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# FINITE $p$-GROUPS $G$ WITH $p > 2$ AND $\mathrm{d}(G) = 2$ HAVING EXACTLY ONE MAXIMAL SUBGROUP WHICH IS NEITHER ABELIAN NOR MINIMAL NONABELIAN

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Abstract. We give here a complete classification (up to isomorphism) of the title groups (Theorem 8 and Theorem 9). The corresponding problem for $p = 2$ was solved in [4].

Let $G$ be a nonabelian finite $p$-group ($p$ prime). If all maximal subgroups of $G$ are abelian, then such groups are minimal nonabelian and they are known long time ago (L. Rédei). If all maximal subgroups of $G$ are abelian or minimal nonabelian and at least one of them is minimal nonabelian, then such $p$-groups are called $\mathrm{A}_2$-groups and they are completely determined in §71 of [2]. It is a surprising fact that it is still possible to classify completely $p$-groups $G$ all of whose maximal subgroups but one are abelian or minimal nonabelian. For 2-groups ($p = 2$) this was done in [4]. Here we classify up to isomorphism such $p$-groups $G$ in case $p > 2$ under the assumption that $\mathrm{d}(G) = 2$, i.e., $G$ is 2-generated (Theorems 8 and 9). In a forthcoming paper we shall also consider the case $\mathrm{d}(G) > 2$.

Our notation is standard (see [1] and [2]). In particular, $\mathrm{S}(p^3)$ denotes for $p > 2$ the nonabelian group of order $p^3$ and exponent $p$ and an $\mathrm{L}_3$-group is a $p$-group $G$ in which $\Omega_1(G)$ is of order $p^3$ and exponent $p$ and $G/\Omega_1(G)$ is cyclic of order $> p$.

We state now all known results which are quoted in the proof of our theorems. Moreover, if these results are quoted from the unpublished book [3], then we also give a proof.

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LEMMA 1 ([1, Lemma 1.1]). *If $G$ is a nonabelian $p$-group with an abelian maximal subgroup, then $|G| = p|\mathrm{Z}(G)||G'|$.*

EXERCISE 1 ([1, Exercise 1.6(a)]). *The number of abelian subgroups of index $p$ in a nonabelian $p$-group $G$ is $0, 1,$ or $p + 1$.*

EXERCISE 2 ([1, Exercise 1.69(a)] (Mann)). *If $A$ and $B$ are distinct maximal subgroups in a $p$-group $G$, then $|G' : (A'B')| \leq p$.*

EXERCISE 3 ([1, Exercise 9.1(c)]). *Let $G$ be a $p$-group of maximal class and order $p^m$. If $p > 2$ and $m > 3$, then $G$ has no cyclic normal subgroup of order $p^2$.*

THEOREM 2 ([1, Theorem 36.1(c)]). *If $G/R$ is metacyclic for some $G$-invariant subgroup $R$ of index $p$ in $G'$, then $G$ is also metacyclic.*

LEMMA 3 ([1, Lemma 36.5]). (a) *If a $p$-group $G$ is two-generator of class 2, then $G'$ is cyclic.*
(b) *If $G$ is a nonabelian two-generator $p$-group, then $G'/\mathrm{K}_3(G)$ is cyclic.*

THEOREM 4 ([1, Theorem A.1.3] (The Hall-Petrescu formula)). *In an arbitrary group $G$, the following formula holds for $x, y \in G$ and any positive integer $n$:*

$$x^n y^n = (xy)^n c_2^{\binom{n}{2}} c_3^{\binom{n}{3}} ... c_n^{\binom{n}{n}},$$

*where $c_i \in \mathrm{K}_i(\langle x, y\rangle)$, $i = 2, ..., n$.*

THEOREM 5 ([2, Theorem 65.7(z)]). *Suppose that $G$ is an $\mathrm{A}_2$ -group of order $> p^4$. If $G'$ is cyclic of order $> p$, then $G$ is metacyclic and $|G'| = p^2$.*

THEOREM 6 ([2, Theorem 69.1]). *If $G$ is a minimal non-metacyclic $p$-group, $p > 2$, then either $G$ is of order $p^3$ and exponent $p$ or $G$ is a group of maximal class and order $3^4$.*

PROPOSITION 7 ([3, Proposition A.40.12] (Berkovich)). *A $p$-group $G$ of order $> p^4$, $p > 2$, has exactly one non-metacyclic maximal subgroup if and only if $G$ is an $\mathrm{L}_3$-group.*

PROOF. Suppose that $G$ has exactly one non-metacyclic maximal subgroup. Assume in addition that $G$ has no normal subgroup of order $p^3$ and exponent $p$. By Theorem 69.3 in [2], $G$ is either metacyclic (which in our case is not possible) or $G$ is a 3-group of maximal class. By Theorem 9.6 in [1], our 3-group $G$ has exactly three subgroups of maximal class and index 3. Since 3-groups of maximal class and order $> 3^3$ are obviously non-metacyclic, we get a contradiction.

Now suppose that $R$ is a $G$-invariant subgroup of order $p^3$ and exponent $p$. Since all maximal subgroups of $G$ that contain $R$ are non-metacyclic, we conclude that $G/R$ is cyclic. Since $G$ has a metacyclic maximal subgroup, it follows that $G$ has no subgroup of order $p^4$ and exponent $p$. Let $H/R$ be a subgroup of order $p$ in $G/R$ so that $\Omega_1(G) \le H$ and $\exp(H) = p^2$. Since an $\mathrm{S}_p$-subgroup of $\mathrm{Aut}(R)$ is of exponent $p$ and $G/R$ is cyclic of order $> p$, we get $H = R\mathrm{C}_H(R)$ and so $H$ is of class $\le 2$. It follows that $\Omega_1(H) = R$ and so $G$ is an $\mathrm{L}_3$-group.

Suppose that $G$ is an $\mathrm{L}_3$-group. Let $M$ be a maximal subgroup of $G$ such that $R \not\le M$. Then $M$ has a cyclic subgroup of index $p$ and so is metacyclic. □

Exercise 4 ([3, Exercise P9]). *Let $H = \langle a, b\rangle$ be a two-generator $p$-group with $|H'| = p$. Then $\Phi(H) = \langle a^p, b^p, [a, b]\rangle$ and $H$ is minimal nonabelian.*

Proof. For any $x, y \in H$, $[x^p, y] = [x, y]^p = 1$ and so $\mho_1(H) \le \mathrm{Z}(H)$ and $\Phi(H) = \langle \mho_1(H), H'\rangle \le \mathrm{Z}(H)$. We get $\Phi(H) = \mathrm{Z}(H)$ and so $H/\mathrm{Z}(H) \cong \mathrm{E}_{p^2}$ implies that $H$ is minimal nonabelian. Set $H_0 = \langle a^p, b^p, [a, b]\rangle \le \Phi(H)$ so that $H/H_0$ is an abelian group generated by two elements of order $p$ and so $H/H_0$ is elementary abelian of order $\le p^2$. Thus $\Phi(H) \le H_0$ and so $H_0 = \Phi(H)$. □

We turn now to a proof of our theorems.

Theorem 8. *Let $G$ be a two-generator $p$-group, $p > 2$, with exactly one maximal subgroup $M$ which is neither abelian nor minimal nonabelian. If $G$ has an abelian maximal subgroup $A$, then we have:*

$$G = \langle h, k \mid [h, k] = v,\ [v, k] = z,\ [v, h] = z^{\rho},$$
$$v^p = z^p = [z, h] = [z, k] = 1,\ h^p = z^{\sigma},\ k^{p^{n+1}} = z^{\tau}\rangle,$$

*where $n \ge 1$ and $\rho, \sigma, \tau$ are integers* mod $p$ *with $\rho \not\equiv 0 \pmod p$.*

*We have $|G| = p^{n+4}$, $G' = \langle v, z\rangle \cong \mathrm{E}_{p^2}$, $\mathrm{Z}(G) = \langle k^p, z\rangle$, $\Phi(G) = \mathrm{Z}(G)G'$, $G' \cap \mathrm{Z}(G) = \langle z\rangle \cong \mathrm{C}_p$, $[G', G] = \langle z\rangle$ and so $G$ is of class 3. Also, $S = \langle v, h\rangle \cong \mathrm{S}(p^3)$ (if $\sigma \equiv 0 \pmod p$ ) or $S \cong \mathrm{M}_{p^3}$ (if $\sigma \not\equiv 0 \pmod p$ ), $S$ is normal in $G$, $G = S\langle k\rangle$, $S \cap \langle k\rangle \le \langle z\rangle$, $G/S \cong \mathrm{C}_{p^{n+1}}$, $M = S\langle k^p\rangle$, $\mathrm{d}(M) = 3$, $M' = \langle z\rangle$, $A = \mathrm{C}_G(G')$, the set of maximal subgroups of $G$ is $\Gamma_1 = \{A, M, M_1, ..., M_{p-1}\}$, where all $M_i$ are minimal nonabelian with $M_1' = ... = M_{p-1}' = \langle z\rangle$ and $G/\mathrm{Z}(G) \cong \mathrm{S}(p^3)$. Finally, $G$ is an $\mathrm{L}_3$-group if and only if $\tau \not\equiv 0 \pmod p$ and in that case $\Omega_1(G) \cong \mathrm{S}(p^3)$ , $G/\Omega_1(G)$ is cyclic of order $p^{n+1}$ ($n \ge 1$) and $\mathrm{Z}(G) = \langle k^p\rangle \cong \mathrm{C}_{p^{n+1}}$ is cyclic.*

Proof. Obviously, $A$ is a unique abelian maximal subgroup of $G$ (otherwise, by Exercise 1.6(a) in [1], all $p + 1$ maximal subgroups of $G$ would be abelian). By a result of A.Mann (see Exercise 1.69(a) in [1]), $|G' : (A'M_1')| \le p$, where $M_1$ is a minimal nonabelian maximal subgroup of $G$ and so $|G'| \le p^2$. But if $|G'| = p$, then this fact together with $\mathrm{d}(G) = 2$ would imply that $G$ is minimal nonabelian, a contradiction. Hence $|G'| = p^2$. From $|G| = p|G'||Z(G)|$ (Lemma 1.1 in [1]) follows $|G : \mathrm{Z}(G)| = p^3$. Set $\Gamma_1 = \{A, M, M_1, ..., M_{p-1}\}$, where all $M_i$ ($i = 1, ..., p-1$) are minimal nonabelian.

We have $\mathrm{Z}(G) \le M_i$ (otherwise $\mathrm{d}(G) = 3$) and so $\mathrm{Z}(G) = \mathrm{Z}(M_i) = \Phi(M_i)$ for all $i = 1, ..., p-1$. Also, $\Phi(M_i) < \Phi(G) < M_i$ and so $\Phi(G)$ is abelian. For each $x \in G - A$, $\mathrm{C}_A(x) = \mathrm{Z}(G)$ and so $x^p \in \mathrm{Z}(G)$. Hence $G/\mathrm{Z}(G)$ is generated by its elements of order $p$ and so $G/\mathrm{Z}(G) \cong \mathrm{S}(p^3)$ because $\mathrm{d}(G) = 2$ and so $G/\mathrm{Z}(G)$ cannot be elementary abelian. This implies $G' \cap \mathrm{Z}(G) \cong \mathrm{C}_p$, $\Phi(G) = \mathrm{Z}(G)G'$ and $G$ is of class 3. Also, $M_i' = M' = G' \cap \mathrm{Z}(G)$ for all $i = 1, ..., p-1$. If $\mathrm{d}(M) = 2$, then $M' \cong \mathrm{C}_p$ would imply that $M$ is minimal nonabelian, a contradiction. Hence we have $\mathrm{d}(M) \ge 3$. In particular, $|M| \ge p^4$ and so $|G| \ge p^5$.

(i) First assume that $G' = \langle v \rangle \cong \mathrm{C}_{p^2}$ is cyclic. Since $\langle v^p \rangle = M_i'$ is not a maximal cyclic subgroup in $M_i > \Phi(G) = \mathrm{Z}(G)G'$, it follows that all $M_i$ ($i = 1, ..., p-1$) are metacyclic. In particular, $|\Omega_1(\Phi(G))| \le p^2$. Suppose that $A$ is also metacyclic so that $M$ (with $\mathrm{d}(M) \ge 3$) is the only non-metacyclic maximal subgroup of $G$. By a result of Y. Berkovich (see A.40.12 in [3]), $G$ is an $\mathrm{L}_3$-group. But then $G' \le \Omega_1(G)$, where $\Omega_1(G)$ is of order $p^3$ and exponent $p$ and so $G' \cong \mathrm{E}_{p^2}$, a contradiction. It follows that $A$ must be non-metacyclic in which case $\Omega_1(A) \not\le \Phi(G)$. Let $a$ be an element of order $p$ in $A - \Phi(G)$ and let $k \in G - A$ be such that $\langle \Phi(G), k \rangle = M_1$. Since $[k, v] \ne 1$, we may replace $k$ with another generator of $\langle k \rangle$ so that we may assume that $[k, v] = v^p$. Since $\langle k, v \rangle' = \langle v^p \rangle \le \mathrm{Z}(G)$, it follows that $\langle k, v \rangle$ is minimal nonabelian and so $\langle k, v \rangle = M_1$. We have (see for example Exercise P9 in [3]),

$$\mathrm{Z}(G) = \Phi(M_1) = \langle k^p, v^p, [k, v] = v^p \rangle = \langle k^p, v^p \rangle.$$

All maximal subgroups of $G$ distinct from $A = \Phi(G)\langle a \rangle$ are $\Phi(G)\langle a^i k \rangle = \mathrm{Z}(G)\langle a^i k,\, v \rangle$, where $i$ is any integer mod $p$. Since $[a^i k,\, v] = [a^i,\, v]^k [k, v] = v^p \in \mathrm{Z}(G)$, it follows that $\langle a^i k, v \rangle$ is minimal nonabelian. Again (see Exercise P9 in [3]),

$$\Phi(\langle a^i k, v \rangle) = \langle (a^i k)^p, v^p, [a^i k, v] = v^p \rangle = \langle (a^i k)^p, v^p \rangle.$$

The factor-group $G/\langle v^p \rangle$ is minimal nonabelian (since $\mathrm{d}(G/\langle v^p \rangle) = 2$ and $(G/\langle v^p \rangle)' \cong \mathrm{C}_p$) and so computing in $G/\langle v^p \rangle$, we get:

$$(a^i k)^p = a^{ip} k^p [k, a^i]^{\binom{p}{2}} x, \text{ where } x \in \langle v^p \rangle.$$

But $a^{ip} = 1$ and $[k, a^i] \in \langle v \rangle$ so that $[k, a^i]^{\binom{p}{2}} \in \langle v^p \rangle$ which gives $(a^i k)^p = k^p y$ for some $y \in \langle v^p \rangle$. By the above,

$$\Phi(\langle a^i k, v \rangle) = \langle k^p y, v^p \rangle = \langle k^p, v^p \rangle = \mathrm{Z}(G)$$

and so $\Phi(G)\langle a^i k \rangle = \mathrm{Z}(G)\langle a^i k, v \rangle = \langle a^i k, v \rangle$ is minimal nonabelian for all $i = 1, .., p-1$. It follows that $G$ is an $\mathrm{A}_2$-group, a contradiction.

(ii) We have proved that $G \cong \mathrm{E}_{p^2}$. Since $[G, G'] = G' \cap \mathrm{Z}(G) \cong \mathrm{C}_p$, we get by the Hall-Petrescu formula (Appendix 1 in [1]) for any $x, y \in G$, $(xy)^p = x^p y^p l$ for some $l \in G' \cap \mathrm{Z}(G)$.

We have $A = \mathrm{C}_G(G')$ and we take an element $k \in G - A$ such that $\Phi(G)\langle k \rangle = M_1$ is a minimal nonabelian maximal subgroup of $G$. Then for any element $v \in G' - \mathrm{Z}(G)$, we have $[v, k] = z$, where $\langle z \rangle = G' \cap \mathrm{Z}(G)$. Since $\langle k, v \rangle' = \langle z \rangle$, $\langle k, v \rangle$ is minimal nonabelian and so $\langle k, v \rangle = M_1$. In particular,

$$\Phi(\langle k, v \rangle) = \langle k^p, v^p, [v, k] \rangle = \langle k^p, z \rangle = \Phi(M_1) = \mathrm{Z}(G).$$

Thus $\langle k^p \rangle$ covers $\mathrm{Z}(G)/\langle z \rangle$, where $|\mathrm{Z}(G)| \geq p^2$. Set $\mathrm{Z}(G)/\langle z \rangle \cong \Phi(G)/G' \cong \mathrm{C}_{p^n}$ with $n \geq 1$ so that $|G| = p^{n+4}$. Consider the abelian group $G/G'$ of rank 2. Since $(G'\langle k \rangle)/G' \cong \mathrm{C}_{p^{n+1}}$, there is a subgroup $S/G'$ of order $p$ such that $G = S\langle k \rangle$ and $S \cap \langle k \rangle \leq \langle z \rangle$. Let $h \in S - G'$ so that $h^p \in \langle z \rangle$ since $h^p \in \mathrm{Z}(G) \cap G' = \langle z \rangle$.

Assume that $S \leq A$ in which case $h \in A - \Phi(G)$ and $G = \langle h, k \rangle$. We may assume $[h, k] = v$ and we examine all maximal subgroups $\Phi(G)\langle h^i k \rangle$ of $G$ ($i$ is any integer mod $p$) which are distinct from $A$. We have $[v, h^i k] = [v, k][v, h^i]^k = [v, k] = z$ and so $\langle v, h^i k \rangle$ is minimal nonabelian. On the other hand,

$$\Phi(\langle v, h^i k \rangle) = \langle v^p = 1, (h^i k)^p = h^{ip} k^p l, [v, h^i k] = z \rangle = \langle k^p, z \rangle = \mathrm{Z}(G),$$

(where $l \in \langle z \rangle$) since $(h^i k)^p = k^p l'$ for some $l' \in \langle z \rangle$. This means that $\Phi(G)\langle h^i k \rangle = \langle v, h^i k \rangle$ and so all these $p$ maximal subgroups of $G$ are minimal nonabelian. But then $G$ is an $\mathrm{A}_2$-group, a contradiction.

We have proved that $S \not\leq A = \mathrm{C}_G(G')$ and so $1 \neq [v, h] \in \langle z \rangle$. Since $G = \langle h, k \rangle$, we may set $[h, k] = v$ and $[v, k] = z$, where $v \in G' - \mathrm{Z}(G)$ and $\langle z \rangle = G' \cap \mathrm{Z}(G)$. Also, $[v, h] = z^\rho$, $h^p = z^\sigma$, and $k^{p^{n+1}} = z^\tau$, where $\rho, \sigma, \tau$ are integers mod $p$ with $\rho \not\equiv 0 \pmod{p}$. Here $S = \langle v, h \rangle \cong \mathrm{S}(p^3)$ or $\mathrm{M}_{p^3}$, $S$ is normal in $G$, $G = S\langle k \rangle$ with $\langle k \rangle \cap S \leq \langle z \rangle$ and $M = S\mathrm{Z}(G) = S\langle k^p \rangle$ with $\mathrm{d}(M) = 3$.

It remains to examine all $p$ maximal subgroups $\Phi(G)\langle h^i k \rangle$ ($i = 0, 1, ..., p-1$) of $G$ which are distinct from $M = \Phi(G)\langle h \rangle = S\langle k^p \rangle$. We compute $[v, h^i k] = [v, k][v, h]^i = z z^{\rho i} = z^{\rho i + 1}$, where the congruence $\rho i + 1 \equiv 0$ (mod $p$) has exactly one solution $i'$ for $i$ (noting that $\rho \not\equiv 0 \pmod{p}$). Hence $A = \Phi(G)\langle h^{i'} k \rangle$ is an abelian maximal subgroup of $G$ and for all other $i \not\equiv i'$ (mod $p$), we see that $\langle v, h^i k \rangle$ is minimal nonabelian and moreover,

$$\Phi(\langle v, h^i k \rangle) = \langle v^p = 1, (h^i k)^p = h^{ip} k^p l, [v, h^i k] = z^{\rho i + 1} \neq 1 \rangle$$

for some $l \in \langle z \rangle$. Hence $\Phi(\langle v, a^i k \rangle) = \langle k^p, z \rangle = \mathrm{Z}(G)$ and so $\Phi(G)\langle h^i k \rangle = \langle v, h^i k \rangle$ is a minimal nonabelian maximal subgroup of $G$. Our theorem is proved. □

Theorem 9. *Let $G$ be a two-generator $p$-group, $p > 2$, with exactly one maximal subgroup $H$ which is neither abelian nor minimal nonabelian. If $G$ has no abelian maximal subgroup, then $\Gamma_1 = \{H, H_1, ..., H_p\}$, where $H_i$ ($i = 1, ..., p$) are non-metacyclic minimal nonabelian, $G' \cong \mathrm{E}_{p^3}$, $W = [G, G'] \cong \mathrm{E}_{p^2}$, $W \leq \mathrm{Z}(G)$ (and so $G$ is of class 3) and $\mathrm{C}_G(G') = \Phi(G)$ is abelian.*

*Moreover,* $\{H', H'_1, ..., H'_p\}$ *is the set of* $p+1$ *subgroups of order* $p$ *in* $W$ *and the following holds.*

(a) *If* $|G| \geq p^6$*, then we have:*

$$G = \langle h, x \mid h^{p^{m+1}} = 1,\ [h,x] = v,\ h^{p^m} = u,\ [v,h] = z,\ [v,x] = u^{\alpha},$$
$$v^p = z^p = [u,x] = [z,h] = [z,x] = 1,\ x^p \in \langle u, z\rangle\rangle,$$

*where* $m \geq 2$ *and* $\alpha$ *is an integer* mod $p$ *with* $\alpha \not\equiv 0 \,(\mathrm{mod}\ p)$*. Here* $|G| = p^{m+4}$*,* $G' = \langle u, z, v\rangle \cong \mathrm{E}_{p^3}$*,* $W = [G, G'] = \langle u, z\rangle \leq \mathrm{Z}(G)$*,* $\mathrm{Z}(G) = \langle h^p\rangle \times \langle z\rangle \cong \mathrm{C}_{p^m} \times \mathrm{C}_p$*,* $\Phi(G) = \mathrm{Z}(G) \times \langle v\rangle$*. Finally,* $H = \Phi(G)\langle x\rangle$*, where in case* $x^p \in W - \langle u\rangle$ *we have* $\mathrm{d}(H) = 3$ *and in case* $x^p \in \langle u\rangle$ *we have* $\mathrm{d}(H) = 4$ *and* $H_i = \langle v, x^i h\rangle$ *(*$i = 1, ..., p$*) is the set of* $p$ *non-metacyclic minimal nonabelian maximal subgroups of* $G$*.*

(b) *If* $|G| = p^5$*, then:*

$$G = \langle h, x \mid h^{p^2} = 1,\ [h,x] = v,\ h^p = u^{\alpha},\ [v,h] = z,\ [v,x] = u,$$
$$v^p = z^p = [u,x] = [z,h] = [z,x] = 1,\ h^p = u^{\beta} z^{\gamma}\rangle,$$

*where* $\alpha, \beta, \gamma$ *are integers* mod $p$ *with* $\beta \not\equiv 0 \,(\mathrm{mod}\ p)$*. We have* $\Phi(G) = G' = \langle u, z, v\rangle \cong \mathrm{E}_{p^3}$ *and* $W = [G, G'] = \langle u, z\rangle = \mathrm{Z}(G) \cong \mathrm{E}_{p^2}$*.*

*If* $p \geq 5$*, then* $\gamma \equiv \alpha$ $(\mathrm{mod}\ p)$*. In that case* $\alpha \equiv 0$ $(\mathrm{mod}\ p)$ *implies* $\mho_1(G) = \langle u\rangle$ *and* $\Omega_1(G) = H \cong \mathrm{S}(p^3) \times \mathrm{C}_p$ *and* $\alpha \not\equiv 0$ $(\mathrm{mod}\ p)$ *implies* $\mho_1(G) = W$*,* $\Omega_1(G) = G'$ *and* $H \cong \mathrm{M}_{p^3} \times \mathrm{C}_p$*. Also, all* $p$ *maximal subgroups* $H_i = G'\langle x^i h\rangle$ *(i integer mod p) are non-metacyclic minimal nonabelian.*

*If* $p = 3$*, then either* $\beta = 1$ *and* $\gamma \not\equiv \alpha$ $(\mathrm{mod}\ 3)$ *or* $\beta = -1$ *and* $\gamma \equiv \alpha$ $(\mathrm{mod}\ 3)$*. In that case* $H = G'\langle x\rangle \cong \mathrm{S}(27) \times \mathrm{C}_3$ *or* $H \cong \mathrm{M}_{27} \times \mathrm{C}_3$ *and all* 3 *maximal subgroups* $H_i = G'\langle x^i h\rangle$ *(i integer mod* 3*) are non-metacyclic minimal nonabelian.*

Proof. We set $\Gamma_1 = \{H, H_1, ..., H_p\}$, where $H_i$ $(i = 1, ..., p)$ are minimal nonabelian. Since $H$ is neither abelian nor minimal nonabelian, $|H| \geq p^4$ and so $|G| \geq p^5$.

First suppose that two distinct minimal nonabelian maximal subgroups of $G$ have the same commutator subgroup, say, $H'_1 = H'_2$. Then considering $G/H'_1$ (see Exercise 1.6(a)), we see that all maximal subgroups of $G/H'_1$ are abelian and so we get $H' = H'_1 = ... = H'_p = \langle z\rangle \cong \mathrm{C}_p$. By a result of A. Mann (see Exercise 1.69(a) in [1]), $|G' : (H'_1 H'_2)| = |G' : H'_1| \leq p$ and so $|G'| \leq p^2$. But if $|G'| = p$, then this fact together with $\mathrm{d}(G) = 2$ implies (see Exercise P9 in [1]) that $G$ is minimal nonabelian, a contradiction. Hence $|G'| = p^2$. Also, $\mathrm{d}(H) \geq 3$ and so $H$ is non-metacyclic. Indeed, if $\mathrm{d}(H) = 2$, then (noting that $|H'| = p$) $H$ would be minimal nonabelian, a contradiction.

Suppose for a moment that $G' = \langle v\rangle \cong \mathrm{C}_{p^2}$ is cyclic. Then $H'_1 = ... = H'_p = \langle v^2\rangle$ and $G' = \langle v\rangle \leq H_i$ so that $H'_i$ is not a maximal cyclic subgroup in $H_i$ and therefore $H_i$ is metacyclic for all $i = 1, ..., p$. By a result of Y.

Berkovich (see A.40.12 in [3]), $G$ is an $\mathrm{L}_3$-group. But in that case, $G'$ is of exponent $p$, a contradiction. We have proved that $G' \cong \mathrm{E}_{p^2}$.

We have that $\Phi(G) = H_1 \cap H_2$ is a maximal normal abelian subgroup of $G$. Taking $h_1 \in H_1 - \Phi(G)$ and $h_2 \in H_2 - \Phi(G)$, we have $\langle h_1, h_2\rangle = G$ and so $s = [h_1, h_2] \in G' - \langle z\rangle$ and $s \notin \mathrm{Z}(G)$. Indeed, if $s \in \mathrm{Z}(G)$, then $G/\langle s\rangle$ would be abelian, a contradiction. In particular, $[G, G'] = \langle z\rangle = H_1'$ and so $G$ is of class 3. Since $s \notin \mathrm{Z}(G)$, we have $s \notin \mathrm{Z}(H_1)$ or $s \notin \mathrm{Z}(H_2)$ and we may assume without loss of generality that $s \notin \mathrm{Z}(H_1)$. Suppose that there is an element $x \in H_1 - \Phi(G)$ such that $x^p \in \langle z\rangle$. Then $G'\langle x\rangle$ is minimal nonabelian of order $p^3$ and so $G'\langle x\rangle = H_1$, contrary to $|G| \geq p^5$. Assume that $\langle z\rangle = H_1'$ is not a maximal cyclic subgroup in $H_1$. Then there is $v \in \Phi(G)$ such that $v^2 = z$. This implies that all $H_i$, $i = 1, ..., p$, are metacyclic. Again by a result of Y. Berkovich (A.40.12 in [3]), $G$ is an $\mathrm{L}_3$-group. This means that $U = \Omega_1(G)$ is of order $p^3$ and exponent $p$ and $G/U$ is cyclic of order $\geq p^2$. We have $G' \leq U$ and so if $U$ is nonabelian, then $\mathrm{C}_G(G')$ covers $G/U$ and $\mathrm{C}_G(G')$ is an abelian maximal subgroup of $G$, a contradiction. If $U$ is elementary abelian, then $|G : \mathrm{C}_G(U)| = p$ since a Sylow $p$-subgroup of $\mathrm{GL}_3(p)$ is isomorphic to $\mathrm{S}(p^3)$ and so is of exponent $p$. But in that case $\mathrm{C}_G(U)$ is an abelian maximal subgroup of $G$, a contradiction. Hence $\langle z\rangle = H_1'$ is a maximal cyclic subgroup in $H_1$ which implies that $H_1$ is non-metacyclic. Noting that $|H_1| \geq p^4$, we get $E = \Omega_1(H_1) = \Omega_1(\Phi(G)) \cong \mathrm{E}_{p^3}$ which also implies that all $H_i$ are non-metacyclic.

By the previous paragraph, $\Omega_1(\langle h_1\rangle) = \langle u\rangle \leq E$ and $u \in E - G'$. We have $H_1 = E\langle h_1\rangle$ and so $H_1$ is a splitting extension of $G'$ by $\langle h_1\rangle$, where $o(h_1) = p^n$, $n \geq 2$. Since $G/G'$ is abelian of rank 2, we get $G = H_1F$ with $H_1 \cap F = G'$ and $|F : G'| = p$. We have $G/F \cong H_1/G' \cong \mathrm{C}_{p^n}$. If $F$ is nonabelian, then $\mathrm{C}_G(G')$ covers $G/F$ and so $\mathrm{C}_G(G')$ is an abelian maximal subgroup of $G$, a contradiction. Hence $F$ is abelian. Assume that $\mho_1(F) \nleq \langle z\rangle$. Then $|\mho_1(F)| = p$ and $G' = \langle z\rangle \times \mho_1(F) \leq \mathrm{Z}(G)$, a contradiction. Hence $\mho_1(F) \leq \langle z\rangle$ and so for an element $x \in F - G'$ we have $x^p \in \langle z\rangle$.

Since $G = \langle h_1, x\rangle$, we may set $[x, h_1] = s \in G' - \langle z\rangle$ and $[s, h_1] = z$, where $H_1' = \langle z\rangle \leq \mathrm{Z}(G)$. Then $x^{h_1} = xs$, $s^{h_1} = sz$ and $s^{h_1^i} = sz^i$ for all $i \geq 1$. We get $x^{h_1^2} = (xs)^{h_1} = (xs)(sz) = xs^2z$ and claim that we have $x^{h_1^i} = xs^iz^{\binom{i}{2}}$ for all $i \geq 2$. Indeed, by induction on $i$,

$$x^{h_1^{i+1}} = (x^{h_1})^{h_1^i} = (xs)^{h_1^i} = (xs^iz^{\binom{i}{2}})(sz^i) = xs^{i+1}z^{\binom{i}{2}+i} = xs^{i+1}z^{\binom{i+1}{2}}.$$

Our formula gives $x^{h_1^p} = xs^pz^{\binom{p}{2}} = x$ and so $F\langle h_1^p\rangle$ is an abelian maximal subgroup of $G$, a contradiction.

We have proved that $H_1' = \langle z_1\rangle$, $H_2' = \langle z_2\rangle$,...,$H_p' = \langle z_p\rangle$ are pairwise distinct subgroups of order $p$ in $G' \cap \mathrm{Z}(G)$. By a result of A. Mann (Exercise 1.69(a) in [1]), $|G' : (H_1'H_2')| \leq p$ and so $|G'| \leq p^3$. Set $W = \langle z_1, ..., z_p\rangle$ so that $W$ is an elementary abelian subgroup of order $\geq p^2$ contained in $G' \cap \mathrm{Z}(G)$

which implies that $G'$ is abelian of exponent $\leq p^2$. We have $G = \langle x, y \rangle$ for some $x, y \in G$. If $[x, y] \in \mathrm{Z}(G)$, then $G/\langle [x, y] \rangle$ is abelian which implies that $G' = \langle [x, y] \rangle$ is cyclic, contrary to the fact that $W \leq G'$. Thus $[x, y] \in G' - W$ which gives $|G'| = p^3$, $W \cong \mathrm{E}_{p^2}$, $\{1\} \neq [G, G'] \leq W \leq \mathrm{Z}(G)$ and so $G$ is of class 3. Let $\langle z_{p+1} \rangle$ be the subgroup of order $p$ in $W$ such that $\langle z_{p+1} \rangle \neq \langle z_i \rangle$ for all $i = 1, ..., p$.

For any fixed $i \in \{1, ..., p\}$ we consider $G/\langle z_i \rangle$, where $H_i/\langle z_i \rangle$ is abelian (and two-generated) and $H_j/\langle z_i \rangle$ is minimal nonabelian for all $j \neq i$, $j \in \{1, ..., p\}$. This implies that $H/\langle z_i \rangle$ must be nonabelian (Exercise 1.6(a) in [1]). If $G/\langle z_i \rangle$ is metacyclic, then a result of N. Blackburn (Theorem 36.1 in [1]) gives that $G$ is also metacyclic, contrary to $\mathrm{E}_{p^2} \cong W \leq G'$. Hence $G/\langle z_i \rangle$ is non-metacyclic. Suppose that $H/\langle z_i \rangle$ is minimal nonabelian. Then $G/\langle z_i \rangle$ is a non-metacyclic $\mathrm{A}_2$-group. If $|G/\langle z_i \rangle| > p^4$, then Theorem 65.7(a) in [2] implies that $G'/\langle z_i \rangle \cong \mathrm{E}_{p^2}$ . Suppose that $|G/\langle z_i \rangle| = p^4$ and $G'/\langle z_i \rangle \cong \mathrm{C}_{p^2}$. In that case each maximal subgroup of $G/\langle z_i \rangle$ is metacyclic and so $G/\langle z_i \rangle$ is minimal non-metacyclic. By Theorem 69.1 in [2], $G/\langle z_i \rangle$ is a group of maximal class and order $3^4$. But in that case $G'/\langle z_i \rangle$ cannot be cyclic (see Exercise 9.1(c)in [1]). We have proved that in any case $G'/\langle z_i \rangle \cong \mathrm{E}_{p^2}$. Assume now that $H/\langle z_i \rangle$ is not minimal nonabelian and we know already that $H/\langle z_i \rangle$ is nonabelian. By Theorem 8, we have again $G'/\langle z_i \rangle \cong \mathrm{E}_{p^2}$. As a consequence we get $[x, y]^p \in \langle z_i \rangle$ for each $i = 1, ..., p$ which implies $[x, y]^p = 1$ and so $G' \cong \mathrm{E}_{p^3}$ is elementary abelian.

By Lemma 36.5(b) in [1], $G'/[G, G']$ is cyclic and so $[G, G'] = W$ (since $[G, G'] \leq W$). Since $(G/W)' \cong \mathrm{C}_p$ and $\mathrm{d}(G/W) = 2$, $G/W$ is minimal nonabelian (Exercise P9 in [3]) and so $H/W$ is abelian which implies $\{1\} \neq H' \leq W$. Suppose that $H' = \langle z_j \rangle$ for some $j \in \{1, ..., p\}$. Then $G/\langle z_j \rangle$ is nonabelian with at least two distinct abelian maximal subgroups $H_j/\langle z_j \rangle$ and $H/\langle z_j \rangle$. But then $(G/\langle z_j \rangle)' \cong \mathrm{C}_p$ (Exercise P1 in [3]), a contradiction. We have proved that $H' = \langle z_{p+1} \rangle$ or $H' = W$.

Suppose that $H' = W \leq \mathrm{Z}(G)$. In that case $\mathrm{d}(H) \geq 3$. Indeed, if $\mathrm{d}(H) = 2$, then $H$ is a two-generator group of class 2 in which case $H'$ must be cyclic (Proposition 36.5(a) in [1]), a contradiction. Consider $G/\langle z_{p+1} \rangle$ with $\mathrm{d}(G/\langle z_{p+1} \rangle) = 2$ and having minimal nonabelian maximal subgroups $H_i/\langle z_{p+1} \rangle$ for all $i = 1, ..., p$. The remaining maximal subgroup $H/\langle z_{p+1} \rangle$ is neither abelian nor minimal nonabelian since $\mathrm{d}(H/\langle z_{p+1} \rangle) \geq 3$. But $(H_i/\langle z_{p+1} \rangle)' = W/\langle z_{p+1} \rangle$ for all $i = 1, ..., p$, contrary to the first part of this proof. Hence we must have $H' = \langle z_{p+1} \rangle$.

We have proved that $H'$, $H_1'$,...,$H_p'$ are $p + 1$ pairwise distinct subgroups of order $p$ in $W$. Since $H$ is not minimal nonabelian, we have $\mathrm{d}(H) \geq 3$ and so $|H| \geq p^4$ and $|G| \geq p^5$. Also, $\Omega_1(H_i) = G' \leq \Phi(G)$ for all $i = 1, ..., p$, where $\Phi(G)$ is abelian. Therefore we have either $\mathrm{C}_G(G') = \Phi(G)$ or $\mathrm{C}_G(G')$ is a maximal subgroup of $G$. In any case there exist two minimal nonabelian maximal subgroups of $G$, say, $H_1$ and $H_2$, such that $G' \not\leq \mathrm{Z}(H_1)$

and $G' \not\leq \mathrm{Z}(H_2)$. Then $H_1 \cap H_2 = \Phi(G)$ and taking some elements $h_1 \in H_1 - \Phi(G)$ and $h_2 \in H_2 - \Phi(G)$, we have $\langle h_1, h_2\rangle = G$ and so $v = [h_1, h_2] \in G' - W$. Indeed, if $v \in W$, then $G/\langle v\rangle$ is abelian, a contradiction. We may set $[v, h_1] = z_1$ and $[v, h_2] = z_2$ so that $H_1' = \langle z_1\rangle$, $H_2' = \langle z_2\rangle$ and $W = \langle z_1\rangle \times \langle z_2\rangle$. All maximal subgroups of $G$ are $H_1 = \Phi(G)\langle h_1\rangle$ and $\Phi(G)\langle h_1^i h_2\rangle$, where $i$ is any integer mod $p$. We compute:

$$[v, h_1^i h_2] = [v, h_2][v, h_1^i]^{h_2} = z_2(z_1^i)^{h_2} = z_1^i z_2 \neq 1,$$

which shows that $\mathrm{C}_G(v) = \Phi(G)$ and so $\mathrm{C}_G(G') = \Phi(G)$. Since $\langle v, h_1\rangle$ (with $[v, h_1] = z_1$) is minimal nonabelian, we have $\langle v, h_1\rangle = H_1 = G'\langle h_1\rangle$. Hence $H_1/G' \cong \mathrm{C}_{p^m}$ is cyclic of order $p^m$, $m \geq 1$, and $h_1^{p^m} \in W - \langle z_1\rangle$. The abelian group $G/G'$ is of rank 2 and so $G/G'$ is of type $(p^m, p)$ and $|G| = p^{m+4}$. Finally, $\Phi(G) = G'\langle h_1^p\rangle = \langle h_1^p\rangle \times \langle v\rangle \times \langle z_1\rangle$ is of type $(p^m, p, p)$.

(i) First suppose that $m \geq 2$. Set $u = h_1^{p^m}$ so that $u \in W - H_1'$, $\mathrm{o}(h_1) = p^{m+1} \geq p^3$ and $\Phi(G)/G'$ is cyclic of order $p^{m-1} \geq p$ since $H_1/G' \cong \mathrm{C}_{p^m}$. Consider any $H_i$ for $2 \leq i \leq p$ so that $H_i \cap H_1 = \Phi(G)$. Let $h_i \in H_i - \Phi(G)$ and $v \in G' - W$ so that $1 \neq [h_i, v] = z_i$ and $H_i' = \langle z_i\rangle$. Since $\langle h_i, v\rangle$ is minimal nonabelian, we have $\langle h_i, v\rangle = H_i = G'\langle h_i\rangle$ and so $H_i/G'$ is also cyclic of order $p^m$. We have $h_i^p \in \Phi(G) - G'$ and $\langle h_i^p\rangle$ covers $\Phi(G)/G'$. It follows $h_i^p = h_1^{\delta p}k$ for some $k \in G'$ and $\delta \not\equiv 0 \pmod p$. Then $h_i^{p^2} = h_1^{\delta p^2}$ and so $\langle h_i^{p^m}\rangle = \langle u\rangle$, where $u = h_1^{p^m}$ and this implies that $u \in W - H_i'$. We have proved that $u \notin H_i'$ for all $i = 1, ..., p$ which forces $\langle u\rangle = H'$.

Since $G/G'$ is abelian of type $(p^m, p)$, $m \geq 2$, we get $G = H_1F$ with $H_1 \cap F = G'$ and $|F : G'| = p$. For the maximal subgroup $\Phi(G)F$ of $G$ we have $(\Phi(G)F)/G' = (\Phi(G)/G') \times (F/G') \cong \mathrm{C}_{p^{m-1}} \times \mathrm{C}_p$ and so $(\Phi(G)F)/G'$ is not cyclic which implies that $\Phi(G)F = H$. Taking $h_1 = h \in H_1 - \Phi(G)$ and $x \in F - \Phi(G)$, we have $\mathrm{o}(x) \leq p^2$, $G = \langle h, x\rangle$ and so $v = [h, x] \in G' - W$. We set again $u = h^{p^m}$ and we know that $H' = \langle u\rangle$. Also set $[v, h] = z_1 = z$, where $H_1' = \langle z\rangle$, $W = \langle u\rangle \times \langle z\rangle$, $v^h = vz$ and $v^{h^j} = vz^j$ for $j \geq 1$. Since $\mathrm{C}_G(G') = \mathrm{C}_G(v) = \Phi(G)$, we have $x^p \in W = \langle u, z\rangle$ and $[v, x] = u^\alpha$ with $\alpha \not\equiv 0 \pmod p$. We have obtained all relations stated in part (a) of our theorem. From $[h, x] = v$, we get $[h^2, x] = [h, x]^h[h, x] = v^h v = (vz)v = v^2 z$. We prove by induction on $j \geq 2$ that $[h^j, x] = v^j z^{\binom{j}{2}}$. Indeed,

$$[h^{j+1}, x] = [hh^j, x] = [h, x]^{h^j}[h^j, x] = v^{h^j}(v^j z^{\binom{j}{2}}) = (vz^j)(v^j z^{\binom{j}{2}}) =$$

$$v^{j+1} z^{j+\binom{j}{2}} = v^{j+1} z^{\binom{j+1}{2}}.$$

In particular, $[h^p, x] = v^p z^{\binom{p}{2}} = 1$ and so $h^p \in \mathrm{Z}(G)$ since $G = \langle h, x\rangle$. We get $\mathrm{Z}(G) = \langle h^p\rangle \times \langle z\rangle \cong \mathrm{C}_{p^m} \times \mathrm{C}_p$ and $\Phi(G) = \mathrm{Z}(G) \times \langle v\rangle$. We have $H = F * \langle h^p\rangle$ with $F \cap \langle h^p\rangle = \langle u\rangle$. If $x^p \in W - \langle u\rangle$, then $F$ is minimal nonabelian and so $\mathrm{d}(H) = 3$. If $x^p \in \langle u\rangle$, then $F = \langle v, x\rangle \times \langle z\rangle$, where $\langle u\rangle = \mathrm{Z}(\langle v, x\rangle)$ and $\langle v, x\rangle \cong \mathrm{S}(p^3)$ or $\mathrm{M}_{p^3}$ and so $\mathrm{d}(H) = 4$.

Finally, we have to check all $p$ maximal subgroups $H_i = \Phi(G)\langle x^i h\rangle$ of $G$ which are distinct from $H$ and we have to show that they are minimal nonabelian. We have

$$[v, x^i h] = [v, h][v, x^i]^h = z(u^{\alpha i})^h = zu^{\alpha i},$$

where $\langle zu^{\alpha i}\rangle$ are pairwise distinct subgroups of order $p$ in $W$ for $i = 1, ..., p$ since $\alpha \not\equiv 0 \pmod p$. Hence $\langle v, x^i h\rangle$ is minimal nonabelian with $\langle v, x^i h\rangle' = \langle zu^{\alpha i}\rangle \neq \langle u\rangle$. By Hall-Petrescu formula (Appendix 1 in [1]), we get for all $r, s \in G$,

$$(rs)^{p^m} = r^{p^m} s^{p^m} c_2^{\binom{p^m}{2}} c_3^{\binom{p^m}{3}},$$

where $c_2 \in G'$ and $c_3 \in W = [G, G']$. But $m \geq 2$ and so $(rs)^{p^m} = r^{p^m} s^{p^m}$. We get $(x^i h)^{p^m} = (x^i)^{p^m} h^{p^m} = h^{p^m} = u$ and so $\mathrm{o}(x^i h) = p^{m+1}$ which together with $\langle v, x^i h\rangle = G'\langle x^i h\rangle$ and $G' \cap \langle x^i h\rangle = \langle u\rangle$ implies that $|\langle v, x^i h\rangle| = p^{m+3}$. But $|G| = p^{m+4}$ and so $\langle v, x^i h\rangle = H_i$ is minimal nonabelian and we are done.

(ii) Suppose that $m = 1$ which implies $|G| = p^5$ and $G' = \Phi(G)$ with $\mathrm{C}_G(G') = G'$. We have $H_1 \cap H = G'$ and since $\Omega_1(H_1) = G'$, we get for an element $h \in H_1 - G'$, $1 \neq h^p \in W = [G, G'] = \mathrm{Z}(G) \cong \mathrm{E}_{p^2}$. Also we have $\mho_1(G) \leq W$. Take an element $x \in H - G'$ so that $x^p \in W$, $G = \langle h, x\rangle$ and $v = [h, x] = G' - W$. Set $[v, x] = u$ so that $1 \neq u \in W$ and $\langle u\rangle = H'$. Then $[v, h] = z \notin \langle u\rangle$ and so $H_1' = \langle z\rangle$ and $W = \langle u\rangle \times \langle z\rangle$. Since $\langle v, x\rangle$ is minimal nonabelian, we have $\langle v, x\rangle \neq H$ which implies $x^p = u^\alpha$ (for some integer $\alpha$ mod $p$) and $H = \langle v, x\rangle \times \langle z\rangle$ , where $\langle v, x\rangle \cong \mathrm{S}(p^3)$ or $\mathrm{M}_{p^3}$. Since $H_1$ is non-metacyclic minimal nonabelian, $\langle z\rangle$ is a maximal cyclic subgroup in $H_1$ which implies $h^p = u^\beta z^\gamma$ with $\beta \not\equiv 0 \pmod p$. All $p$ maximal subgroups $H_i = G'\langle x^i h\rangle$ ($i$ is any integer mod $p$) of $G$ which are distinct from $H$ must be minimal nonabelian. Since $[v, x^i h] = [v, h][v, x^i]^h = z(u^i)^h = zu^i \neq 1$ and $\langle v, x^i h\rangle$ is minimal nonabelian with $\langle v, x^i h\rangle' = \langle u^i z\rangle$ and so we must have $H_i = \langle v, x^i h\rangle$, we get $\langle (x^i h)^p\rangle \neq \langle u^i z\rangle$ or equivalently

$$(*) \qquad \langle (x^i h)^p, u^i z\rangle = \langle u, z\rangle$$

for all integers $i$ mod $p$.

(ii1) First we assume $p \geq 5$ in which case $G$ is regular. By Hall-Petrescu formula (Appendix 1 in [1]), we have in our case for all $r, s \in G$,

$$(rs)^p = r^p s^p c_2^{\binom{p}{2}} c_3^{\binom{p}{3}},$$

where $c_2 \in G'$ and $c_3 \in W = [G, G']$ and so $(rs)^p = r^p s^p$. Hence $(x^i h)^p = x^{pi} h^p = u^{\alpha i}(u^\beta z^\gamma) = u^{\alpha i + \beta} z^\gamma$. Our condition $(*)$ is equivalent with:

$$\begin{vmatrix} \alpha i + \beta & \gamma \\ i & 1 \end{vmatrix} = (\alpha - \gamma)i + \beta \not\equiv 0 \pmod p$$

for all integers $i$ mod $p$, where we know that $\beta \not\equiv 0$ (mod). This is equivalent with $\alpha - \gamma \equiv 0 \pmod p$ and so $\gamma \equiv \alpha \pmod p$. If $\alpha \equiv 0 \pmod p$, then

$\mho_1(G) = \langle u \rangle$ and $\Omega_1(G) = H \cong \mathrm{S}(p^3) \times \mathrm{C}_p$. If $\alpha \not\equiv 0 \pmod p$, then $\mho_1(G) = W$ and $\Omega_1(G) = G'$ so that $H \cong \mathrm{M}_{p^3} \times \mathrm{C}_p$.

(ii2) Finally, we suppose $p = 3$. In that case the Hall-Petrescu formula gives for all $r, s \in G$, $(rs)^3 = r^3 s^3 c_3$, where $c_3 \in W$. This is not a sufficient information because we have to know exactly the element $c_3$. Using the usual commutator identities (see §7, p. 98) together with $xy = yx[x, y]$ we compute exactly $(rs)^3$. We get:

$$\begin{aligned}(rs)^2 &= r(sr)s = r(rs[s,r])s = r^2 s(s[s,r][s,r,s]) = r^2 s^2 [s,r][s,r,s],\\ (rs)^3 &= r^2 s^2 [s,r][s,r,s] \cdot rs = r^2 s^2 \cdot r[s,r]s[s,r,r][s,r,s]\\ &= r^2(s^2 r)s[s,r][s,r,s][s,r,r][s,r,s].\end{aligned}$$

But we have:

$$\begin{aligned}r^2(s^2 r)s[s,r] &= r^2(rs^2[s^2,r])s[s,r] = r^3 s^3 [s^2,r][s^2,r,s][s,r]\\ &= r^3 s^3 [s,r][s,r]^s [s^2,r,s][s,r]\\ &= r^3 s^3 [s,r]([s,r][s,r,s])[s^2,r,s][s,r] = r^3 s^3 [s,r,s][s^2,r,s]\end{aligned}$$

and so

$$(1)\qquad (rs)^3 = r^3 s^3 [s,r,s][s^2,r,s][s,r,s][s,r,r][s,r,s] = r^3 s^3 [s^2,r,s][s,r,r].$$

Also we get:

$$[s^2, r] = [s,r]^s [s,r] = [s,r][s,r,s][s,r] = [s,r]^2 [s,r,s]$$

and so:

$$[s^2,r,s] = [[s,r]^2[s,r,s], s] = [[s,r]^2, s] = [s,r,s]^{[s,r]}[s,r,s] = [s,r,s]^2,$$

and so we have by (1):

$$(rs)^3 = r^3 s^3 [s,r,s]^2 [s,r,r] = r^3 s^3 [s,r,s]^{-1}[s,r,r] = r^3 s^3 [s,[s,r]][[s,r],r],$$

and so we have obtained the formula:

$$(**)\qquad (rs)^3 = r^3 s^3 [s,[s,r]]\,[[s,r],r].$$

Since $\langle h^p, z \rangle = \langle u^\beta z^\gamma, z \rangle = \langle u, z \rangle$ (noting that $\beta \not\equiv 0 \pmod 3$),we have to use our condition $(*)$ only for $i = 1, 2$. By $(**)$, $(xh)^3 = x^3 h^3 [h,[h,x]]\,[[h,x],x] = u^\alpha u^\beta z^\gamma [h,v][v,x] = u^{\alpha+\beta+1} z^{\gamma-1}$, and so from $\langle u^{\alpha+\beta+1} z^{\gamma-1}, uz \rangle = \langle u, z \rangle$, we get

$$(2)\qquad \begin{vmatrix} \alpha+\beta+1 & \gamma-1 \\ 1 & 1 \end{vmatrix} = \alpha+\beta-\gamma-1 \not\equiv 0 \pmod 3.$$

We compute $[h, x^2] = [h,x][h,x]^x = vv^x = v(vu) = v^2 u$ and so by $(**)$,

$$(x^2 h)^3 = x^6 h^3 [h, v^2 u][v^2 u, x^2] = u^{2\alpha}(u^\beta z^\gamma) z^{-2} u = u^{-\alpha+\beta+1} z^{\gamma+1}.$$

From our condition $(*)$ for $i = 2$, we get $\langle u^{-\alpha+\beta+1}z^{\gamma+1}, u^2 z\rangle = \langle u, z\rangle$, or equivalently:

$$\begin{vmatrix} -\alpha+\beta+1 & \gamma+1 \\ -1 & 1 \end{vmatrix} = -\alpha+\beta+\gamma-1 \not\equiv 0 \pmod 3. \tag{3}$$

Now, (2) and (3) hold if and only if:

$$(\alpha+\beta-\gamma-1)(-\alpha+\beta+\gamma-1) \not\equiv 0 \pmod 3.$$

This is equivalent with:

$$((\beta-1)+(\alpha-\gamma))((\beta-1)-(\alpha-\gamma)) \not\equiv 0 \pmod 3 \quad \text{or}$$
$$(\beta-1)^2-(\alpha-\gamma)^2 \not\equiv 0 \pmod 3.$$

Hence if $\beta = 1$, then $\gamma \not\equiv \alpha \pmod 3$ and if $\beta = -1$, then $\gamma \equiv \alpha \pmod 3$.

We have obtained the groups stated in part (b) of our theorem which is now completely proved. □

## References

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