-1}$ we have $\sigma_\lambda(h)\in\mathfrak{D}_{r-1}$.*

Proof. Write $\lambda=b+e\pi^2$ where $b\in O$, $e\in O_E$, and let $h=\frac{1-a\pi^{r-1}}{1+a\pi^{r-1}}D^1_n\in H_{r-1}$. Then by expanding the fraction on the right hand side we have:

$$\begin{aligned}(3.14)\qquad \sigma_\lambda(h) &= \frac{1-a\lambda\pi^{r-1}}{1+a\lambda\pi^{r-1}}D^1_n\\ &= 1-2ab\pi^{r-1}-2ae\pi^{r+1}+2a^2\pi^{2(r-1)}\,(\mathrm{mod}\,P^n_D).\end{aligned}$$

Set $\mu=1-2ab\pi^{r-1}+2(a+b)^2\pi^{2(r-1)}\,(\mathrm{mod}\,P^n_D)$. Then if $r>3$ we can rewrite 3.14 as follows:

$$\begin{aligned}\sigma_\lambda(h) &= \mu-2ae\pi^{r+1}-2ab\pi^{2(r-1)}-2b^2\pi^{2(r-1)}\,(\mathrm{mod}\,P^n_D)\\ (3.15)\qquad &= \mu\left(1-\left(2a\bar{\mu}e-2ab\bar{\mu}\pi^{(r-3)}-2b^2\bar{\mu}\pi^{(r-3)}\right)\pi^{r+1}\right)(\mathrm{mod}\,P^n_D).\end{aligned}$$

Now note that we have $\mu\in H_{r-1}$, and the rest of 3.15 is in the $D^1_{r+1}/D^1_n\subset D^1_r/D^1_n$. So $\sigma_\lambda(h)\in\mathfrak{D}_{r-1}$. If $r=2$, then we have:

$$\begin{aligned}\sigma_\lambda(h) &= \mu-2b(a+b)\pi^2\\ &= \mu\left(1-2b\bar{\mu}(a+b)\pi^2\right).\end{aligned}$$

So again $\sigma_\lambda(h)\in\mathfrak{D}_1=\mathfrak{D}_{r-1}$. □

Corollary 3.21. *Notations are as in Lemmas 3.18, 3.20. Then $E^1_1\mathfrak{D}_{r-1}\rtimes_\sigma E^1_1$ is a subgroup of $E^1D^1_{r-1}/D^1_n\rtimes_\sigma E^1$.*

Proof. Arguing as in [7] and Lemma 3.20 yields the result. □

Corollary 3.22. *Notations are as in Lemma 3.20 Let $\alpha\in D^\circ$ has order $\upsilon_D(\alpha)=-n-1$, where n is a positive odd integer such that $r=\frac{n+1}{2}$ is even. Then for any $\lambda\in E^1_1$ and $h\in H_{r-1}$we have $\chi_\alpha(\sigma_\lambda(h))=1$.*

Proof. Using 3.14 in Lemma 3.20 and definition of χ_α we get:

$$\begin{aligned}\chi_\alpha\left(\sigma_\lambda\left(h\right)\right) &= \chi\left[Tr\alpha\left(-2ab\pi^{r-1}-2ae\pi^{r+1}+2a^2\pi^{2(r-1)}\right)\right]\\ &= \chi\left(-2abTr\alpha\pi^{r-1}\right)\chi\left(-2aTr\alpha e\pi^{r+1}\right)\chi\left(2a^2\pi^{2(r-1)}Tr\alpha\right)\\ &= \chi\left(0\right)\chi\left(0\right)\chi\left(0\right)=1.\end{aligned}$$

□

Lemma 3.23. *Let α, n and r be as in Corollary 3.22. For any $\psi \in \Psi\left(\alpha\right)$ and any character ξ of E^1 define:*

$$\widetilde{\psi}_{(\alpha,\xi)} : E_1^1\mathfrak{D}_{r-1}\rtimes_\sigma E_1^1 \to \mathbb{C}^\times,$$

by

$$\widetilde{\psi}_{(\alpha,\xi)}\left(xh,\lambda\right)=\psi_{(\alpha,\xi)}\left(x,\lambda\right), \quad x\in E_1^1D_r^1/D_n^1,\ h\in H_{r-1},\ \lambda\in E_1^1.$$

Then $\widetilde{\psi}_{(\alpha,\xi)}$ is a well-defined character of $E_1^1\mathfrak{D}_{r-1}\rtimes_\sigma E_1^1$.

Proof. Arguing as in [7] and Corollary 3.22 gives the result. □

If Γ is a group and Γ_1 and Γ_2 are subgroups of Γ write $[\Gamma:\Gamma_1]$ for the number of left Γ_1-cosets in Γ and $[\Gamma_1:\Gamma:\Gamma_2]$ for the number of $\left(\Gamma_1,\Gamma_2\right)-$double cosets in Γ. Arguing as in [7], and using Lemmas 3.18 and 3.19 one can prove the following:

Lemma 3.24. *Let α, n and r be as in Corollary 3.22. Then:*

1. $\left[E^1\mathfrak{D}_{r-1}\rtimes_\sigma E_1^1 : E_1^1D_r^1/D_n^1\rtimes_\sigma E_1^1\right]=q\left(\frac{q+1}{2}\right)$.
2. $\left[E^1D_{r-1}/D_n^1\rtimes_\sigma E_1^1 : E_1^1\mathfrak{D}_{r-1}\rtimes_\sigma E_1^1\right]=q\left(\frac{q+1}{2}\right)$.
3. $\left[E^1:E_1^1\right]=\frac{q+1}{2}$.
4. $\left[E^1D_r^1/D_n^1\rtimes_\sigma E^1 : E^1D_{r-1}^1/D_n^1\rtimes_\sigma E^1 : E^1D_r^1/D_n^1\rtimes_\sigma E^1\right]=2q-1$.
5. $\left[E_1^1D_r^1/D_n^1\rtimes_\sigma E_1^1 : E^1D_{r-1}^1/D_n^1\rtimes_\sigma E^1 : E_1^1D_r^1/D_n^1\rtimes_\sigma E_1^1\right]=q^2\left(\frac{q+1}{2}\right)^2$.

Lemma 3.25. *Let α, n and r be as in Corollary 3.22. For any $\psi \in \Psi\left(\alpha\right)$ and any character ξ of E^1 set*

$$\tau_{(\alpha,\varphi,\xi)}=Ind\left(E^1D_{r-1}^1/D_n^1\rtimes_\sigma E_1^1, E_1^1\mathfrak{D}_{r-1}\rtimes_\sigma E_1^1, \widetilde{\psi}_{(\alpha,\xi)}\right).$$

Then $\tau_{(\alpha,\varphi,\xi)}$ is an irreducible representation of $E^1D_{r-1}^1/D_n^1\rtimes_\sigma E_1^1$ having dimension $q\left(\frac{q+1}{2}\right)$.

Proof. One can show that the stabilizer of $\widetilde{\psi}_{(\alpha,\xi)}$ in $E^1D_{r-1}^1/D_n^1\rtimes_\sigma E_1^1$ is $E_1^1\mathfrak{D}_{r-1}\rtimes_\sigma E_1^1$. Now the result follows from [1, Theorem $(45.2)'$]. □

Lemma 3.26. *Notation is as in Lemma 3.25. For any $\psi \in \Psi\left(\alpha\right)$ and any character ξ of E^1 let $\psi'_{(\alpha,\xi)}$ be the restriction of $\psi_{(\alpha,\xi)}$ to the $E_1^1D_r^1/D_n^1\rtimes_\sigma E_1^1$ and set:*

$$\tau'_{(\alpha,\psi,\xi)}=Ind\left(E^1D_{r-1}\rtimes_\sigma E_1^1, E_1^1D_r^1/D_n^1\rtimes_\sigma E_1^1, \psi'_{(\alpha,\xi)}\right).$$

Then $\tau'_{(\alpha,\psi,\xi)}$ *is a direct sum of* $q\left(\frac{q+1}{2}\right)$ *copies of* $\tau_{(\alpha,\psi,\xi)}$.

PROOF. Since $\widetilde{\psi}_{(\alpha,\xi)} = \psi'_{(\alpha,\xi)}$ on the $E_1^1 D_r^1/D_n^1 \rtimes_\sigma E_1^1$, so $\tau'_{(\alpha,\psi,\xi)}$ is equivalent to $\left[E^1\mathfrak{D}_{r-1}\rtimes_\sigma E_1^1 : E_1^1 D_r^1/D_n^1\rtimes_\sigma E_1^1\right]$ copies of $\tau_{(\alpha,\psi,\xi)}$. Now part 1 of Lemma 3.24 implies the result. ∎

LEMMA 3.27. *Let η be the character of the following representation:*

$$Ind\left(E^1 D_{r-1}^1/D_n^1\rtimes_\sigma E^1, E_1^1 D_r^1/D_n^1\rtimes_\sigma E_1^1, \psi_{(\alpha,\xi)}\right).$$

Then $\zeta = \frac{2}{q(q+1)}\eta$ *is the character of an irreducible representation, say* $\rho'(\alpha,\psi,\xi)$, *of* $E^1 D_{r-1}^1/D_n^1\rtimes_\sigma E^1$ *whose restriction to* $E^1 D_{r-1}^1/D_n^1\rtimes_\sigma E_1^1$ *is* $\tau_{(\alpha,\psi,\xi)}$.

PROOF. Let (,) denote the usual scalar product on $L^2\left(E^1 D_{r-1}^1/D_r^1\rtimes_\sigma E^1\right)$. By Lemma 3.24 and Mackey's Theorem we get:

$$(\zeta,\zeta) = \frac{4}{q^2(q+1)^2}(\eta,\eta) = \frac{4}{q^2(q+1)^2}q^2\left(\frac{q+1}{2}\right)^2 = 1.$$

Since $\eta(1) = q^2\left(\frac{q+1}{2}\right)^2$, thus $\zeta(1) = q\left(\frac{q+1}{2}\right)$. So ζ is the character of an irreducible representation of $E^1 D_{r-1}^1/D_n^1\rtimes_\sigma E^1$ having dimension $q\left(\frac{q+1}{2}\right)$. Call this representation $\rho'(\alpha,\psi,\xi)$. The multiplicity of $\rho'(\alpha,\psi,\xi)$ in $\tau_{(\alpha,\psi,\xi)}$ is:

$$\begin{aligned}\left(\zeta,\chi_{\tau_{(\alpha,\psi,\xi)}}\right) &= \frac{2}{q(q+1)}\left(\eta,\chi_{\tau_{(\alpha,\psi,\xi)}}\right) = \frac{2}{q(q+1)}\left(\mathrm{Re}\,\eta, Ind\chi_{\tau_{(\alpha,\psi,\xi)}}\right)\\ &= \frac{2}{q(q+1)}\left[E^1\mathfrak{D}_{r-1}\rtimes_\sigma E_1^1 : E_1^1 D_r^1/D_n^1\rtimes_\sigma E_1^1\right]\\ &= \frac{2}{q(q+1)}\cdot q\left(\frac{q+1}{2}\right) = 1,\end{aligned}$$

where $\chi_{\tau_{(\alpha,\psi,\xi)}}$ is the character of $\tau_{(\alpha,\psi,\xi)}$. Now by Frobenius reciprocity, the restriction of $\rho'(\alpha,\psi,\xi)$ to $E^1 D_{r-1}^1/D_n^1\rtimes_\sigma E_1^1$ is $\tau_{(\alpha,\psi,\xi)}$. ∎

LEMMA 3.28. *For any positive integer $r \geq 1$ we have*

$$\left(E^1 D_{r-1}^1\rtimes_\sigma E^1\right)/E^1 D_{r-1}^1\rtimes_\sigma\{1\} \cong E^1.$$

PROOF. Define $f : E^1 D_{r-1}^1\rtimes_\sigma E^1 \to E^1$ by $f(x,e) = e$, for any $x \in E^1 D_{r-1}^1, e \in E^1$. Then one can show that f is an onto homomorphism with $\ker f = E^1 D_{r-1}^1\rtimes_\sigma\{1\}$. ∎

THEOREM 3.29. *Let $\alpha \in D^\circ$ with $v_D(\alpha) = -n-1$ where n is a positive integer and $r = \left[\frac{n+1}{2}\right]$.*

1. *Let n be even. For any $\varphi \in \Phi(\alpha)$ and any character ξ of $E^1_{\left[\frac{r+1}{2}\right]}$ set:*

$$\rho_{(\alpha,\varphi,\xi)} = Ind\left(U(2), St(\chi_\alpha), \varphi_{(\alpha,\xi)}\right).$$

Then $\rho_{(\alpha,\varphi,\xi)}$ is an irreducible representation of $U(2)$.

2. *Let n and $r = \left[\frac{n+1}{2}\right]$ be odd. Then we know ([7]) $E^1 D^1_{r-1}$, and $E^1 D^1_r$ have the same characters. So for any $\psi \in \Psi(\alpha)$ and any character ξ of E^1 set*

$$\rho_{(\alpha,\psi,\xi)} = Ind\left(U(2), St(\chi_\alpha), \psi_{(\alpha,\xi)}\right).$$

Then $\rho_{(\alpha,\psi,\xi)}$ is an irreducible representation of $U(2)$.

3. *Let n be odd but $r = \left[\frac{n+1}{2}\right]$ is even. For any $\psi \in \Psi(\alpha)$ and any character ξ of E^1, let $\rho'(\alpha,\psi,\xi)$ be as in Lemma 3.27. Set:*

$$\rho_{(\alpha,\psi,\xi)} = Ind\left(U(2), St(\chi_\alpha), \rho'(\alpha,\psi,\xi)\right).$$

Then $\rho_{(\alpha,\psi,\xi)}$ is an irreducible representation of $U(2)$.

Proof. 1. The result follows from [1, Theorem $(45.2)'$].
2. The result follows from [1, Theorem $(45.2)'$].
3. Since $\rho'(\alpha,\psi,\xi)$ is an extension of $\tau_{(\alpha,\psi,\xi)}$, so by [1, Theorem 51.7] and Lemma 3.28 any irreducible component of the following representation

$$Ind\left(E^1 D^1_{r-1} \rtimes_\sigma E^1, E^1_1 D^1_r \rtimes_\sigma \{1\}, \tau_{(\alpha,\psi,\xi)}\right).$$

is equivalent to $\rho'(\alpha,\psi,\xi) \otimes \gamma$, for some character γ of E^1. On the other hand by [1, Theorem 38.5] we have $\rho'(\alpha,\psi,\xi) \otimes \gamma \cong \rho'(\alpha,\psi,\xi\gamma)$. Now apply Clifford Theorem.

∎

4. Characters, one-Dimensional Representations of $U(2)$

In this section we parametrize all one-dimensional representations (characters) of $U(2)$. Further, we classify all smooth irreducible representations of $U(2)$.

Lemma 4.1. *The commutator group of $U(2)$, $[U(2), U(2)]$, is equal to $D^1_1 \rtimes_\sigma \{1\} \cong D^1_1$.*

Proof. Let $x = (\delta, e), y = (\delta', e') \in U(2)$ where $\delta, \delta' \in D^1$, and $e, e' \in E^1$. Then:

$$\begin{aligned} xyx^{-1}y^{-1} &= (\delta, e)(\delta', e')\left(\sigma_{\bar{e}}\left(\bar{\delta}\right), e^{-1}\right)\left(\sigma_{\overline{e'}}\left(\overline{\delta'}\right), e'^{-1}\right) \\ &= \left(\delta' \sigma_{e'}\left(\bar{\delta}\right)\sigma_e(\delta')\delta, 1\right). \end{aligned}$$

So $xyx^{-1}y^{-1} \in D^1$, and hence $[U(2), U(2)] \subset D^1 \rtimes_\sigma \{1_{E^1}\}$. On the other hand Lemma 4.2 below shows that $U(2) \diagup \left(D^1_1 \rtimes_\sigma \{1_{E^1}\}\right)$ is abelian and this implies that $D^1_1 \rtimes_\sigma \{1_{E^1}\} \subset [U(2), U(2)]$. ∎

Lemma 4.2. *$U(2) \diagup \left(D^1_1 \rtimes_\sigma \{1_{E^1}\}\right) \cong \left(D^1/D^1_1\right) \rtimes_\sigma E^1$, where the right hand side is the external semidirect product via σ.*

PROOF. Define $f : U(2) \to \left(D^1 / D^1_1\right) \rtimes_\sigma E^1$, by $f(\delta, e) = \left(\delta D^1_1, e\right)$. Then f is a homomorphism because:

$$\begin{aligned} f\left((\delta,e)(\delta',e')\right) &= f\left(\sigma_e(\delta')\,\delta, ee'\right) = \left(\sigma_e(\delta')\,\delta D^1_1, ee'\right) \\ &= \left(\delta D^1_1, e\right)\left(\delta' D^1_1, e'\right) = f(\delta,e)\, f(\delta',e'). \end{aligned}$$

f obviously is onto and one can check that $\ker f = D^1_1 \rtimes_\sigma \{1_{E^1}\}$. □

LEMMA 4.3. *D^1 / D^1_1 is a cyclic group of order $q+1$ and we will denote it by μ_{q+1}.*

PROOF. See [7]. □

PROPOSITION 4.4. *For any character of μ_{q+1}, say η, and any character ξ of E^1,*

$$\eta_\xi : \mu_{q+1} \rtimes_\sigma E^1 \to \mathbb{C}^\times,$$

defined by

$$\eta_\xi(x,\lambda) = \eta(x)\,\xi(\lambda),$$

is a character of $U(2)$. Conversely any character of $U(2)$ is in this form.

PROOF. Since μ_{q+1} and E^1 are abelian it is easy to see that η_ξ is a character of $\mu_{q+1} \rtimes_\sigma E^1$. Now by inflation we can define η_ξ on $U(2)$. Conversely let Ω be a character of $U(2)$. Then for any $x, y \in U(2)$ we have:

$$\Omega\left(xyx^{-1}y^{-1}\right) = \Omega(x)\,\Omega(y)\,\Omega\left(x^{-1}\right)\Omega\left(y^{-1}\right) = 1.$$

i.e., $\Omega_{|D^1_1 \rtimes_\sigma \{1_{E^1}\}} = 1$. So Ω is a character of $\mu_{q+1} \rtimes_\sigma E^1$. Let $\eta = \Omega_{|\mu_{q+1} \rtimes_\sigma \{1_{E^1}\}}$ and $\xi = \Omega_{|\{1_{D^1}\} \rtimes_\sigma E^1}$. Then one can show that $\Omega = \eta_\xi$. □

THEOREM 4.5. *Any irreducible representation of $U(2)$ is either a character or is one of those determined by Theorem 3.29.*

PROOF. Let ρ be an irreducible representation of $U(2)$. Since the family $\left\{D^1_n \rtimes_\sigma \{1\}\right\}_{n \geq 1}$ is a system of unity neighborhoods in $D^1 \rtimes_\sigma \{1\}$, there is a least integer $n \geq 1$ such that the restriction of ρ to $D^1_n \rtimes_\sigma \{1\}$, $\rho_{|D^1_n \rtimes_\sigma \{1\}}$, is trivial. Now we have the following cases:

1. $n = 1$. Then the restriction of ρ to $D^1_1 \rtimes_\sigma \{1_{E^1}\}, \rho_{|D^1_1 \rtimes_\sigma \{1_{E^1}\}}$, is trivial. So we can look at ρ as an irreducible representation of $U(2) / \left(D^1_1 \rtimes_\sigma \{1_{E^1}\}\right)$. But $U(2) / \left(D^1_1 \rtimes_\sigma \{1_{E^1}\}\right)$ is abelian because by Lemma 4.2 $D^1_1 \rtimes_\sigma \{1_{E^1}\}$ is the commutator group of $U(2)$, so by Proposition 4.4 ρ is a character of $U(2)$.
2. $n = 2k+1$, k a positive integer ≥ 1. Then $\rho \cong \rho(\alpha, \psi, \xi)$ where $\alpha \in D^\circ$ with $\upsilon(\alpha) = -n-1$, ψ is a character of E^1 occurring in $\rho_{|E^1 \rtimes_\sigma \{1_{E^1}\}}$ and ξ is a character of E^1 occurring in $\rho_{|\{1_{D^1}\} \rtimes_\sigma E^1}$.

3. $n = 2k$, k a positive integer ≥ 1. Then $\rho \cong \rho(\alpha, \varphi, \xi)$ where $\alpha \in D^\circ$ with $v(\alpha) = -n-1$, φ is a character of L^1 occurring in $\rho_{|L^1 \rtimes_\sigma \{1_{E^1}\}}$ where $L = F(\alpha)$ and ξ is a character of $E^1_{\left[\frac{r+1}{2}\right]}$ occurring in $\rho_{|\{1_{D^1}\} \rtimes_\sigma E^1_{\left[\frac{r+1}{2}\right]}}$.

$\square$

Theorem 4.6. *Irreducible representations of $U(2)$ enjoy the following equivalencies:*

1. *Any irreducible representations of $U(2)$ determined by Theorem 3.29, never is equivalent to a character.*
2. *Two characters Ω_1, Ω_2 of $U(2)$ are equivalent if and only if $\Omega_1 = \Omega_2$.*
3. *Any two irreducible representations $\rho_1 = \rho_1(\alpha_1, \psi_1, \xi_1)$ and $\rho_2 = \rho_2(\alpha_2, \psi_2, \xi_2)$ of $U(2)$ determined by Theorem 3.29 are equivalent if and only if:*
   - $\xi_1 = \xi_2$
   - *If $\psi_{1\alpha_1}$ is any irreducible representation of D^1 occurring in $\rho_{1|D^1 \rtimes_\sigma \{1\}}$ and $\psi_{2\alpha_2}$ is any irreducible representation of D^1 occurring in $\rho_{2|D^1 \rtimes_\sigma \{1\}}$, then $\psi_{1\alpha_1} \cong \psi_{2\alpha_2}$.*

Proof. Parts 1 and 2 are clear. For last part of part 3, from [7] we know that two irreducible representations $\rho(\alpha, \varphi)$, and $\rho(\alpha', \varphi')$ of D^1 are equivalent if and only if:

- they have same conductor, n,
- there exists $g \in D^1$such that $\alpha' - g\alpha g^{-1} \in P_D^{n-r}$ where $r = \left[\frac{n+1}{2}\right]$,
- $\varphi'(e') = \varphi\left(geg^{-1}\right)$ for $e' \in E' = F(\alpha')$, and $e \in E = F(\alpha)$,
- $E' = gEg^{-1}$.

This is because if we consider the restriction of $\rho(\alpha, \varphi)$, and $\rho(\alpha', \varphi')$ to D_r^1 where $r = \left[\frac{n+1}{2}\right]$, then Clifford's Theorem ([1]) gives the result. For more detail see [7]. $\square$

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