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ON SQUARES OF IRREDUCIBLE CHARACTERS

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Abstract. We study the finite groups G with a faithful irreducible character whose square is a linear combination of algebraically conjugate irreducible characters of G. In conclusion, we offer another proof of one theorem of Isaacs-Zisser.

There are a few papers treating the finite groups possessing an irreducible character whose powers are linear combinations of appropriate irreducible characters, for example, [BC] and [IZ]. Our note is inspired by these two papers, especially, the second one.

In what follows, G is a finite group. We use standard notation of finite group theory (see [BZ]). Recall that if $\chi \in \mathrm{Irr}(G)$, then the generalized character $\chi^{(2)}$ (see [BZ, Chapter 4]) is defined as follows:

$$\chi^{(2)}(g) = \chi(g^2)\,(g \in G).$$

Next, $\mathrm{Char}(G)$ denotes the set of characters of a group G and, if θ is a generalized character of G, then

$$\mathrm{Irr}(\theta) = \{\chi \in \mathrm{Irr}(G) \mid \langle \theta, \chi \rangle \neq 0\}.$$

The quasikernel $\mathrm{Z}(\chi)$ of $\chi \in \mathrm{Char}(G)$ is defined as follows:

$$\mathrm{Z}(\chi) = \{g \in G \mid |\chi(g)| = \chi(1)\}.$$

It is known that $\mathrm{Z}(\chi)$ is a normal subgroup of G containing $\ker(\chi)$ and $\mathrm{Z}(G/\ker(\chi)) \leq \mathrm{Z}(\chi)/\ker(\chi)$ with equality if, in addition, $\chi \in \mathrm{Irr}(G)$. In what follows, we use freely results stated in this paragraph.

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Let $\chi \in \operatorname{Irr}(G)$. It is known that

$$\operatorname{Irr}(\chi^{(2)}) \subseteq \operatorname{Irr}(\chi^2). \tag{1}$$

Indeed, $\chi^{(2)} = \chi^2 - 2\bigwedge^2\chi$ (see formula (22) in [BZ, §4.6]) so it suffices to show that $\operatorname{Irr}(\bigwedge^2\chi) \subseteq \operatorname{Irr}(\chi^2)$. Next, $\chi^2 = \bigwedge^2\chi + \theta$, where θ is the exterior square of χ (see [BZ, Lemma 4.16, formula (17)]) so $\operatorname{Irr}(\bigwedge^2\chi) \subseteq \operatorname{Irr}(\chi^2)$, as desired. Therefore, if $\operatorname{Irr}(\chi^2) = \{\psi_1, \ldots, \psi_n\}$ and

$$\chi^2 = \sum_{j=1}^{n} a_j\psi_j, \text{ where all } a_j \text{ are positive integers,} \tag{2}$$

then, by (1), we have

$$\chi^{(2)} = \sum_{j=1}^{n} b_j\psi_j, \text{ where all } b_i \text{ are integers.} \tag{3}$$

Set

$$a = a_1 + \cdots + a_n, \quad b = b_1 + \cdots + b_n. \tag{4}$$

Let ϵ be a $|G|$-th primitive root of 1, $\mathcal{G} = \operatorname{Gal}(Q(\epsilon)/Q)$, where Q is the field of rational numbers. The group $\mathcal{G}$ acts in the natural way on the set $\operatorname{Irr}(G)$ as follows: if $\chi \in \operatorname{Irr}(G)$ and $\sigma \in \mathcal{G}$, then $(\sigma\chi)(g) = \sigma(\chi(g))$ for all $g \in G$ (see [BZ, Chapter 3]). Characters $\psi, \psi' \in \operatorname{Irr}(G)$ are said to be *algebraically conjugate* if $\psi' = \sigma\psi$ for some $\sigma \in \mathcal{G}$. In that case, as it is easy to check, $\psi(1) = \psi'(1)$, $\ker(\psi) = \ker(\psi')$ and $\mathrm{Z}(\psi) = \mathrm{Z}(\psi')$.[1] In what follows we will retain the notation introduced above and in this paragraph.

Definition 1. *A group G with an irreducible character χ possesses a property $\mathcal{A}$, if it satisfies the following conditions:*

($\mathcal{A}1$) *$|G|$ is even.*
($\mathcal{A}2$) *χ is faithful.*
($\mathcal{A}3$) *$n \geq 2$ and ψ_j $(j = 1, \ldots, n)$ are algebraically conjugate with $\psi = \psi_1$ (see decomposition (2)), i.e., $\psi_j = \sigma_j\psi$ for some $\sigma_j \in \mathcal{G}$ $(j = 1, \ldots, n)$.*
($\mathcal{A}4$) *χ and $\psi_1, \ldots, \psi_n$ satisfy (2) and (3).*

Our main result is the following

Theorem 2. *If the group G satisfies condition $\mathcal{A}$, then the following assertions hold (here, as in part ($\mathcal{A}3$) of the definition, $\psi = \psi_1$):*

(a) *G is nonabelian and $\chi(1) > 1$.*
(b) *G has only one involution u.*
(c) $\ker(\psi) = \langle u\rangle$.
(d) *$\mathrm{Z}(\psi)$ is abelian.*
(e) *Sylow 2-subgroups of G are cyclic.*

[1] Let us prove the second equality. Take $g \in \ker(\psi)$. We have $\psi'(g) = \sigma(\psi(g)) = \sigma(\psi(1)) = \psi(1)$ so $\ker(\psi) \leq \ker(\psi')$. Since $\sigma^{-1} \in \mathcal{G}$, the reverse inclusion holds as well.

(f) $G = P \cdot N$, *a semidirect product, where* $P \in \mathrm{Syl}_2(G)$ *and* $\{1\} < N \lhd G$.
(g) $\chi(1) = \psi(1)$ *and* $b = 1$.
(h) $\chi_N \in \mathrm{Irr}(N)$.
(i) *All* $(\psi_j)_N$ *are irreducible and nonreal for* $j = 1, \ldots, n$.
(j) *If* $w \in G$ *with* $o(w) = 8$, *then* $\psi(w) = m\sqrt{-1}$, *where* m *is an integer dividing* $\chi(1)$.

Proof. (a) It follows from (2) and (4) that $\chi(1)^2 = a\psi(1)$ so $\chi(1) > 1$ since $a \geq n > 1$. Therefore, G is nonabelian.

(b) Let u be an involution in G. Since $\psi(u)$ is a rational integer, it follows that $\psi_j(u) = \psi(u)$ since the ψ_j's are algebraically conjugate. Therefore, $\chi^{(2)}(u) = b\psi(u)$ (see (3) and (4)). Since $\chi^{(2)}(u) = \chi(u^2) = \chi(1)$, we get

$$\chi(1) = b\psi(u). \tag{5}$$

On the other hand, $\chi(1) = \chi^{(2)}(1) = b\psi(1)$ so, taking into account that $b \neq 0$, we get $\psi(u) = \psi(1)$, by (5), i.e., $u \in \ker(\psi) = \ker(\psi_j)$ for $j = 1, \ldots, n$. Therefore, it follows from (2) that $\ker(\chi^2) = \ker(\psi)$ so $u \in \ker(\chi^2)$, i.e., $\chi(u)^2 = \chi(1)^2$. In that case, $\chi(u) = \pm\chi(1)$, and we obtain

$$\chi(u) = -\chi(1) \tag{6}$$

since our character χ is faithful. It follows that $u \in \mathrm{Z}(\chi) = \mathrm{Z}(G)$ since χ is faithful, i.e., $\mathrm{Z}(G)$ contains all involutions of G. Since $\mathrm{Z}(G)$ is cyclic (χ is faithful), we conclude that u is the unique involution in G, and (b) is proven.

(c) As we have proved in (b) (see the sentence after formula (5)), $u \in \ker(\psi)$. It suffices to show that u is the unique nonidentity element of $\ker(\psi)$. Take $x \in \ker(\psi)^\#$. Then $\psi_j(x) = \psi(1)$ for all j so, by (2), $x \in \ker(\chi^2)$ so that, again by (2), we have

$$\chi(1)^2 = \chi(x)^2 = a\psi(1), \text{ and hence } \chi(x) = -\chi(1)$$

since χ is faithful. In particular, $x \in \mathrm{Z}(\chi)$ so that $\chi(x^2) = \chi(1)$ and $x^2 \in \ker(\chi) = \{1\}$, i.e., x is an involution. It follows from this and (b), that $x = u$. Thus $\ker(\psi) = \{1, u\} = \langle u \rangle$, as required.

(d) It follows from (c), that $\mathrm{Z}(\psi)$ is abelian since $\mathrm{Z}(\psi)/\ker(\psi)$ is cyclic (in view of irreducibility of ψ) and $|\ker(\psi)| = 2$.

(e) Let $P \in \mathrm{Syl}_2(G)$. By (b), P is either cyclic or generalized quaternion. Assume, by way of contradiction, that P is generalized quaternion. Take in P an element v of order 4. Then, by (b),

$$v^2 = u. \tag{7}$$

Let $j \in \{1, \ldots, n\}$. Then, by $(\mathcal{A}2)$, $\psi_j = \sigma_j\psi$ so that $\psi_j(v) = (\sigma_j\psi)(v) = \sigma_j(\psi(v))$. We have $\sigma_j\epsilon = \epsilon^{\nu_j}$ for some rational integer ν_j such that $\mathrm{GCD}(\nu_j, |G|) = 1$ (recall that ϵ is the primitive $|G|$-th root of 1 chosen above); then ν_j is odd in view of $(\mathcal{A}1)$. Setting $\nu_j = 2\lambda_j + 1$ and taking into account that $\psi(v)$ is a sum of powers of ϵ, we get

$$\psi_j(v) = \sigma_j(\psi(v)) = \psi(v^{\nu_j}) = \psi(v^{2\lambda_j+1}) = \psi(u^{\lambda_j}v) = \psi(v)$$

since $u \in \ker(\psi)$, by (c). Thus

$$\psi_j(v) = \psi(v) \quad (j = 1, \ldots, n). \tag{8}$$

Then, by (6), (8), (3) and (4), we obtain

$$-\chi(1) = \chi(u) = \chi(v^2) = \chi^{(2)}(v) = \sum_{j=1}^{n} b_j \psi_j(v) = b\psi(v).$$

On the other hand, $\chi(1) = \chi^{(2)}(1) = b\psi(1)$ so $\psi(v) = -\psi(1)$. It follows from this and (8), that

$$\psi_j(v) = -\psi(1) \quad (j = 1, \ldots, n). \tag{9}$$

It follows from (9) that, if T is a representation of G affording the character ψ_j, then $T(v) = -I_{\psi(1)}$, where $I_{\psi(1)}$ is a $\psi(1) \times \psi(1)$ identity matrix. Therefore, by (2), we get

$$\chi(v)^2 = \sum_{j=1}^{n} a_j \psi_j(v) = -a\psi(1) = -\chi(1)^2,$$

and we conclude that

$$\chi(v) = ci\chi(1), \tag{10}$$

where $c = \pm 1$ and $i = \sqrt{-1}$. It follows from $|\chi(v)| = \chi(1)$ that $v \in \mathrm{Z}(\chi) = \mathrm{Z}(G)$, a contradiction, since the center of P, which is a generalized quaternion group, has order 2. Thus, $P \in \mathrm{Syl}_2(G)$ is cyclic.

(f) By (e), G is 2-nilpotent so $G = P \cdot N$, and $N > \{1\}$ since G is nonabelian.

(g) Since $G/N \cong P$ is cyclic, then χ is not ramified over N (Burnside; see [BZ, Exercise 7 in Chapter 7]) so we get the following Clifford decomposition:

$$\chi_N = \sum_{k=1}^{l} \phi_k. \tag{11}$$

It follows from (11) that

$$(\chi^{(2)})_N = \sum_{k=1}^{l} \phi_k^{(2)}. \tag{12}$$

Since $|N|$ is odd, it follows that $\phi_k^{(2)}$ are distinct irreducible characters of N for all k, and $(\chi^{(2)})_N$ is a character of N.[2] By (9), we have

$$(\chi^{(2)})_N = \sum_{j=1}^{n} b_j (\psi_j)_N.$$

Let $\phi_1 = \phi$. Since

$$1 = \langle (\chi^{(2)})_N, \phi^{(2)} \rangle = \sum_{j=1}^{n} b_j \langle (\psi_j)_N, \phi^{(2)} \rangle,$$

we get $\langle (\psi_s)_N, \phi^{(2)} \rangle \neq 0$ for some $s \in \{1, \ldots, n\}$. This means that $\phi^{(2)} \in \mathrm{Irr}((\psi_s)_N)$. By Clifford's theorem, $\mathrm{Irr}((\psi_s)_N)$ is a G-orbit of $\phi^{(2)}$, i.e.,

$$\mathrm{Irr}((\psi_s)_N) = \{\phi_1^{(2)}, \ldots, \phi_l^{(2)}\}, \text{ where } \phi_1 = \phi.$$

We conclude that $(\psi_s)_N = \sum_{k=1}^{l} \phi_k^{(2)}$ so, by (12), $(\chi^{(2)})_N = (\psi_s)_N$. It follows, in particular, that $\chi(1) = \psi_s(1) = \psi(1)$. Since $\chi(1) = \chi^{(2)}(1) = b\psi(1)$, we get $b = 1$, completing the proof.

(h) It follows from (2) and (3) that

$$\chi^2 - \chi^{(2)} = \sum_{j=1}^{n} (a_j - b_j)\psi_j.$$

Since

$$\frac{1}{2}(\chi^2 - \chi^{(2)}) = \bigwedge\nolimits^2 \chi \in \mathrm{Char}(G)$$

(see formula (22) in [BZ, §4.6]), we get $a_j \equiv b_j \pmod 2$ for $j = 1, \ldots, n$. Summing up over all j, one obtains $a \equiv b \pmod 2$ so a is odd since $b = 1$, by (g). As we have noticed in the proof of (a), $\chi(1)^2 = a\psi(1)$. Therefore, since $\chi(1) = \psi(1)$, by (g), we have $\chi(1) = a$, and hence $\chi(1)$ is odd. It follows from (11) that $l = |\mathrm{Irr}(\chi_N)| = |G : \mathrm{I}_G(\phi)|$, where $\mathrm{I}_G(\phi)$ is the inertia group of ϕ in G, and $\chi(1) = l\phi(1)$ so l is odd. Since $\mathrm{I}_G(\phi) \geq N$ and $|G : N| = |P|$ is a power of 2, we get $l = 1$, i.e., $\chi_N \in \mathrm{Irr}(N)$, proving (h).

(i) Since $l = 1$, there exists $s \in \{1, \ldots, n\}$ such that $(\psi_s)_N = \phi^{(2)} \in \mathrm{Irr}(N)$. Since all ψ_j are algebraically conjugate, we get $(\psi_j)_N \in \mathrm{Irr}(N)$. If $(\psi_j)_N$ is real for some j, it is the principal character 1_G of G since $|N|$ is odd. It follows that $\psi(1) = \psi_j(1) = 1$. Then, by (g), $\chi(1) = 1$, a contradiction. Thus, all ψ_j are not real.

(j) Let $w \in G$ be of order 8. Setting $v = w^2$, we get $o(v) = 4$. By (b), $v^2 = u$. It follows from (10) that $ci\chi(1) = \chi(v) = \chi(w^2) = \chi^{(2)}(w)$ so, in view

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[2] Indeed, assume that $\phi_i^{(2)} = \phi_j^{(2)}$. Then, for $x \in N$ we have $\phi_i(x^2) = \phi_i^{(2)}(x) = \phi_j^{(2)} = \phi_j(x^2)$, and so $\phi_i = \phi_j$ since $\{x^2 \mid x \in N\} = N$. This proves the first assertion. Now the second assertion is obvious.

of (9), we get

$$\sum_{j=1}^{n} b_j\psi_j(w) = \chi^{(2)}(w) = ci\chi(1) \tag{13}$$

(here $i=\sqrt{-1}$). Recall, that $\psi_j(w)=\psi(w^{\nu_j})$, where ν_j is odd integer (see the proof of (e)). It follows from $o(w)=8$ that $\nu_j\in\{1,3,5,7\}$. Since $u\in\ker(\psi)$ and $\psi(v)=-\psi(1)$ (see (9)), we have (consider a representation of G affording the character ψ; see two line after formula (9))

$$\psi(w^3)=\psi(vw)=-\psi(w),$$
$$\psi(w^5)=\psi(uw)=\psi(w),$$
$$\psi(w^7)=\psi(uvw)=-\psi(w).$$

Thus, $\psi_j(w)\in\{\psi(w),-\psi(w)\}$. Setting $\psi_j(w)=c_j\psi(1)$, where $c_j=\pm1$, one can rewrite (13) in the form

$$ci\chi(1)=d\psi(w), \tag{14}$$

where $d=\sum_{j=1}^{n} b_jc_j$ is a rational integer. Note that

$$\overline{\psi(w)}=\psi(w^{-1})=\psi(w^7)=-\psi(w).$$

Therefore, $\psi(w)=im$, where m is a real number. Now, (14) yields $\chi(1)=cdm$. Therefore, $m=c\cdot\frac{\chi(1)}{d}$ is rational. Since $m=-i\psi(w)$ is an algebraic integer, it follows that m is a rational integer so m divides $\chi(1)$. This completes the proof of our theorem. □

Now we are ready to offer another proof of the following

THEOREM 3 (Isaacs-Zisser [IZ]). *Let $G>\{1\}$ be a group and suppose that there is a faithful $\chi\in\mathrm{Irr}(G)$ such that*

$$\chi^2=a\psi+b\bar{\psi}, \tag{15}$$

where a,b are positive integers and $\psi\in\mathrm{Irr}(G)$. Then G is a direct product of a cyclic 2-group of order not exceeding 4 and a group of odd order.

PROOF. Since $\psi_1=\psi$ and $\psi_2=\bar{\psi}_1$ are algebraically conjugate, one can apply Theorem 2 to our group G. By that theorem, P is cyclic. It remains to prove that $|P|\le 4$ and P is normal in G. We claim that $|\mathrm{Irr}(\chi^{(2)})|=1$ (recall that $\chi^{(2)}: x\mapsto\chi(x^2)$). Otherwise, $\chi^{(2)}=b_1\psi_1+b_2\psi_2$, where b_1,b_2 are nonzero rational integers. It follows that

$$(\chi^{(2)})_N=b_1(\psi_1)_N+b_2(\psi_2)_N$$

(recall that $(\chi^{(2)})_N,(\psi_1)_N,(\psi_2)_N$ are irreducible, by Theorem 2(h,i)). We have $(\chi^{(2)})_N=(\psi_s)_N$ for some $s\in\{1,2\}$ (see the proof of Theorem 2(g)). Using this and the equality $b_1+b_2=1$, we get $(\psi_1)_N=(\psi_2)_N$. It follows that $(\psi_1)_N=\overline{(\psi_1)_N}$, i.e., character $(\psi_1)_N$ is real, contrary to Theorem 2(i). Thus, one of numbers b_1, b_2 equals 0 so our claim is proven, i.e., $\chi^{(2)} \in \{\psi_1, \psi_2\}$. Assume, for definiteness, that $\chi^{(2)} = \psi_1$. Thus,

$$\chi^2 = a_1\psi_1 + a_2\psi_2 \text{ and } \chi^{(2)} = \psi_1.$$

As above, set $\psi = \psi_1$. Assume that $|P| > 4$. Take in P an element w of order 8. Set $v = w^2$. Then, by (10),

$$\psi(w) = \chi^{(2)}(w) = \chi(v) = ci\chi(1), \text{ where } c = \pm 1.$$

Since χ and ψ have the same degree, we get $\psi(w) = ci\psi(1)$ so $|\psi(w)| = \psi(1)$ whence $w \in \mathrm{Z}(\psi)$. Let $H \in \mathrm{Syl}_2(\mathrm{Z}(\psi))$; then H is normal in G so $HN = H \times N$ and w centralizes N. Since w centralizes $P \in \mathrm{Syl}_2(G)$, by Theorem 2(e), we see that $w \in \mathrm{Z}(G)$ since $G = P \cdot N$, and so $|\chi(w)| = \chi(1)$. Then equalities

$$\psi_2(w) = \overline{\psi_1(w)} = -\psi(w) \text{ and } |\psi(w)| = \chi(1)$$

imply

$$\begin{aligned}\chi(1)^2 = |\chi(w)|^2 &= |a_1\psi(w) + a_2\psi_2(w)| = |(a_1 - a_2)\psi(w)| \\ &= |a_1 - a_2||\psi(w)| = |a_1 - a_2|\chi(1).\end{aligned}$$

Hence $\chi(1) = |a_1 - a_2|$. On the other hand, $\chi(1) = a = a_1 + a_2$. Thus $a_1 + a_2 = |a_1 - a_2|$, a contradiction since $a_1, a_2 > 0$. Thus $|P| \leq 4$.

Let us prove that the subgroup P is normal in G. In view of Theorem 2(b), one may assume that $|P| = 4$ and $P = \langle v \rangle$. Then, by (9), $\psi(v) = -\psi(1)$ so $v \in \mathrm{Z}(\psi)$ and hence $P \leq \mathrm{Z}(\psi)$. Since P is characteristic in $\mathrm{Z}(\psi)$ (indeed, P is a Sylow 2-subgroup of the abelian group $\mathrm{Z}(\psi)$; see Theorem 2(d)), it follows that P is normal in G, and the proof is complete. ∎

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