GLASNIK MATEMATIČKI
Vol. 45(65)(2010), 475 – 503

VECTOR-VALUED INEQUALITIES ON HERZ SPACES AND CHARACTERIZATIONS OF HERZ–SOBOLEV SPACES WITH VARIABLE EXPONENT

Mitsuo Izuki
Hokkaido University, Japan

Abstract. Our first aim in this paper is to prove the vector-valued inequalities for some sublinear operators on Herz spaces with variable exponent. As an application, we obtain some equivalent norms and wavelet characterization of Herz–Sobolev spaces with variable exponent.

1. Introduction

The origin of Herz spaces is the study of characterization of functions and multipliers on the classical Hardy spaces ([1, 8]). By virtue of many authors' works Herz spaces have became one of the remarkable classes of function spaces in harmonic analysis now. One of the important problems on the spaces is boundedness of sublinear operators satisfying proper conditions. Hernández, Li, Lu and Yang ([7, 17, 19]) have proved that if a sublinear operator T is bounded on $L^p(\mathbb{R}^n)$ and satisfies the size condition

$$|Tf(x)| \leq C \int_{\mathbb{R}^n} |x-y|^{-n} |f(y)| \, dy$$

for all $f \in L^1(\mathbb{R}^n)$ with compact support and a.e. $x \notin \operatorname{supp} f$, then T is bounded on both of the homogeneous Herz space $\dot{K}_p^{\alpha,q}(\mathbb{R}^n)$ and the non-homogeneous Herz space $K_p^{\alpha,q}(\mathbb{R}^n)$. This result is extended to the weighted case by Lu, Yabuta and Yang ([18]), and to the vector-valued case by Tang and Yang ([25]). As an application of [18], noting that the Hardy–Littlewood maximal operator M is a sublinear operator satisfying the assumption above,

2010 *Mathematics Subject Classification.* 42B20, 42B35, 42C40, 46B15.
Key words and phrases. Herz–Sobolev space with variable exponent, wavelet, vector-valued inequality.

Nakai, Tomita and Yabuta ([22]) have proved the density of the set of all infinitely differentiable functions with compact support in weighted Herz spaces. On the other hand, Lu and Yang ([20]) have initially introduced Herz-type Sobolev spaces and Bessel potential spaces, and proved the equivalence of them using boundedness of sublinear operators on Herz spaces. Later Xu and Yang ([27, 28]) generalize the result on Herz-type Sobolev spaces to the case of Herz-type Triebel–Lizorkin spaces.

Many results on wavelet characterization of various function spaces including Herz spaces are well-known now (cf. [5, 6, 9, 10, 12, 14, 21]). The first wavelet characterization of Herz spaces is proved by Hernández, Weiss and Yang ([6]). The author and Tachizawa ([12]) have obtained the characterizations of weighted Herz spaces applying the boundedness of sublinear operators ([18]). Recently the author ([9]) and Kopaliani ([14]) have independently proved wavelet characterization of Lebesgue spaces with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ by virtue of the extrapolation theorem due to Cruz-Uribe, Fiorenza, Martell and Pérez ([2]). Herz spaces with variable exponent are initially defined by the author in the paper [10] where he uses the Haar functions to obtain wavelet characterization of those spaces by virtue of the result on $L^{p(\cdot)}(\mathbb{R}^n)$ ([9,14]). He also gives wavelet characterization of non-homogeneous Herz–Sobolev spaces with variable exponent in terms of wavelets with proper smoothness and compact support ([11]). But the homogeneous case is not considered in [10, 11] where he follows an argument applicable only to the non-homogeneous case using local properties of wavelets.

In the present paper we will prove the vector-valued inequalities for sublinear operators on Herz spaces with variable exponent $\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$ and $K^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$. Additionally we apply the result to obtain Littlewood-Paley type and wavelet characterization of Herz–Sobolev spaces with variable exponent $\dot{K}^{\alpha,q}_{p(\cdot)}W^s(\mathbb{R}^n)$ and $K^{\alpha,q}_{p(\cdot)}W^s(\mathbb{R}^n)$. We note that our method is applicable to both the homogeneous and the non-homogeneous cases.

Let us explain the outline of this article. We first define function spaces with variable exponent in Section 2. We will state properties of variable exponent in Section 3. In Section 4 we prove the vector-valued inequalities for sublinear operators on $\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$ and $K^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$. Applying the results, we give Littlewood–Paley-type characterization of $\dot{K}^{\alpha,q}_{p(\cdot)}W^s(\mathbb{R}^n)$ and $K^{\alpha,q}_{p(\cdot)}W^s(\mathbb{R}^n)$ in Section 5. We additionally prove wavelet characterization of them in terms of the Meyer scaling function and the Meyer wavelets in Section 6.

Throughout this paper $|S|$ denotes the Lebesgue measure and χ_S means the characteristic function for a measurable set $S \subset \mathbb{R}^n$. The set of all nonnegative integers is denoted by $\mathbb{N}_0$. A symbol C always means a positive constant independent of the main parameters and may change from one occurrence to another. The Fourier transform of a function f on $\mathbb{R}^n$ is denoted by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\, dx.$$

2. Definition of function spaces with variable exponent

In this section we define some function spaces with variable exponent. Let E be a measurable set in $\mathbb{R}^n$ with $|E| > 0$.

Definition 2.1. *Let $p(\,\cdot\,) : E \to [1,\infty)$ be a measurable function.*

1. *The Lebesgue space with variable exponent $L^{p(\,\cdot\,)}(E)$ is defined by*

$$L^{p(\,\cdot\,)}(E) := \{f \text{ is measurable} : \rho_p(f/\lambda) < \infty \text{ for some constant } \lambda > 0\},$$

where $\rho_p(f) := \int_E |f(x)|^{p(x)}\, dx$.

2. *The space $L^{p(\,\cdot\,)}_{\mathrm{loc}}(E)$ is defined by*

$$L^{p(\,\cdot\,)}_{\mathrm{loc}}(E) := \left\{ f \,:\, f \in L^{p(\,\cdot\,)}(K) \text{ for all compact subsets } K \subset E \right\}.$$

The Lebesgue space $L^{p(\,\cdot\,)}(E)$ is a Banach space with the norm defined by

$$\|f\|_{L^{p(\,\cdot\,)}(E)} := \inf \{\lambda > 0 : \rho_p(f/\lambda) \le 1\}.$$

Now we define two classes of exponent functions. Given a function $f \in L^1_{\mathrm{loc}}(E)$, the Hardy–Littlewood maximal operator M is defined by

$$Mf(x) := \sup_{r>0} r^{-n} \int_{B(x,r)\cap E} |f(y)|\, dy \quad (x \in E),$$

where $B(x,r) := \{y \in \mathbb{R}^n \,:\, |x-y| < r\}$. We also use the following notation.

$$p_- := \operatorname{ess\,inf}\{p(x) : x \in E\}, \quad p_+ := \operatorname{ess\,sup}\{p(x) : x \in E\}.$$

Definition 2.2.

$$\mathcal{P}(E) := \{p(\,\cdot\,) : E \to [1,\infty) \,:\, p_- > 1 \text{ and } p_+ < \infty\}. \tag{2.1}$$

$$\mathcal{B}(E) := \left\{ p(\,\cdot\,) \in \mathcal{P}(E) \,:\, M \text{ is bounded on } L^{p(\,\cdot\,)}(E) \right\}. \tag{2.2}$$

Later we will state some properties of variable exponent.

Now we define Herz and Herz–Sobolev spaces with variable exponent. We use the following notation in order to define them. Let $l \in \mathbb{Z}$.

$$B_l := \left\{ x \in \mathbb{R}^n \,:\, |x| \le 2^l \right\}. \tag{2.3}$$

$$R_l := B_l \setminus B_{l-1}. \tag{2.4}$$

$$\chi_l := \chi_{R_l}. \tag{2.5}$$

$$\tilde{\chi}_l := \chi_{R_l} \text{ if } l \ge 1 \text{ and } \tilde{\chi}_0 := \chi_{B_0}. \tag{2.6}$$

We first define Herz spaces with variable exponent by analogy with the definition of the usual Herz spaces (cf. [6, 7, 17, 19]).

Definition 2.3. *Let $\alpha \in \mathbb{R}$, $0 < q < \infty$ and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$.*

1. *The homogeneous Herz space $\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$ is defined by*
$$\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n) := \left\{ f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) \,:\, \|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} < \infty \right\},$$
where
$$\begin{aligned} \|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} &:= \left\| \left\{ 2^{\alpha l} \|f\chi_l\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right\}_{l=-\infty}^{\infty} \right\|_{\ell^q(\mathbb{Z})} \\ &= \left(\sum_{l=-\infty}^{\infty} 2^{\alpha l q} \|f\chi_l\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^{1/q}. \end{aligned}$$

2. *The non-homogeneous Herz space $K^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$ is defined by*
$$K^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n) := \left\{ f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n) \,:\, \|f\|_{K^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} < \infty \right\},$$
where
$$\begin{aligned} \|f\|_{K^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} &:= \left\| \left\{ 2^{\alpha l} \|f\tilde{\chi}_l\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right\}_{l=0}^{\infty} \right\|_{\ell^q(\mathbb{N}_0)} \\ &= \left(\sum_{l=0}^{\infty} 2^{\alpha l q} \|f\tilde{\chi}_l\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^{1/q}. \end{aligned}$$

The Herz spaces $\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$ and $K^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$ are quasi-norm spaces with $\|\cdot\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)}$ and $\|\cdot\|_{K^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)}$ respectively.

Next we define Herz–Sobolev spaces with variable exponent.

Definition 2.4. *Let $\alpha \in \mathbb{R}$, $0 < q < \infty$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $s \in \mathbb{N}_0$.*

1. *The homogeneous Herz–Sobolev space $\dot{K}^{\alpha,q}_{p(\cdot)}W^s(\mathbb{R}^n)$ is the space of all functions $f \in \dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$ satisfying weak derivatives $D^\gamma f \in \dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$ for all $\gamma \in {\mathbb{N}_0}^n$ with $|\gamma| \le s$.*
2. *The non-homogeneous Herz–Sobolev space $K^{\alpha,q}_{p(\cdot)}W^s(\mathbb{R}^n)$ is the space of all functions $f \in K^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$ satisfying weak derivatives $D^\gamma f \in K^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$ for all $\gamma \in {\mathbb{N}_0}^n$ with $|\gamma| \le s$.*

The Herz–Sobolev spaces for constant p are initially defined by Lu and Yang ([20]). Both of $\dot{K}^{\alpha,q}_{p(\cdot)}W^s(\mathbb{R}^n)$ and $K^{\alpha,q}_{p(\cdot)}W^s(\mathbb{R}^n)$ are also quasi-norm spaces with
$$\begin{aligned} \|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}W^s(\mathbb{R}^n)} &:= \sum_{|\gamma|\le s} \|D^\gamma f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)}, \\ \|f\|_{K^{\alpha,q}_{p(\cdot)}W^s(\mathbb{R}^n)} &:= \sum_{|\gamma|\le s} \|D^\gamma f\|_{K^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} \end{aligned}$$
respectively.

3. Properties of variable exponent

Cruz-Uribe, Fiorenza and Neugebauer ([3]) and Nekvinda ([23]) proved the following sufficient conditions independently. We remark that Nekvinda ([23]) gave a more general condition in place of (3.2).

Proposition 3.1. *Suppose that E is an open set. If $p(\cdot) \in \mathcal{P}(E)$ satisfies*

$$|p(x)-p(y)| \leq \frac{-C}{\log(|x-y|)} \quad \text{if } |x-y| \leq 1/2, \tag{3.1}$$

$$|p(x)-p(y)| \leq \frac{C}{\log(e+|x|)} \quad \text{if } |y| \geq |x|, \tag{3.2}$$

where $C>0$ is a constant independent of x and y, then we have $p(\cdot) \in \mathcal{B}(E)$.

Below $p'(\cdot)$ means the conjugate exponent of $p(\cdot)$, namely $1/p(x) + 1/p'(x) = 1$ holds. Then $\mathcal{Y}$ denotes all families of disjoint and open cubes in $\mathbb{R}^n$. The following propositions are due to Diening ([4, Theorem 8.1 and Lemma 5.5]). We remark that Diening has proved general results on Musielak–Orlicz spaces. We describe them for Lebesgue spaces with variable exponent.

Proposition 3.2. *Suppose $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Then the following conditions are equivalent.*

(I) *There exists a constant $C>0$ such that for all $Y \in \mathcal{Y}$ and all $f \in L^{p(\cdot)}(\mathbb{R}^n)$,*

$$\Big\| \sum_{Q \in Y} |f|_Q \chi_Q \Big\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \, \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \tag{3.3}$$

(II) *$p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$.*

(III) *$p'(\cdot) \in \mathcal{B}(\mathbb{R}^n)$.*

(IV) *There exists a constant $p_0 \in (1, p_-)$ such that $p_0^{-1} p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$.*

Proposition 3.3. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. If $p(\cdot)$ satisfies condition (I) in Proposition 3.2, then there exist two positive constants δ and C such that for all $Y \in \mathcal{Y}$, all non-negative numbers t_Q and all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ with $f_Q := \frac{1}{|Q|}\int_Q f(x)\,dx \neq 0$ $(Q \in Y)$,*

$$\Big\| \sum_{Q \in Y} t_Q \left|\frac{f}{f_Q}\right|^{\delta} \chi_Q \Big\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \, \Big\| \sum_{Q \in Y} t_Q \chi_Q \Big\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \tag{3.4}$$

Propositions 3.2 and 3.3 lead the following corollary.

Corollary 3.4. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. If $p(\cdot)$ satisfies condition (I) in Proposition 3.2, then there exists a positive constant C such that for all balls*

B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,

$$\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leq C\,\frac{|B|}{|S|}, \tag{3.5}$$

$$\frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|S|}{|B|}\right)^{\delta}, \tag{3.6}$$

where δ is the constant appearing in (3.4).

Proof. Take a ball B and a measurable subset $S \subset B$ arbitrarily. We first show (3.5). By Proposition 3.2, $p(\cdot)$ belongs to $\mathcal{B}(\mathbb{R}^n)$. Thus M satisfies the weak $(p(\cdot), p(\cdot))$ inequality, i.e., for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and all $\lambda > 0$ we have

$$\lambda\,\|\chi_{\{Mf(x)>\lambda\}}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\,\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

If we take $\lambda \in (0, |S|/|B|)$ arbitrarily, then we get

$$\lambda\,\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq \lambda\,\|\chi_{\{M(\chi_S)(x)>\lambda\}}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\,\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

namely

$$\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leq C\lambda^{-1}.$$

Since λ is arbitrary, we obtain (3.5). Next we prove (3.6). We can take a open cube Q_B so that $B \subset Q_B \subset \sqrt{n}B$. Putting $f = \chi_S$ and $Y = \{Q_B\}$ in (3.4), we get

$$\frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{Q_B}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|S|}{|Q_B|}\right)^{\delta}.$$

By virtue of $B \subset Q_B \subset \sqrt{n}B$ and (3.5), we see that

$$\begin{aligned}
\frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} &= \frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{Q_B}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \cdot \frac{\|\chi_{Q_B}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \\
&\leq \frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{Q_B}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \cdot \frac{\left\|\chi_{\sqrt{n}B}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \\
&\leq C\left(\frac{|S|}{|Q_B|}\right)^{\delta} \cdot \frac{|\sqrt{n}B|}{|B|} = C\left(\frac{|S|}{|Q_B|}\right)^{\delta} \leq C\left(\frac{|S|}{|B|}\right)^{\delta}.
\end{aligned}$$

$\square$

The next lemma describes the generalized Hölder inequality and the duality of $L^{p(\cdot)}(E)$. The proof is found in [15].

Lemma 3.5. *Suppose $p(\cdot) \in \mathcal{P}(E)$. Then the following hold.*

1. *For all* $f \in L^{p(\,\cdot\,)}(E)$ *and all* $g \in L^{p'(\,\cdot\,)}(E)$ *we have*

$$\int_E |f(x)g(x)|\,dx \le r_p\,\|f\|_{L^{p(\,\cdot\,)}(E)}\|g\|_{L^{p'(\,\cdot\,)}(E)}, \tag{3.7}$$

where $r_p := 1 + 1/p_- - 1/p_+$.

2. *For all* $f \in L^{p(\,\cdot\,)}(E)$ *we have*

$$\|f\|_{L^{p(\,\cdot\,)}(E)} \le \sup\left\{\int_E |f(x)g(x)|\,dx \,:\, \|g\|_{L^{p'(\,\cdot\,)}(E)} \le 1\right\}. \tag{3.8}$$

In particular, $L^{p(\,\cdot\,)}(E)$ *coincides with the dual space of* $L^{p'(\,\cdot\,)}(E)$ *and the norm* $\|f\|_{L^{p(\,\cdot\,)}(E)}$ *is equivalent to the value*

$$\sup\left\{\int_E |f(x)g(x)|\,dx \,:\, \|g\|_{L^{p'(\,\cdot\,)}(E)} \le 1\right\}.$$

REMARK 3.6. Below we write $u_r(\,\cdot\,) := r^{-1}p(\,\cdot\,)$ for positive constant $r > 0$.

1. If $p(\,\cdot\,) \in \mathcal{B}(\mathbb{R}^n)$, then Corollary 3.4 implies that there exists a positive constant δ_1 such that for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,

$$\frac{\|\chi_S\|_{L^{p(\,\cdot\,)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\,\cdot\,)}(\mathbb{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_1}. \tag{3.9}$$

Thus we have that for all $r > 0$,

$$\frac{\|\chi_S\|_{L^{u_r(\,\cdot\,)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{u_r(\,\cdot\,)}(\mathbb{R}^n)}} = \frac{\|\chi_S\|^{r^{-1}}_{L^{p(\,\cdot\,)}(\mathbb{R}^n)}}{\|\chi_B\|^{r^{-1}}_{L^{p(\,\cdot\,)}(\mathbb{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{r^{-1}\delta_1}. \tag{3.10}$$

2. If $p(\,\cdot\,) \in \mathcal{P}(\mathbb{R}^n)$ satisfies (3.1) and (3.2) in Proposition 3.1, then $p'(\,\cdot\,)$, $u_r(\,\cdot\,)$ and $u'_r(\,\cdot\,)$ also belong to $\mathcal{P}(\mathbb{R}^n)$ and satisfy (3.1) and (3.2) for all $0 < r < p_-$. Because they are in $\mathcal{B}(\mathbb{R}^n)$, we can take constants δ_2, $\delta(r) > 0$ such that for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,

$$\frac{\|\chi_S\|_{L^{p'(\,\cdot\,)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'(\,\cdot\,)}(\mathbb{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_2}, \tag{3.11}$$

$$\frac{\|\chi_S\|_{L^{u'_r(\,\cdot\,)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{u'_r(\,\cdot\,)}(\mathbb{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta(r)}. \tag{3.12}$$

3. If $p(\,\cdot\,) \in \mathcal{B}(\mathbb{R}^n)$, then Proposition 3.2 implies that $p'(\,\cdot\,)$, $u_{p_0} \in \mathcal{B}(\mathbb{R}^n)$. Thus (3.11) holds. In addition (3.10) and (3.12) are true with $r = p_0$.

Applying Lemma 3.5 we obtain the following.

LEMMA 3.7. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. If $p(\cdot)$ satisfies condition (I) in Proposition 3.2, then there exists a constant $C > 0$ such that for all balls B in $\mathbb{R}^n$,*

$$\frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \le C.$$

PROOF. The assumption shows that

$$\||f|_Q \chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C \ \|f\,\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

for all cube Q and all $f \in L^{p(\cdot)}(\mathbb{R}^n)$. Using (3.8) we obtain

$$\begin{aligned}
&\frac{1}{|Q|}\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_Q\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\
&\le \frac{1}{|Q|}\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sup\left\{\int_{\mathbb{R}^n}|f(x)\chi_Q(x)|\,dx \,:\, \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le 1\right\} \\
&= \sup\left\{|f|_Q\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \,:\, \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le 1\right\} \\
&\le C \sup\left\{\|f\,\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \,:\, \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le 1\right\} \le C.
\end{aligned}$$

For each ball B we can take a cube Q_B such that $n^{-1/2}Q_B \subset B \subset Q_B$. Thus we get

$$\begin{aligned}
&\frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\
&\qquad \le C\cdot\frac{1}{|Q_B|}\|\chi_{Q_B}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{Q_B}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \le C.
\end{aligned}$$

□

4. VECTOR-VALUED INEQUALITIES FOR SUBLINEAR OPERATORS

In this section we prove the vector-valued inequalities for some sublinear operators on Herz spaces with variable exponent under proper assumptions.

THEOREM 4.1. *Let $1 < r < \infty$, $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $0 < q < \infty$ and $-n\delta_1 < \alpha < n\delta_2$, where $\delta_1, \delta_2 > 0$ are the constants appearing in (3.9) and (3.11). Suppose that T is a sublinear operator satisfying the vector-valued inequality on $L^{p(\cdot)}(\mathbb{R}^n)$*

$$\left\|\left(\sum_{h=1}^{\infty}|Tg_h|^r\right)^{1/r}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\left\|\left(\sum_{h=1}^{\infty}|g_h|^r\right)^{1/r}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \tag{4.1}$$

for all sequences of functions $\{g_h\}_{h=1}^{\infty}$ satisfying $\|\|\{g_h\}_h\|_{\ell^r}\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \infty$, and size condition

$$|Tf(x)| \le C\int_{\mathbb{R}^n}|x-y|^{-n}|f(y)|\,dy \tag{4.2}$$

for all $f \in L^1(\mathbb{R}^n)$ with compact support and a.e. $x \notin \operatorname{supp} f$. Then we have the vector-valued inequality on $\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$

$$\left\| \left(\sum_{h=1}^{\infty} |Tf_h|^r \right)^{1/r} \right\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} \le C \left\| \left(\sum_{h=1}^{\infty} |f_h|^r \right)^{1/r} \right\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} \tag{4.3}$$

for all sequences of functions $\{f_h\}_{h=1}^{\infty}$ satisfying $\|\|\{f_h\}_h\|_{\ell^r}\|_{\dot{K}^{\alpha,q}_{p(\cdot)}} < \infty$. Moreover T also satisfies the same vector-valued inequality as (4.3) on $K^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$.

REMARK 4.2.
1. There do exist some operators satisfying vector-valued inequality (4.1) provided $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, for example, the Hardy–Littlewood maximal operator and singular integral operators (see [2]).
2. Theorem 4.1 is known when $p(\cdot)$ equals to a constant $p \in (1, \infty)$. Hernández, Li, Lu and Yang ([7,17,19]) have proved (4.3) on the usual Herz spaces provided $-n/p < \alpha < n/p'$.

In order to prove Theorem 4.1, we additionally introduce the next lemma which is well-known as the generalized Minkowski inequality.

LEMMA 4.3. *If $1 < r < \infty$, then there exists a constant $C > 0$ such that for all sequences of functions $\{f_h\}_{h=1}^{\infty}$ satisfying $\|\|\{f_h\}_h\|_{\ell^r}\|_{L^1(\mathbb{R}^n)} < \infty$,*

$$\left\{ \sum_{h=1}^{\infty} \left(\int_{\mathbb{R}^n} |f_h(y)|\, dy \right)^r \right\}^{1/r} \le C \int_{\mathbb{R}^n} \left\{ \sum_{h=1}^{\infty} |f_h(y)|^r \right\}^{1/r} dy. \tag{4.4}$$

PROOF OF THEOREM 4.1. Our method is based on [18, Proof of Theorem 1] and [25]. We give the proof for the homogeneous case while the non-homogeneous case is similar.

Because T is sublinear, we have that

$$\begin{aligned} &\|\|\{Tf_h\}_h\|_{\ell^r}\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} \\ &\quad = \left\{ \sum_{j=-\infty}^{\infty} 2^{\alpha q j} \, \|\chi_j \|\{Tf_h\}_h\|_{\ell^r}\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \right\}^{1/q} \\ &\quad \le \left\{ \sum_{j=-\infty}^{\infty} 2^{\alpha q j} \left\| \chi_j \sum_{l=-\infty}^{\infty} \|\{T(f_h \chi_l)\}_h\|_{\ell^r} \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \right\}^{1/q}, \end{aligned}$$

for every $\{f_h\}_{h=1}^{\infty}$ with $\| \|\{f_h\}_h\|_{\ell^r} \|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} < \infty$. Thus we can decompose as follows.

$$
\begin{aligned}
&\| \|\{Tf_h\}_h\|_{\ell^r} \|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} \\
&\leq \quad C \left\{ \sum_{j=-\infty}^{\infty} 2^{\alpha q j} \left(\sum_{l=-\infty}^{j-2} \|\chi_j \|\{T(f_h\chi_l)\}_h\|_{\ell^r} \|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \right\}^{1/q} \\
&\qquad + C \left\{ \sum_{j=-\infty}^{\infty} 2^{\alpha q j} \left(\sum_{l=j-1}^{j+1} \|\chi_j \|\{T(f_h\chi_l)\}_h\|_{\ell^r} \|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \right\}^{1/q} \\
&\qquad + C \left\{ \sum_{j=-\infty}^{\infty} 2^{\alpha q j} \left(\sum_{l=j+2}^{\infty} \|\chi_j \|\{T(f_h\chi_l)\}_h\|_{\ell^r} \|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \right\}^{1/q} \\
&=: \quad E_1 + E_2 + E_3.
\end{aligned}
$$

For convenience below we denote $F := \|\{f_h\}_h\|_{\ell^r}$. Since T satisfies (4.1), we can easily obtain

$$E_2 \leq C \, \|F\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} .$$

We consider the estimates of E_1 and E_3.

For each $j \in \mathbb{Z}$ and $l \leq j-2$ and a.e. $x \in R_j$, size condition (4.2), generalized Minkowski's inequality (4.4) and generalized Hölder's inequality (3.7) imply

$$
\begin{aligned}
\|\{T(f_h\chi_l)(x)\}_h\|_{\ell^r} &\leq C \left\| \left\{ \int_{R_l} |x-y|^{-n} |f_h(y)| \, dy \right\}_h \right\|_{\ell^r} \\
&\leq C \, 2^{-jn} \left\| \left\{ \int_{R_l} |f_h(y)| \, dy \right\}_h \right\|_{\ell^r} \\
&\leq C \, 2^{-jn} \|F\chi_l\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_l\|_{L^{p'(\cdot)}(\mathbb{R}^n)}. \qquad (4.5)
\end{aligned}
$$

On the other hand, Proposition 3.2 and Lemma 3.7 lead

$$
\begin{aligned}
&2^{-jn} \, \|\chi_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \, \|\chi_l\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\
&\quad \leq \quad 2^{-jn} \left\|\chi_{B_j}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_{B_l}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\
&\quad \leq \quad C \, 2^{-jn} \left\{ |B_j| \left\|\chi_{B_j}\right\|^{-1}_{L^{p'(\cdot)}(\mathbb{R}^n)} \right\} \|\chi_{B_l}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\
&\quad = \quad C \, \frac{\|\chi_{B_l}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\left\|\chi_{B_j}\right\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \leq C \, 2^{n\delta_2 (l-j)}. \qquad (4.6)
\end{aligned}
$$

Applying (4.5) and (4.6) to E_1, we get

$$\begin{aligned} E_1 &\le C\Bigg\{\sum_{j=-\infty}^{\infty} 2^{\alpha qj}\Bigg(\sum_{l=-\infty}^{j-2} \|\chi_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\qquad \times 2^{-jn}\|F\chi_l\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_l\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\Bigg)^q\Bigg\}^{1/q} \\ &\le C\left\{\sum_{j=-\infty}^{\infty}\left(\sum_{l=-\infty}^{j-2} 2^{\alpha l}\|F\chi_l\|_{L^{p(\cdot)}(\mathbb{R}^n)} 2^{b(l-j)}\right)^q\right\}^{1/q}, \end{aligned} \tag{4.7}$$

where $b := n\delta_2 - \alpha > 0$. Similarly we obtain

$$E_3 \le C\left\{\sum_{j=-\infty}^{\infty}\left(\sum_{l=j+2}^{\infty} 2^{\alpha l}\|F\chi_l\|_{L^{p(\cdot)}(\mathbb{R}^n)} 2^{d(j-l)}\right)^q\right\}^{1/q}, \tag{4.8}$$

where $d := n\delta_1 + \alpha > 0$. To continue calculations for (4.7) and (4.8), we consider the two cases "$1 < q < \infty$" and "$0 < q \le 1$".

If $1 < q < \infty$, then we use Hölder's inequality and obtain

$$\begin{aligned} E_1 &\le C\Bigg\{\sum_{j=-\infty}^{\infty}\left(\sum_{l=-\infty}^{j-2} 2^{\alpha ql}\|F\chi_l\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} 2^{bq(l-j)/2}\right) \\ &\qquad \times \left(\sum_{l=-\infty}^{j-2} 2^{bq'(l-j)/2}\right)^{q/q'}\Bigg\}^{1/q} \\ &= C\left\{\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{j-2} 2^{\alpha ql}\|F\chi_l\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} 2^{bq(l-j)/2}\right\}^{1/q} \\ &= C\left\{\sum_{l=-\infty}^{\infty} 2^{\alpha ql}\|F\chi_l\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \sum_{j=l+2}^{\infty} 2^{bq(l-j)/2}\right\}^{1/q} \\ &= C\left\{\sum_{l=-\infty}^{\infty} 2^{\alpha ql}\|F\chi_l\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right\}^{1/q} \\ &= C\,\|F\|_{\dot K^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

If $0 < q \le 1$, then we get

$$\begin{aligned}
E_1 &\le C\left\{\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{j-2} 2^{\alpha q l}\|F\chi_l\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q 2^{bq(l-j)}\right\}^{1/q} \\
&= C\left\{\sum_{l=-\infty}^{\infty} 2^{\alpha q l}\|F\chi_l\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \sum_{j=l+2}^{\infty} 2^{bq(l-j)}\right\}^{1/q} \\
&= C\left\{\sum_{l=-\infty}^{\infty} 2^{\alpha q l}\|F\chi_l\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q\right\}^{1/q} \\
&= C\,\|F\|_{\dot{K}_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)}\,.
\end{aligned}$$

Applying the same calculations to (4.8), we obtain

$$E_3 \le C\,\|F\|_{\dot{K}_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)}\,.$$

We have finished the proof. ∎

Theorem 4.1 leads further results. The Hardy–Littlewood maximal operator M satisfies the vector-valued inequality on $L^{p(\cdot)}(\mathbb{R}^n)$ provided $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. The next proposition is [2, Corollary 2.1].

PROPOSITION 4.4. *If $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and $1 < r < \infty$, then we have*

$$\left\|\left(\sum_{h=1}^{\infty}|Mg_h|^r\right)^{1/r}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\left\|\left(\sum_{h=1}^{\infty}|g_h|^r\right)^{1/r}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

for all sequences of functions $\{g_h\}_{h=1}^{\infty}$ satisfying $\|\|\{g_h\}_h\|_{\ell^r}\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \infty$.

Because the Hardy–Littlewood maximal operator M is sublinear and satisfies the size condition

$$Mf(x) \le C\int_{\mathbb{R}^n}|x-y|^{-n}|f(y)|\,dy,$$

we immediately the following.

COROLLARY 4.5. *Suppose that α, q and $p(\cdot)$ satisfy the same assumptions as Theorem 4.1. Then we have*

$$\left\|\left(\sum_{h=1}^{\infty}|Mf_h|^r\right)^{1/r}\right\|_{\dot{K}_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)} \le C\left\|\left(\sum_{h=1}^{\infty}|f_h|^r\right)^{1/r}\right\|_{\dot{K}_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)} \tag{4.9}$$

for all sequences of functions $\{f_h\}_{h=1}^{\infty}$ satisfying $\|\|\{f_h\}_h\|_{\ell^r}\|_{\dot{K}_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)} < \infty$. In addition, M also satisfies the same vector-valued inequality as (4.9) on $K_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)$.

In particular M becomes a bounded operator on Herz spaces with variable exponent. Thus the same argument as [22] leads the density of the set of all differentiable functions with compact support $C_c^\infty(\mathbb{R}^n)$.

THEOREM 4.6. *Let $s \in \mathbb{N}_0$ and suppose that α, q and $p(\,\cdot\,)$ satisfy the same assumptions as Theorem 4.1. Then $C_c^\infty(\mathbb{R}^n)$ is dense in $\dot{K}^{\alpha,q}_{p(\,\cdot\,)}W^s(\mathbb{R}^n)$ and in $K^{\alpha,q}_{p(\,\cdot\,)}W^s(\mathbb{R}^n)$.*

5. Characterizations of Herz–Sobolev spaces with variable exponent

In this section we will give some equivalent norms on Herz–Sobolev spaces with variable exponent.

5.1. *Characterizations of Herz spaces.*

We first define a class of systems $\Phi(\mathbb{R}^n)$ and Herz-type Triebel–Lizorkin spaces with variable exponent. Using them, we will characterize Herz spaces with variable exponent.

DEFINITION 5.1. *The set $\Phi(\mathbb{R}^n)$ consists of all systems $\{\varphi_j\}_{j=0}^\infty \subset \mathcal{S}(\mathbb{R}^n)$ satisfying the following conditions.*

(i) $\operatorname{supp}\varphi_0 \subset \{x \in \mathbb{R}^n : |x| \le 2\}$.
(ii) $\operatorname{supp}\varphi_j \subset \{x \in \mathbb{R}^n : 2^{j-1} \le |x| \le 2^{j+1}\}$ *if* $j \in \mathbb{N}_0$.
(iii) *For every $\gamma \in {\mathbb{N}_0}^n$ there exists a constant $c_\gamma > 0$ such that $|D^\gamma\varphi_j(x)| \le c_\gamma 2^{-j|\gamma|}$ for all $j \in \mathbb{N}_0$ and all $x \in \mathbb{R}^n$.*
(iv) $\sum_{j=0}^\infty \varphi_j(x) \equiv 1$ *on* $\mathbb{R}^n$.

DEFINITION 5.2. *Suppose $s \in \mathbb{N}_0$, $0 < q < \infty$, $p(\,\cdot\,) \in \mathcal{P}(\mathbb{R}^n)$ and $\alpha \in \mathbb{R}$. Take $\varphi = \{\varphi_j\}_{j=0}^\infty \in \Phi(\mathbb{R}^n)$. Then define*

$$\begin{aligned}\dot{K}^{\alpha,q}_{p(\,\cdot\,)}F^{s,\varphi}_2(\mathbb{R}^n) &:= \left\{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{\dot{K}^{\alpha,q}_{p(\,\cdot\,)}F^{s,\varphi}_2(\mathbb{R}^n)} < \infty\right\},\\ K^{\alpha,q}_{p(\,\cdot\,)}F^{s,\varphi}_2(\mathbb{R}^n) &:= \left\{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{K^{\alpha,q}_{p(\,\cdot\,)}F^{s,\varphi}_2(\mathbb{R}^n)} < \infty\right\},\end{aligned}$$

where

$$\begin{aligned}\|f\|_{\dot{K}^{\alpha,q}_{p(\,\cdot\,)}F^{s,\varphi}_2(\mathbb{R}^n)} &:= \left\|\left(\sum_{j=0}^\infty \left|2^{sj}\mathcal{F}^{-1}[\varphi_j\hat{f}]\right|^2\right)^{1/2}\right\|_{\dot{K}^{\alpha,q}_{p(\,\cdot\,)}(\mathbb{R}^n)},\\ \|f\|_{K^{\alpha,q}_{p(\,\cdot\,)}F^{s,\varphi}_2(\mathbb{R}^n)} &:= \left\|\left(\sum_{j=0}^\infty \left|2^{sj}\mathcal{F}^{-1}[\varphi_j\hat{f}]\right|^2\right)^{1/2}\right\|_{K^{\alpha,q}_{p(\,\cdot\,)}(\mathbb{R}^n)}.\end{aligned}$$

Now we give characterizations of Herz spaces with variable exponent.

Lemma 5.3. *Let $1 < q < \infty$, $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and $\alpha \in \mathbb{R}$ with $|\alpha| < n\min\{\delta_1, \delta_2\}$, where $\delta_1, \delta_2 > 0$ are the constants appearing in (3.9) and (3.11). Then we have*

$$\dot{K}^{\alpha,q}_{p(\cdot)}F^{0,\varphi}_2(\mathbb{R}^n) = \dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n), \quad K^{\alpha,q}_{p(\cdot)}F^{0,\varphi}_2(\mathbb{R}^n) = K^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$$

with equivalent norms. In particular $\dot{K}^{\alpha,q}_{p(\cdot)}F^{0,\varphi}_2(\mathbb{R}^n)$ and $K^{\alpha,q}_{p(\cdot)}F^{0,\varphi}_2(\mathbb{R}^n)$ are independent of the choice of $\varphi = \{\varphi_j\}_{j=0}^{\infty} \in \Phi(\mathbb{R}^n)$.

In order to prove Lemma 5.3, we introduce weighted Lebesgue spaces, Muckenhoupt's A_{p_1} class and the extrapolation theorem. A non-negative and locally integrable function is said to be a weight.

Definition 5.4. *Let w be a weight and $p_1 \in (1,\infty)$ a constant. The weighted Lebesgue space $L^{p_1}_w(\mathbb{R}^n)$ is defined by*

$$L^{p_1}_w(\mathbb{R}^n) := \left\{ f \text{ is measurable} : \|f\|_{L^{p_1}_w(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|f\|_{L^{p_1}_w(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |f(x)|^{p_1} w(x)\, dx \right)^{1/p_1}.$$

Now we define Muckenhoupt's A_{p_1} weights.

Definition 5.5. *The class of weights A_{p_1} consists of all weights w satisfying $w^{-1/(p_1-1)} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and*

$$A_{p_1}(w) := \sup_{Q:\text{cube}} \frac{1}{|Q|}\int_Q w(x)\, dx \left(\frac{1}{|Q|}\int_Q w^{-1/(p_1-1)}(x)\, dx \right)^{p_1-1} < \infty.$$

The next lemma is the extrapolation theorem due to Cruz-Uribe, Fiorenza, Martell and Pérez ([2, Corollary 1.11]). They have proved the boundedness of many important operators on variable Lebesgue spaces by applying the result, provided that M is bounded.

Lemma 5.6. *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and $\mathcal{A}$ be a family of ordered pairs of non-negative and measurable functions (f,g). Suppose that there exists a constant $p_1 \in (1, p_-)$ such that for all $w \in A_{p_1}$ and all $(f,g) \in \mathcal{A}$ with $f \in L^{p_1}_w(\mathbb{R}^n)$,*

$$\|f\|_{L^{p_1}_w(\mathbb{R}^n)} \le C\, \|g\|_{L^{p_1}_w(\mathbb{R}^n)},$$

where $C > 0$ is a constant depending only on n, p_1 and $A_{p_1}(w)$. Then it follows that for all $(f,g) \in \mathcal{A}$ such that $f \in L^{p(\cdot)}(\mathbb{R}^n)$,

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C'\, \|g\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where $C' > 0$ is a constant independent of (f,g).

We also describe the duality of Herz spaces with variable exponent. The duality for usual Herz spaces is proved in [7]. By virtue of Lemma 3.5, the same argument as [7] leads the next lemma.

LEMMA 5.7. *Suppose $\alpha \in \mathbb{R}$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $1 < q < \infty$. Then $\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$ coincides with the dual space of $\dot{K}^{-\alpha,q'}_{p'(\cdot)}(\mathbb{R}^n)$. Moreover the quasi-norm $\|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)}$ is equivalent to the value*

$$\sup\left\{\left|\int_{\mathbb{R}^n} f(x)g(x)\,dx\right| \,:\, \|g\|_{\dot{K}^{-\alpha,q'}_{p'(\cdot)}(\mathbb{R}^n)} \le 1\right\}.$$

The same duality is also true for $K^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$.

PROOF OF LEMMA 5.3. Define

$$Tf := \left(\sum_{j=0}^{\infty}\left|\mathcal{F}^{-1}[\varphi_j \hat{f}]\right|^2\right)^{1/2}.$$

Take a constant exponent $p_1 \in (1,\infty)$ and $w \in A_{p_1}$ arbitrarily. By virtue of [16], we have

$$C^{-1}\|f\|_{L^{p_1}_w(\mathbb{R}^n)} \le \|Tf\|_{L^{p_1}_w(\mathbb{R}^n)} \le C\,\|f\|_{L^{p_1}_w(\mathbb{R}^n)}$$

for all $f \in C_c^\infty(\mathbb{R}^n)$. Using Lemma 5.6, we get

$$C^{-1}\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le \|Tf\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\,\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \tag{5.1}$$

Because $C_c^\infty(\mathbb{R}^n)$ is dense in $L^{p(\cdot)}(\mathbb{R}^n)$, (5.1) also holds for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$. On the other hand, the operator T satisfies size condition (4.2) (cf. [27, Proof of Theorem 2.1]). Thus Theorem 4.1 leads the estimate

$$\|Tf\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} \le C\,\|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} \tag{5.2}$$

for all $f \in \dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$.

Next we show

$$\|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} \le C\,\|Tf\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)}$$

for all $f \in \dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$. By virtue of Theorem 4.6 and Lemma 5.7, it suffices to prove

$$\left|\int_{\mathbb{R}^n} f(x)g(x)\,dx\right| \le C\,\|Tf\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} \tag{5.3}$$

for all $f,\, g \in C_c^\infty(\mathbb{R}^n)$ with $\|g\|_{\dot{K}^{-\alpha,q'}_{p'(\cdot)}(\mathbb{R}^n)} \le 1$. Using the Plancherel formula and the properties of $\{\varphi_j\}_{j=0}^\infty$, we obtain

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} f(x)g(x)\,dx\right| &= \left|\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\xi)\,d\xi\right| \\
&= \left|\int_{\mathbb{R}^n} \sum_{j=0}^\infty \varphi_j(\xi)\hat{f}(\xi)\sum_{l=0}^\infty \varphi_l(\xi)\hat{g}(\xi)\,d\xi\right| \\
&= \left|\int_{\mathbb{R}^n} \varphi_0(\xi)\hat{f}(\xi)\left\{\varphi_0(\xi)+\varphi_1(\xi)\right\}\hat{g}(\xi)\,d\xi\right. \\
&\quad \left. + \int_{\mathbb{R}^n} \sum_{j=1}^\infty \varphi_j(\xi)\hat{f}(\xi)\left\{\varphi_{j-1}(\xi)+\varphi_j(\xi)+\varphi_{j+1}(\xi)\right\}\hat{g}(\xi)\,d\xi\right| \\
&\le \sum_{j=0}^\infty \left|\int_{\mathbb{R}^n} \varphi_j(\xi)\hat{f}(\xi)\varphi_j(\xi)\hat{g}(\xi)\,d\xi\right| + \sum_{j=1}^\infty \left|\int_{\mathbb{R}^n} \varphi_j(\xi)\hat{f}(\xi)\varphi_{j-1}(\xi)\hat{g}(\xi)\,d\xi\right| \\
&\quad + \sum_{j=0}^\infty \left|\int_{\mathbb{R}^n} \varphi_j(\xi)\hat{f}(\xi)\varphi_{j+1}(\xi)\hat{g}(\xi)\,d\xi\right|. \qquad (5.4)
\end{aligned}$$

By the Cauchy–Schwarz inequality and generalized Hölder inequality we have

$$\begin{aligned}
\sum_{j=0}^\infty \left|\int_{\mathbb{R}^n} \varphi_j(\xi)\hat{f}(\xi)\varphi_j(\xi)\hat{g}(\xi)\,d\xi\right| &= \sum_{j=0}^\infty \left|\int_{\mathbb{R}^n} \mathcal{F}^{-1}[\varphi_j\hat{f}](x)\mathcal{F}^{-1}[\varphi_j\hat{g}](x)\,dx\right| \\
&\le C\,\|Tf\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)}\,\|Tg\|_{\dot{K}^{-\alpha,q'}_{p'(\cdot)}(\mathbb{R}^n)}. \qquad (5.5)
\end{aligned}$$

Applying (5.2), we get

$$\|Tg\|_{\dot{K}^{-\alpha,q'}_{p'(\cdot)}(\mathbb{R}^n)} \le C\,\|g\|_{\dot{K}^{-\alpha,q'}_{p'(\cdot)}(\mathbb{R}^n)} \le C. \qquad (5.6)$$

Combining (5.4), (5.5) and (5.6), we have (5.3). The non-homogeneous case follows by the same argument. ∎

5.2. *Characterizations of Herz–Sobolev spaces.*

We need preparations in order to get equivalent norms of Herz–Sobolev spaces with variable exponent. In this subsection we refer to [26, 28].

THEOREM 5.8. *Let $d_j > 0$, $\{\Omega_j\}_{j=0}^\infty$ be a sequence of compact sets of $\mathbb{R}^n$ defined by $\Omega_j := \{x \in \mathbb{R}^n : |x| \le d_j\}$ for each $j \in \mathbb{N}_0$, $0 < q < \infty$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 < r < p_-$ and $-nr^{-2}\delta_1 < \alpha < nr^{-1}\delta(r)$, where δ_1, $\delta(r) > 0$ are the constants appearing in (3.9) and (3.12). Suppose the following (I) or (II).*

(I) *$r = p_0$, $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and $p_- < 2$, where $p_0 \in (1, p_-)$ is the constant appearing in Proposition 3.2.*

(II) *$p(\cdot)$ satisfies (3.1) and (3.2) in Proposition 3.1.*

Then there exists a constant $C > 0$ such that

$$\left\| \left\| \left\{ \sup_{\xi \in \mathbb{R}^n} \frac{f_j(\cdot - \xi)}{1 + (d_j|\xi|)^{n/r}} \right\}_{j=-\infty}^{\infty} \right\|_{\ell^2(\mathbb{Z})} \right\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} \le C \left\| \|\{f_j\}_j\|_{\ell^2} \right\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} \tag{5.7}$$

for all sequences of functions $\{f_j\}_{j=0}^{\infty}$ satisfying $\left\| \|\{f_j\}_j\|_{\ell^2} \right\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} < \infty$ and $\operatorname{supp} f_j \subset \Omega_j$ for each $j \in \mathbb{N}_0$. The same inequality as (5.7) also holds for the non-homogeneous case.

Proof. Take $\{f_j\}_{j=0}^{\infty}$ satisfying $\left\| \|\{f_j\}_j\|_{\ell^2} \right\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} < \infty$ and $\operatorname{supp} f_j \subset \Omega_j$ for each $j \in \mathbb{N}_0$ arbitrarily. Denote $g_j(x) := f_j(d_j^{-1}x)$. Because $\hat{g}_j(\xi) = {d_j}^n \hat{f}_j(d_j\xi)$ and $\operatorname{supp} \hat{g}_j \subset \{|\xi| \le 1\}$, we get

$$\frac{|g_j(x - \xi)|}{1 + |\xi|^{n/r}} \le C \left\{ M(|g_j|^r)(x) \right\}^{1/r} \tag{5.8}$$

for a.e. $x, \xi \in \mathbb{R}^n$ (cf. [26, p. 22]). By $M(|f_j|^r)(x) = M(|g_j(d_j \cdot)|^r)(x) = M(|g_j)|^r)(d_j x)$, (5.8) leads

$$\frac{|f_j(x - \xi)|}{1 + (d_j|\xi|)^{n/r}} \le C \left\{ M(|f_j|^r)(x) \right\}^{1/r}.$$

Denote $u_r(\cdot) := r^{-1}p(\cdot)$. Note that $1 < 2/r < \infty$ and $-n\delta_1 r^{-1} < r\alpha < n\delta(r)$. Applying Corollary 4.5 we obtain

$$\begin{aligned}
&\left\| \left\| \left\{ \sup_{\xi \in \mathbb{R}^n} \frac{f_j(\cdot - \xi)}{1 + (d_j|\xi|)^{n/r}} \right\}_j \right\|_{\ell^2} \right\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} \\
&\quad \le \quad C \left\| \left\| \left\{ M(|f_j|^r)^{1/r} \right\}_j \right\|_{\ell^2} \right\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} \\
&\quad = \quad C \left\| \left\| \{ M(|f_j|^r) \}_j \right\|_{\ell^{2/r}} \right\|^{1/r}_{\dot{K}^{r\alpha,q/r}_{u_r(\cdot)}(\mathbb{R}^n)} \\
&\quad \le \quad C \left\| \left\| \{ {f_j}^r \}_j \right\|_{\ell^{2/r}} \right\|^{1/r}_{\dot{K}^{r\alpha,q/r}_{u_r(\cdot)}(\mathbb{R}^n)} \\
&\quad = \quad C \left\| \left\| \{ f_j \}_j \right\|_{\ell^2} \right\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Namely we have proved (5.7). The non-homogeneous case is obtained by the same argument as above. ∎

DEFINITION 5.9. *Let $L \in \mathbb{N}_0$. $\mathcal{A}_L(\mathbb{R}^n)$ consists of all systems $\{\varphi_j\}_{j=0}^{\infty} \subset \mathcal{S}(\mathbb{R}^n)$ of functions with compact supports such that*

$$\begin{aligned} C(\{\varphi_j\}_j) \quad &:= \quad \sup_{x\in\mathbb{R}^n} |x|^L \sum_{|\gamma|\le L} |D^\gamma \varphi_0(x)| \\ &\qquad + \sup_{x\neq 0,\, j\in\mathbb{N}_0} (|x|^L + |x|^{-L}) \sum_{|\gamma|\le L} |D^\gamma(\varphi_j(2^j\,\cdot\,))(x)| \\ &< \quad \infty. \end{aligned}$$

Note that $\Phi(\mathbb{R}^n) \subset \mathcal{A}_L(\mathbb{R}^n)$ for all $L \in \mathbb{N}_0$. The next lemma is due to [26, p. 53]. Let $r > 0$, $L \in \mathbb{N}$, $\{\varphi_j\}_{j=0}^{\infty} \in \mathcal{A}_L(\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$. We define the maximal function for each $j \in \mathbb{N}_0$,

$$(\varphi_j^* f)(x) := \sup_{y\in\mathbb{R}^n} \frac{\left|\mathcal{F}^{-1}[\varphi_j \hat{f}](x-y)\right|}{1+|2^j y|^{n/r}}. \tag{5.9}$$

LEMMA 5.10. *Let $r > 0$, $s \in \mathbb{R}$ and $L \in \mathbb{N}$ such that $L > |s|+3n/r+n+2$. Then there exists a constant $C > 0$ such that*

$$\begin{aligned} &\left\{ \sum_{j=0}^{\infty} \left(2^{js} \sup_{0<\tau<1} (\varphi_j^{\tau *} f)(x) \right)^2 \right\}^{1/2} \\ &\qquad \le C \sup_{0<\tau<1} C(\{\varphi_j^\tau\}_j) \left\{ \sum_{j=0}^{\infty} \left(2^{js} (\varphi_j^* f)(x) \right)^2 \right\}^{1/2} \end{aligned}$$

for all $\{\varphi_j\}_{j=0}^{\infty} \in \Phi(\mathbb{R}^n)$, all $\{\varphi_j^\tau\}_{j=0}^{\infty} \in \mathcal{A}_L(\mathbb{R}^n)$ with $0 < \tau < 1$, all $f \in \mathcal{S}'(\mathbb{R}^n)$ and all $x \in \mathbb{R}^n$.

Applying Lemma 5.10 and Theorem 5.8 with $f_j = \mathcal{F}^{-1}\left[\varphi_j \hat{f}\right]$ and $d_j = 2^{j+2}$, we immediately obtain the following lemma.

LEMMA 5.11. *Let $0 < q < \infty$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 < r < p_-$, $-nr^{-2}\delta_1 < \alpha < nr^{-1}\delta(r)$, where δ_1, $\delta(r) > 0$ are the constants appearing in (3.9) and (3.12), $s \in \mathbb{N}_0$ and $L \in \mathbb{N}$ with $L > s + 3n/r + n + 2$. Suppose (I) or (II) in Theorem 5.8. Then there exists a constant $C > 0$ such that*

$$\begin{aligned} &\left\| \left\{ \sum_{j=0}^{\infty} \left(2^{js} \sup_{0<\tau<1} (\varphi_j^{\tau *} f) \right)^2 \right\}^{1/2} \right\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} \\ &\qquad \le C \sup_{0<\tau<1} C(\{\varphi_j^\tau\}_j) \|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)} F_2^{s,\varphi}(\mathbb{R}^n)} \end{aligned} \tag{5.10}$$

for all $\varphi = \{\varphi_j\}_{j=0}^{\infty} \in \Phi(\mathbb{R}^n)$, all $\{\varphi_j^\tau\}_{j=0}^{\infty} \in \mathcal{A}_L(\mathbb{R}^n)$ with $0 < \tau < 1$ and all $f \in \dot{K}^{\alpha,q}_{p(\cdot)} F_2^{s,\varphi}(\mathbb{R}^n)$. The same inequality as (5.10) also holds for the non-homogeneous case.

The next theorem is the Fourier multiplier theorem for Herz–Sobolev spaces with variable exponent.

THEOREM 5.12. *Let* $\varphi = \{\varphi_j\}_{j=0}^{\infty} \in \Phi(\mathbb{R}^n)$, $0 < q < \infty$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 < r < p_-$, $-nr^{-2}\delta_1 < \alpha < nr^{-1}\delta(r)$, *where* δ_1, $\delta(r) > 0$ *are the constants appearing in (3.9) and (3.12),* $s \in \mathbb{N}_0$ *and* $L \in \mathbb{N}$ *with* $L > s + 3n/r + n + 2$. *Suppose (I) or (II) in Theorem 5.8. Then there exists a constant* $C > 0$ *such that*

$$\left\|\mathcal{F}^{-1}\left[m\hat{f}\right]\right\|_{\dot{K}^{\alpha,q}_{p(\cdot)}F^{s,\varphi}_2(\mathbb{R}^n)} \le C\, m_L\, \|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}F^{s,\varphi}_2(\mathbb{R}^n)} \tag{5.11}$$

for all $\{\varphi_j\}_{j=0}^{\infty} \in \Phi(\mathbb{R}^n)$, *all* $m \in C^{\infty}(\mathbb{R}^n)$ *with* $m_L < \infty$ *and all* $f \in \dot{K}^{\alpha,q}_{p(\cdot)}F^{s,\varphi}_2(\mathbb{R}^n)$, *where*

$$m_L := \sup_{|\gamma| \le L} \sup_{x \in \mathbb{R}^n} (1 + |x|^2)^{|\gamma|/2} \left|D^{\gamma} m(x)\right|.$$

The same inequality as (5.11) also holds for the non-homogeneous case.

A function m is said to be a Fourier multiplier for $\dot{K}^{\alpha,q}_{p(\cdot)}F^{s,\varphi}_2(\mathbb{R}^n)$ if (5.11) holds for all $f \in \dot{K}^{\alpha,q}_{p(\cdot)}F^{s,\varphi}_2(\mathbb{R}^n)$.

PROOF. For all $m \in C^{\infty}(\mathbb{R}^n)$ with $m_L < \infty$ and all $f \in \dot{K}^{\alpha,q}_{p(\cdot)}F^{s,\varphi}_2(\mathbb{R}^n)$, we have

$$\begin{aligned}
&\left\|\mathcal{F}^{-1}\left[m\hat{f}\right]\right\|_{\dot{K}^{\alpha,q}_{p(\cdot)}F^{s,\varphi}_2(\mathbb{R}^n)} \\
&= \left\|\left(\sum_{j=0}^{\infty}\left|\mathcal{F}^{-1}\left[\varphi_j\mathcal{F}\left[\mathcal{F}^{-1}\left[m\hat{f}\right]\right]\right]\right|^2\right)^{1/2}\right\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} \\
&= \left\|\left(\sum_{j=0}^{\infty}\left|\mathcal{F}^{-1}\left[\varphi_j m\hat{f}\right]\right|^2\right)^{1/2}\right\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} \\
&\le \left\|\left\{\sum_{j=0}^{\infty}\left((\varphi_j m)^* f\right)^2\right\}^{1/2}\right\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Because $\{\varphi_j m\}_{j=0}^{\infty} \in \mathcal{A}_L(\mathbb{R}^n)$, Lemma 5.11 leads

$$\begin{aligned}
\left\|\mathcal{F}^{-1}\left[m\hat{f}\right]\right\|_{\dot{K}^{\alpha,q}_{p(\cdot)}F^{s,\varphi}_2(\mathbb{R}^n)} &\le C\, C(\{\varphi_j m\}_j)\, \|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}F^{s,\varphi}_2(\mathbb{R}^n)} \\
&\le C\, m_L\, \|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}F^{s,\varphi}_2(\mathbb{R}^n)}.
\end{aligned}$$

The non-homogeneous case follows by the same argument. ∎

Now we have some equivalent norms of Herz–Sobolev spaces with variable exponent.

THEOREM 5.13. *Let $\varphi = \{\varphi_j\}_{j=0}^{\infty} \in \Phi(\mathbb{R}^n)$, $1 < q < \infty$, $p(\,\cdot\,) \in \mathcal{P}(\mathbb{R}^n)$ and $\alpha \in \mathbb{R}$. Suppose the following (I) or (II).*

(I) *$p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $p_- < 2$, $(p')_- < 2$ and $-n\min\{p_0^{-2}\delta_1, \delta_2\} < \alpha < n\min\{\delta(p_0)p_0^{-1}, \delta_1, \delta_2\}$, where $\delta_1, \delta_2 > 0$ and $p_0 \in (1, p_-)$ are the constants appearing in (3.9), (3.11) and Proposition 3.2, respectively, and $\delta(p_0) > 0$ appears in (3.12) with $r = p_0$.*

(II) *$p(\cdot)$ satisfies (3.1) and (3.2) in Proposition 3.1, and $|\alpha| < n\min\{\delta_1, \delta_2\}$.*

Then the following four values

$$\|f\|_{\dot{K}^{\alpha,q}_{p(\,\cdot\,)}W^s(\mathbb{R}^n)}, \qquad \sum_{|\gamma|=0,s} \|D^\gamma f\|_{\dot{K}^{\alpha,q}_{p(\,\cdot\,)}(\mathbb{R}^n)},$$
$$\|f\|_{\dot{K}^{\alpha,q}_{p(\,\cdot\,)}F_2^{s,\varphi}(\mathbb{R}^n)}, \qquad \left\|\mathcal{F}^{-1}\left[(1+|x|^2)^{s/2}\hat{f}\right]\right\|_{\dot{K}^{\alpha,q}_{p(\,\cdot\,)}(\mathbb{R}^n)}$$

are equivalent on $\dot{K}^{\alpha,q}_{p(\,\cdot\,)}W^s(\mathbb{R}^n)$.

The same equivalence also holds for the non-homogeneous case.

In particular $\|f\|_{\dot{K}^{\alpha,q}_{p(\,\cdot\,)}F_2^{s,\varphi}(\mathbb{R}^n)}$ and $\|f\|_{K^{\alpha,q}_{p(\,\cdot\,)}F_2^{s,\varphi}(\mathbb{R}^n)}$ are independent of the choice of $\varphi = \{\varphi_j\}_{j=0}^{\infty} \in \Phi(\mathbb{R}^n)$.

REMARK 5.14. Theorem 5.13 is proved by Xu and Yang ([27, 28]) for constant p in the setting of Herz-type Triebel–Lizorkin spaces provided $-n/p < \alpha < n/p'$.

PROOF. Below we denote $I^s f := \mathcal{F}^{-1}\left[(1+|x|^2)^{s/2}\hat{f}\right]$. For every $f \in \dot{K}^{\alpha,q}_{p(\,\cdot\,)}W^s(\mathbb{R}^n)$ it obviously follows that

$$\sum_{|\gamma|=0,s} \|D^\gamma f\|_{\dot{K}^{\alpha,q}_{p(\,\cdot\,)}(\mathbb{R}^n)} \le \|f\|_{\dot{K}^{\alpha,q}_{p(\,\cdot\,)}W^s(\mathbb{R}^n)}.$$

We also note that Theorem 4.6 shows the density of $C_c^\infty(\mathbb{R}^n)$ in $\dot{K}^{\alpha,q}_{p(\,\cdot\,)}W^s(\mathbb{R}^n)$. Thus we have only to prove

$$\|I_s f\|_{\dot{K}^{\alpha,q}_{p(\,\cdot\,)}(\mathbb{R}^n)} \le C\,\|f\|_{\dot{K}^{\alpha,q}_{p(\,\cdot\,)}F_2^{s,\varphi}(\mathbb{R}^n)}, \tag{5.12}$$

$$\|f\|_{\dot{K}^{\alpha,q}_{p(\,\cdot\,)}F_2^{s,\varphi}(\mathbb{R}^n)} \le C\,\|I_s f\|_{\dot{K}^{\alpha,q}_{p(\,\cdot\,)}(\mathbb{R}^n)}, \tag{5.13}$$

$$\|f\|_{\dot{K}^{\alpha,q}_{p(\,\cdot\,)}W^s(\mathbb{R}^n)} \le C\,\|I_s f\|_{\dot{K}^{\alpha,q}_{p(\,\cdot\,)}(\mathbb{R}^n)}, \tag{5.14}$$

$$\|I_s f\|_{\dot{K}^{\alpha,q}_{p(\,\cdot\,)}(\mathbb{R}^n)} \le C \sum_{|\gamma|=0,s} \|D^\gamma f\|_{\dot{K}^{\alpha,q}_{p(\,\cdot\,)}(\mathbb{R}^n)} \tag{5.15}$$

for all $f \in C_c^\infty(\mathbb{R}^n)$.

We write $\varphi_j^s(x) := 2^{js}(1+|x|^2)^{-s/2}\varphi_j(x)$. Then we see that $\varphi^s := \{\varphi_j^s\}_{j=0}^{\infty} \in \mathcal{A}_L(\mathbb{R}^n)$ for every $L \in \mathbb{N}$.

• Proof of (5.12): By Lemma 5.3 we have

$$\begin{aligned}\|I_s f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} &\le C\, \|I_s f\|_{\dot{K}^{\alpha,q}_{p(\cdot)} F_2^{0,\varphi^s}(\mathbb{R}^n)} \\ &= C\, \left\| \left\| \left\{ \mathcal{F}^{-1} \left[\varphi_j^s \mathcal{F}[I_s f] \right] \right\}_j \right\|_{\ell^2} \right\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} \\ &= C\, \left\| \left\| \left\{ \mathcal{F}^{-1} \left[2^{js}(1+|x|^2)^{-s/2} \varphi_j \cdot (1+|x|^2)^{s/2} \hat{f} \right] \right\}_j \right\|_{\ell^2} \right\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} \\ &= C\, \|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)} F_2^{s,\varphi}(\mathbb{R}^n)}\,.\end{aligned}$$

• Proof of (5.13): We define $\left(\varphi_j^s\right)^* f$ by (5.9) with $r := p_0$ if we suppose (I), $r := 1$ if we suppose (II). Using Lemmas 5.11 and 5.3 we obtain

$$\begin{aligned}\|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)} F_2^{s,\varphi}(\mathbb{R}^n)} &= \|I_{-s}\,(I_s f)\|_{\dot{K}^{\alpha,q}_{p(\cdot)} F_2^{s,\varphi}(\mathbb{R}^n)} \\ &= \left\| \left\| \left\{ 2^{sj} \mathcal{F}^{-1} \left[(1+|x|^2)^{-s/2} \varphi_j \mathcal{F}[I_s f] \right] \right\}_j \right\|_{\ell^2} \right\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} \\ &= \left\| \left\| \left\{ \mathcal{F}^{-1} \left[\varphi_j^s \mathcal{F}[I_s f] \right] \right\}_j \right\|_{\ell^2} \right\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} \\ &\le \left\| \left\| \left\{ \left(\varphi_j^s\right)^* (I_s f) \right\}_j \right\|_{\ell^2} \right\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} \\ &\le C\, \|I_s f\|_{\dot{K}^{\alpha,q}_{p(\cdot)} F_2^{0,\varphi^s}(\mathbb{R}^n)} \le C\, \|I_s f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)}\,.\end{aligned}$$

• Proof of (5.14): Note that $x^\gamma (1+|x|^2)^{-s/2}$ is a Fourier multiplier for $\dot{K}^{\alpha,q}_{p(\cdot)} F_2^{0,\varphi}(\mathbb{R}^n)$ for every $|\gamma| \le s$. By virtue of Lemma 5.3 and Theorem 5.12, we get

$$\begin{aligned}\|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)} W^s(\mathbb{R}^n)} &= \sum_{|\gamma|\le s} \|D^\gamma f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} \le C \sum_{|\gamma|\le s} \left\| \mathcal{F}^{-1} \left[x^\gamma \hat{f} \right] \right\|_{\dot{K}^{\alpha,q}_{p(\cdot)} F_2^{0,\varphi}(\mathbb{R}^n)} \\ &= C \sum_{|\gamma|\le s} \left\| \mathcal{F}^{-1} \left[x^\gamma (1+|x|^2)^{-s/2} \cdot \mathcal{F}[I_s f] \right] \right\|_{\dot{K}^{\alpha,q}_{p(\cdot)} F_2^{0,\varphi}(\mathbb{R}^n)} \\ &\le C \sum_{|\gamma|\le s} \|I_s f\|_{\dot{K}^{\alpha,q}_{p(\cdot)} F_2^{0,\varphi}(\mathbb{R}^n)} \le C\, \|I_s f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)}\,.\end{aligned}$$

• Proof of (5.15): We can take Fourier multipliers $\rho_1(x), \dots, \rho_n(x)$ for $\dot{K}^{\alpha,q}_{p(\cdot)} F_2^{0,\varphi}(\mathbb{R}^n)$ such that

$$G(x) := 1 + \sum_{h=1}^{n} \rho_h(x) {x_h}^s \ge C\, (1+|x|^2)^{s/2}$$

(see [26, p. 60]). Define $H(x) := (1+|x|^2)^{s/2} G(x)^{-1}$, then $H(x)$ is also a Fourier multiplier for $\dot{K}^{\alpha,q}_{p(\cdot)} F_2^{0,\varphi}(\mathbb{R}^n)$. Therefore using Lemma

5.3 and Theorem 5.12 again, we obtain

$$
\begin{aligned}
&\|I_s f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} \\
&\quad \le C\, \|I_s f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}F^{0,\varphi}_2(\mathbb{R}^n)} \\
&\quad = C\, \left\| \mathcal{F}^{-1}\left[H \cdot \mathcal{F}\left[\mathcal{F}^{-1}\left[G\hat{f} \right]\right]\right]\right\|_{\dot{K}^{\alpha,q}_{p(\cdot)}F^{0,\varphi}_2(\mathbb{R}^n)} \\
&\quad \le C\, \left\| \mathcal{F}^{-1}\left[G\hat{f} \right]\right\|_{\dot{K}^{\alpha,q}_{p(\cdot)}F^{0,\varphi}_2(\mathbb{R}^n)} \\
&\quad \le C \left\{ \|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}F^{0,\varphi}_2(\mathbb{R}^n)} + \sum_{h=1}^{n} \left\| \mathcal{F}^{-1}\left[\rho_h(x) {x_h}^s \hat{f} \right]\right\|_{\dot{K}^{\alpha,q}_{p(\cdot)}F^{0,\varphi}_2(\mathbb{R}^n)} \right\} \\
&\quad \le C \left\{ \|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}F^{0,\varphi}_2(\mathbb{R}^n)} + \sum_{h=1}^{n} \left\| \mathcal{F}^{-1}\left[\rho_h(x) \mathcal{F}\left[\frac{\partial^s}{\partial {x_h}^s} f \right]\right]\right\|_{\dot{K}^{\alpha,q}_{p(\cdot)}F^{0,\varphi}_2(\mathbb{R}^n)} \right\} \\
&\quad \le C \left\{ \|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}F^{0,\varphi}_2(\mathbb{R}^n)} + \sum_{h=1}^{n} \left\| \frac{\partial^s}{\partial {x_h}^s} f \right\|_{\dot{K}^{\alpha,q}_{p(\cdot)}F^{0,\varphi}_2(\mathbb{R}^n)} \right\} \\
&\quad \le C \sum_{|\gamma|=0,s} \|D^\gamma f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} .
\end{aligned}
$$

Consequently we have proved the theorem for $\dot{K}^{\alpha,q}_{p(\cdot)}W^s(\mathbb{R}^n)$. The case of $K^{\alpha,q}_{p(\cdot)}W^s(\mathbb{R}^n)$ is proved by the same argument. ∎

6. Wavelet characterization of Herz–Sobolev spaces

Based on the fundamental wavelet theory (cf. [21]), we can construct functions φ, ψ^1, $\psi^2, \dots, \psi^{2^n-1}$ satisfying the following.

1. φ, $\psi^l \in \mathcal{S}(\mathbb{R}^n)$ for every $l = 1, 2, \dots, 2^n - 1$.
2. The sequence
$$\left\{ \varphi_{0,k},\, \psi^l_{j,k} \,:\, l = 1, 2, \dots, 2^n - 1,\, j \in \mathbb{N}_0,\, k \in \mathbb{Z}^n \right\}$$
forms an orthonormal basis in $L^2(\mathbb{R}^n)$.
3. $\int_{\mathbb{R}^n} x^\gamma \psi^l(x)\, dx = 0$ for all $\gamma \in {\mathbb{N}_0}^n$ and every $l = 1, 2, \dots, 2^n - 1$.
4. φ is real-valued and band-limited with $\operatorname{supp}\hat{\varphi} \subset \left[-\frac{4}{3}\pi, \frac{4}{3}\pi\right]^n$.
5. Each ψ^l is real-valued and band-limited with $\operatorname{supp}\hat{\psi^l} \subset \left\{\xi \in \mathbb{R}^n : a^{-1} \le |\xi| \le a\right\}$ for some constant $a > \frac{8\pi\sqrt{n}}{3}$.

The function φ is called the Meyer scaling function and each ψ^l is called the Meyer wavelet in terms of multiresolution analysis.

Using the Meyer scaling function φ and the Meyer wavelets $\{\psi^l : l = 1, 2, \ldots, 2^n - 1\}$ we define

$$\begin{aligned}
\mathcal{I}_0 f &:= \left(\sum_{k \in \mathbb{Z}^n} |\langle f, \varphi_{0,k} \rangle \chi_{0,k}|^2 \right)^{1/2} \\
\mathcal{I}_1 f &:= \left(\sum_{l=1}^{2^n-1} \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^n} \left| 2^{js} \langle f, \psi^l_{j,k} \rangle \chi_{j,k} \right|^2 \right)^{1/2} \\
\mathcal{U} f &:= \mathcal{I}_0 f + \mathcal{I}_1 f,
\end{aligned}$$

where

$$\chi_{j,k} := 2^{jn/2} \chi_{Q_{j,k}}, \; Q_{j,k} := \prod_{h=1}^{n} \left[2^{-j} k_h, 2^{-j}(k_h + 1) \right) \; (k = (k_1, \ldots, k_n) \in \mathbb{Z}^n),$$

and $\langle f, g \rangle$ denotes the L^2-inner product, namely $\langle f, g \rangle := \int_{\mathbb{R}^n} f(x) \overline{g(x)} \, dx$.

In terms of the square function $\mathcal{U} f$, we have the following wavelet characterization of Herz–Sobolev spaces with variable exponent.

THEOREM 6.1. *Let $s \in \mathbb{N}_0$, $1 < q < \infty$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\alpha \in \mathbb{R}$. Suppose the following (I) or (II).*

(I) *$p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $p_- < 2$, $(p')_- < 2$ and $|\alpha| < n \min\{\delta(p_0) p_0^{-1}, p_0^{-2} \delta_1, \delta_2\}$, where δ_1, $\delta_2 > 0$ and $p_0 \in (1, p_-)$ are the constants appearing in (3.9), (3.11) and Proposition 3.2, respectively, and $\delta(p_0) > 0$ appears in (3.12) with $r = p_0$.*

(II) *$p(\cdot)$ satisfies (3.1) and (3.2) in Proposition 3.1, and $|\alpha| < n \min\{\delta_1, \delta_2\}$.*

Then we have that for all $f \in K^{\alpha,q}_{p(\cdot)} W^s(\mathbb{R}^n)$,

$$C^{-1} \|f\|_{K^{\alpha,q}_{p(\cdot)} W^s(\mathbb{R}^n)} \le \|\mathcal{U} f\|_{K^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} \le C \, \|f\|_{K^{\alpha,q}_{p(\cdot)} W^s(\mathbb{R}^n)}. \tag{6.1}$$

Additionally if α satisfies $\alpha < \frac{n(p')_-}{q'(p')_+}$, then the same wavelet characterization as (6.1) also holds for $\dot{K}^{\alpha,q}_{p(\cdot)} W^s(\mathbb{R}^n)$.

REMARK 6.2. Before the proof, we have to check that the wavelet coefficients $\{\langle f, \varphi_{0,k} \rangle, \langle f, \psi^l_{j,k} \rangle\}_{l,j,k}$ are well-defined in Theorem 6.1. We first consider the homogeneous case. If $\mathcal{S}(\mathbb{R}^n) \subset \dot{K}^{-\alpha,q'}_{p'(\cdot)}(\mathbb{R}^n)$ is true, then

$$|\langle f, g \rangle| \le C \, \|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} \|g\|_{\dot{K}^{-\alpha,q'}_{p'(\cdot)}(\mathbb{R}^n)} < \infty$$

holds for all $f \in \dot{K}^{\alpha,q}_{p(\cdot)} W^s(\mathbb{R}^n)$ and all $g \in \mathcal{S}(\mathbb{R}^n)$. Thus we have only to show that $\mathcal{S}(\mathbb{R}^n) \subset \dot{K}^{-\alpha,q'}_{p'(\cdot)}(\mathbb{R}^n)$. Take $g \in \mathcal{S}(\mathbb{R}^n)$ arbitrarily and denote $m_0 := \frac{(p')_+}{(p')_-}$. Because $|g(x)| \le C \, (1 + |x|)^{-K}$ for any positive number $K >$

$-\alpha q' + nm_0$, we get

$$\|g\,\chi_{R_l}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \le C\,(1+2^{l-1})^{-K}\,\|\chi_{R_l}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.$$

Hence we obtain the estimate

$$\begin{aligned}
&\|g\|^{q'}_{\dot{K}^{-\alpha,q'}_{p'(\cdot)}(\mathbb{R}^n)} \\
&\le C\sum_{l=-\infty}^{\infty} 2^{-\alpha l q'}(1+2^{l-1})^{-K}\,\|\chi_{R_l}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\
&\le C\sum_{l=1}^{\infty} 2^{-\alpha l q'}(1+2^{l-1})^{-K} 2^{lnm_0} + C\sum_{l=-\infty}^{0} 2^{-\alpha l q'}(1+2^{l-1})^{-K} 2^{lnm_0^{-1}} \\
&\le C\sum_{l=1}^{\infty} 2^{l(-\alpha q' + nm_0 - K)} + C\sum_{l=-\infty}^{0} 2^{l(-\alpha q' + nm_0^{-1})}.
\end{aligned}$$

By virtue of $K > -\alpha q' + nm_0$ and $\alpha < \frac{n}{q' m_0}$, both of the two series are finite. Therefore we have proved $\mathcal{S}(\mathbb{R}^n) \subset \dot{K}^{-\alpha,q'}_{p'(\cdot)}(\mathbb{R}^n)$.

In the non-homogeneous case, the similar calculations imply $\mathcal{S}(\mathbb{R}^n) \subset K^{-\alpha,q'}_{p'(\cdot)}(\mathbb{R}^n)$ without the assumption $\alpha < \frac{n}{q' m_0}$.

We will prove Theorem 6.1 for the non-homogeneous case while the homogeneous case is similar. In order to prove the theorem, we need the following two lemmas.

Lemma 6.3. *Let $\gamma \in \mathbb{N}_0{}^n$ with $|\gamma| \le s$. Then we have that for all $f \in K^{\alpha,q}_{p(\cdot)}W^s(\mathbb{R}^n)$,*

$$\mathcal{I}_0(D^\gamma f)(x) \le C\,Mf(x).$$

Lemma 6.4. *Let $N \in \mathbb{N}_0$ and $\gamma \in \mathbb{N}_0{}^n$ such that $|\gamma| \le N$. Then we have that for all $f \in K^{\alpha,q}_{p(\cdot)}W^{N-|\gamma|}(\mathbb{R}^n)$,*

$$\left\| \left(\sum_{l=1}^{2^n-1} \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}^n} \left| 2^{j(N-|\gamma|)} \langle f, (D^\gamma \psi^l)_{j,k}\rangle \chi_{j,k}\right|^2 \right)^{1/2} \right\|_{K^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} \le C\,\|f\|_{K^{\alpha,q}_{p(\cdot)}W^{N-|\gamma|}(\mathbb{R}^n)}.$$

Proof of Lemma 6.3. We see that

$$\begin{aligned}
\mathcal{I}_0(D^\gamma f)(x) &= \left(\sum_{k\in\mathbb{Z}^n} |\langle f, (D^\gamma\varphi)_{0,k}\rangle \chi_{0,k}(x)|^2 \right)^{1/2} \\
&\le \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}^n} |\chi_{0,k}(x)(D^\gamma\varphi)_{0,k}(y)|\,|f(y)|\,dy.
\end{aligned}$$

By virtue of [13, Lemma 2.8], we can take a bounded and radial decreasing function $H \in L^1(\mathbb{R}^n)$ so that

$$\sum_{k\in\mathbb{Z}^n} |\chi_{0,k}(x)(D^\gamma\varphi)_{0,k}(y)| \le H(x-y).$$

In addition, we get

$$\int_{\mathbb{R}^n} H(x-y)|f(y)|\,dy \le C\,Mf(x)$$

(cf. [24, p. 63]). Therefore we have proved the lemma. ∎

Proof of Lemma 6.4. Denote $\phi_j^l(y) := 2^{jn}(D^\gamma\psi^l)(-2^j y)$ for $j \in \mathbb{Z}$ and $l = 1, 2, \ldots, 2^n - 1$. Then we have

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^n} \left|\langle f, (D^\gamma\psi^l)_{j,k}\rangle\chi_{j,k}(x)\right|^2 \\
&\quad= \sum_{k\in\mathbb{Z}^n} \left|\int_{\mathbb{R}^n} f(z)2^{-jn/2}\phi_j^l(2^{-j}k-z)\,dz\right|^2 \chi_{j,k}(x)^2 \\
&\quad= \sum_{k\in\mathbb{Z}^n} \left|\phi_j^l * f(2^{-j}k)\right|^2 \chi_{Q_{j,k}}(x) \\
&\quad\le \sum_{k\in\mathbb{Z}^n} \sup_{y\in Q_{j,k}} \left|\phi_j^l * f(y)\right|^2 \chi_{Q_{j,k}}(x) \\
&\quad\le \sup_{|z|\le 2^{-j}\sqrt{n}} \left|\phi_j^l * f(x-z)\right|^2 \\
&\quad\le \left\{\sup_{|z|\le 2^{-j}\sqrt{n}} \frac{\left|\phi_j^l * f(x-z)\right|}{(1+|2^j z|)^{n/r}}\right\}^2 \cdot \sup_{|z|\le 2^{-j}\sqrt{n}} (1+|2^j z|^{n/r})^2 \\
&\quad\le C\,(\mathcal{F}[\phi_j^l]^* f)(x)^2,
\end{aligned}$$

where $\mathcal{F}[\phi_j^l]^* f$ is defined by (5.9) with $r := p_0$ if we suppose (I), $r := 1$ if we suppose (II). Hence by virtue of Lemma 5.11 and Theorem 5.12, we get

$$\begin{aligned}
&\left\|\left(\sum_{l=1}^{2^n-1}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n} \left|2^{j(N-|\gamma|)}\langle f, (D^\gamma\psi^l)_{j,k}\rangle\chi_{j,k}\right|^2\right)^{1/2}\right\|_{K^{\alpha,q}_{p(\,\cdot\,)}(\mathbb{R}^n)} \\
&\quad\le C\sum_{l=1}^{2^n-1}\left\|\left(\sum_{j=0}^{\infty}\left|2^{j(N-|\gamma|)}(\mathcal{F}[\phi_j^l]^* f)\right|^2\right)^{1/2}\right\|_{K^{\alpha,q}_{p(\,\cdot\,)}(\mathbb{R}^n)} \\
&\quad\le C\,\|f\|_{K^{\alpha,q}_{p(\,\cdot\,)}W^{N-|\gamma|}(\mathbb{R}^n)}.
\end{aligned}$$

∎

Proof of Theorem 6.1. By virtue of Lemma 6.3, we obtain $\mathcal{I}_0 f(x) \le C\, Mf(x)$ for all $f \in K_{p(\cdot)}^{\alpha,q} W^s(\mathbb{R}^n)$. Because $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, we get

$$\|\mathcal{I}_0 f\|_{K_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)} \le C\, \|f\|_{K_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)} \le C\, \|f\|_{K_{p(\cdot)}^{\alpha,q} W^s(\mathbb{R}^n)}\,.$$

Additionally if we apply Lemma 6.4 with $N = s$ and $\gamma = (0, \dots, 0)$, then we immediately obtain

$$\|\mathcal{I}_1 f\|_{K_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)} \le C\, \|f\|_{K_{p(\cdot)}^{\alpha,q} W^s(\mathbb{R}^n)}\,.$$

Therefore we have

$$\|\mathcal{U} f\|_{K_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)} \le \|\mathcal{I}_0 f\|_{K_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)} + \|\mathcal{I}_1 f\|_{K_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)} \le C\, \|f\|_{K_{p(\cdot)}^{\alpha,q} W^s(\mathbb{R}^n)}\,.$$

Next we prove

$$\|f\|_{K_{p(\cdot)}^{\alpha,q} W^s(\mathbb{R}^n)} \le C\, \|\mathcal{U} f\|_{K_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)}\,. \tag{6.2}$$

We follow a duality argument (cf. [5, Chapter 6]) applying Lemma 5.7. Note that $C_c^\infty(\mathbb{R}^n)$ is dense in $K_{p(\cdot)}^{\alpha,q} W^s(\mathbb{R}^n)$ and in $K_{p'(\cdot)}^{-\alpha,q'}(\mathbb{R}^n)$ by virtue of Theorem 4.6. Hence it suffices to show that for all f, $g \in C_c^\infty(\mathbb{R}^n)$ with $\|g\|_{K_{p'(\cdot)}^{-\alpha,q'}(\mathbb{R}^n)} \le 1$ and all $\gamma \in {\mathbb{N}_0}^n$ with $|\gamma| \le s$,

$$\left| \int_{\mathbb{R}^n} D^\gamma f(x) g(x)\, dx \right| \le C\, \|\mathcal{U} f\|_{K_{p(\cdot)}^{\alpha,q}(\mathbb{R}^n)}\,.$$

Because f, $D^\gamma g \in L^2(\mathbb{R}^n)$, we obtain the wavelet expansions

$$\begin{aligned} f &= \sum_{k\in\mathbb{Z}^n} \langle f, \varphi_{0,k}\rangle \varphi_{0,k} + \sum_{l=1}^{2^n-1} \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}^n} \langle f, \psi_{j,k}^l\rangle \psi_{j,k}^l, \\ D^\gamma g &= \sum_{k\in\mathbb{Z}^n} \langle D^\gamma g, \varphi_{0,k}\rangle \varphi_{0,k} + \sum_{l=1}^{2^n-1} \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}^n} \langle D^\gamma g, \psi_{j,k}^l\rangle \psi_{j,k}^l. \end{aligned}$$

Thus we have

$$\begin{aligned} &\left| \int_{\mathbb{R}^n} D^\gamma f(x) g(x)\, dx \right| = \left| \int_{\mathbb{R}^n} f(x) D^\gamma g(x)\, dx \right| \\ &\le \left| \sum_{k\in\mathbb{Z}^n} \langle f, \varphi_{0,k}\rangle \langle D^\gamma g, \varphi_{0,k}\rangle \right| + \left| \sum_{l=1}^{2^n-1} \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}^n} \langle f, \psi_{j,k}^l\rangle \langle D^\gamma g, \psi_{j,k}^l\rangle \right|. \end{aligned} \tag{6.3}$$

Now we estimate the first sum. Using the Cauchy–Schwarz inequality and the generalized Hölder inequality, we get

$$\begin{aligned}\left|\sum_{k\in\mathbb{Z}^n}\langle f,\varphi_{0,k}\rangle\langle D^\gamma g,\varphi_{0,k}\rangle\right| &\leq \sum_{k\in\mathbb{Z}^n}|\langle f,\varphi_{0,k}\rangle\langle D^\gamma g,\varphi_{0,k}\rangle|\cdot\int_{\mathbb{R}^n}\chi_{0,k}(x)\,dx\\ &\leq \int_{\mathbb{R}^n}\mathcal{I}_0 f(x)\mathcal{I}_0(D^\gamma g)(x)\,dx\\ &\leq C\,\|\mathcal{I}_0 f\|_{K^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)}\,\|\mathcal{I}_0(D^\gamma g)\|_{K^{-\alpha,q'}_{p'(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Applying Lemma 6.3, we see that

$$\|\mathcal{I}_0(D^\gamma g)\|_{K^{-\alpha,q'}_{p'(\cdot)}(\mathbb{R}^n)}\leq C\,\|Mg\|_{K^{-\alpha,q'}_{p'(\cdot)}(\mathbb{R}^n)}\leq C\,\|g\|_{K^{-\alpha,q'}_{p'(\cdot)}(\mathbb{R}^n)}\leq C.$$

Therefore we have the estimate

$$\left|\sum_{k\in\mathbb{Z}^n}\langle f,\varphi_{0,k}\rangle\langle D^\gamma g,\varphi_{0,k}\rangle\right|\leq C\,\|\mathcal{I}_0 f\|_{K^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)}. \tag{6.4}$$

Next we estimate the second sum of (6.3). Using the Cauchy–Schwarz inequality and the generalized Hölder inequality again, we obtain

$$\begin{aligned}&\left|\sum_{l=1}^{2^n-1}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n}\langle f,\psi^l_{j,k}\rangle\langle D^\gamma g,\psi^l_{j,k}\rangle\right|\\ &\leq \sum_{l=1}^{2^n-1}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n}\left|\langle f,\psi^l_{j,k}\rangle\langle D^\gamma g,\psi^l_{j,k}\rangle\right|\cdot\int_{\mathbb{R}^n}\chi_{j,k}(x)^2\,dx\\ &= \int_{\mathbb{R}^n}\sum_{l=1}^{2^n-1}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n}\left|\langle f,\psi^l_{j,k}\rangle\chi_{j,k}(x)\langle g,2^{j|\gamma|}(D^\gamma\psi^l)_{j,k}\rangle\chi_{j,k}(x)\right|\,dx\\ &\leq \int_{\mathbb{R}^n}\sum_{l=1}^{2^n-1}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n}\left|2^{js}\langle f,\psi^l_{j,k}\rangle\chi_{j,k}(x)\cdot\langle g,(D^\gamma\psi^l)_{j,k}\rangle\chi_{j,k}(x)\right|\,dx\\ &\leq \int_{\mathbb{R}^n}\mathcal{I}_1 f(x)\cdot\left(\sum_{l=1}^{2^n-1}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n}\left|\langle g,(D^\gamma\psi^l)_{j,k}\rangle\chi_{j,k}(x)\right|^2\right)^{1/2}dx\\ &\leq C\,\|\mathcal{I}_1 f\|_{K^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)}\\ &\qquad\times\left\|\left(\sum_{l=1}^{2^n-1}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n}\left|\langle g,(D^\gamma\psi^l)_{j,k}\rangle\chi_{j,k}(x)\right|^2\right)^{1/2}\right\|_{K^{-\alpha,q'}_{p'(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Thus by virtue of Lemma 6.4 we get

$$\begin{aligned}\left|\sum_{l=1}^{2^n-1}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n}\langle f,\psi^l_{j,k}\rangle\langle D^\gamma g,\psi^l_{j,k}\rangle\right| &\le C\,\|\mathcal{I}_1 f\|_{K^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)}\,\|g\|_{K^{-\alpha,q'}_{p'(\cdot)}(\mathbb{R}^n)}\\ &\le C\,\|\mathcal{I}_1 f\|_{K^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)}. \qquad (6.5)\end{aligned}$$

Combing (6.4) and (6.5), we obtain (6.2).

Consequently we have proved the theorem. ∎

Remark 6.5. In [11] a wavelet characterization of different type of $K^{\alpha,q}_{p(\cdot)}W^s(\mathbb{R}^n)$ without the restriction on α in Theorem 6.1 is given. Let $s\in\mathbb{N}$, $1<q<\infty$, $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$ and $\alpha\in\mathbb{R}$. Using a C^{s+1}-smooth and compactly supported scaling function φ of a multiresolution analysis and an associated wavelets $\{\psi^l : l=1,2,\dots,2^n-1\}$ it was proved that the quasi-norm $\|f\|_{K^{\alpha,q}_{p(\cdot)}W^s(\mathbb{R}^n)}$ is equivalent to

$$\left\|\left(\sum_{k\in\mathbb{Z}^n}|\langle f,\varphi_{J,k}\rangle\chi_{J,k}|^2\right)^{1/2}+\left(\sum_{l=1}^{2^n-1}\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}^n}\left|2^{js}\langle f,\psi^l_{j,k}\rangle\chi_{j,k}\right|^2\right)^{1/2}\right\|_{K^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)},$$

where J is a sufficiently large integer depending only on s. In order to prove above he does not use boundedness of sublinear operators but local properties of wavelets. But this argument is not applicable to the homogeneous case.

Acknowledgements.

The author is thankful to Professor Naohito Tomita for his advices. He is also grateful to the referees whose valuable comments improved the first manuscript.

References

[1] A. Baernstein II and E. T. Sawyer, *Embedding and multiplier theorems for $H^p(\mathbb{R}^n)$,* Mem. Amer. Math. Soc. **53** (1985), no. 318.

[2] D. Cruz-Uribe, SFO, A. Fiorenza, J. M. Martell and C. Pérez, *The boundedness of classical operators on variable L^p spaces,* Ann. Acad. Sci. Fenn. Math. **31** (2006), 239–264.

[3] D. Cruz-Uribe, A. Fiorenza and C. J. Neugebauer, *The maximal function on variable L^p spaces,* Ann. Acad. Sci. Fenn. Math. **28** (2003), 223–238, and **29** (2004), 247–249.

[4] L. Diening, *Maximal functions on Musielak-Orlicz spaces and generalized Lebesgue spaces,* Bull. Sci. Math. **129** (2005), 657–700.

[5] E. Hernández and G. Weiss, A first course on wavelets, CRC Press, Boca Raton, 1996.

[6] E. Hernández, G. Weiss and D. Yang, *The φ-transform and wavelet characterizations of Herz-type spaces,* Collect. Math. **47** (1996), 285–320.

[7] E. Hernández and D. Yang, *Interpolation of Herz spaces and applications,* Math. Nachr. **205** (1999), 69–87.

[8] C. S. Herz, *Lipschitz spaces and Bernstein's theorem on absolutely convergent Fourier transforms,* J. Math. Mech. **18** (1968/1969), 283–323.

[9] M. Izuki, *Wavelets and modular inequalities in variable L^p spaces,* Georgian Math. J. **15** (2008), 281–293.

[10] M. Izuki, *Herz and amalgam spaces with variable exponent, the Haar wavelets and greediness of the wavelet system,* East J. Approx. **15** (2009), 87–109.

[11] M. Izuki, *Wavelet characterization of Herz–Sobolev spaces with variable exponent,* submitted.

[12] M. Izuki and K. Tachizawa, *Wavelet characterizations of weighted Herz spaces,* Sci. Math. Jpn. **67** (2008), 353–363.

[13] S. E. Kelly, M. A. Kon and L. A. Raphael, *Local convergence for wavelet expansions,* J. Funct. Anal. **126** (1994), 102–138.

[14] T. S. Kopaliani, *Greediness of the wavelet system in $L^{p(t)}(\mathbb{R})$ spaces,* East J. Approx. **14** (2008), 59–67.

[15] O. Kováčik and J. Rákosník, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$,* Czechoslovak Math. J. **41(116)** (1991), 592–618.

[16] D. S. Kurtz, *Littlewood-Paley and multiplier theorems on weighted L^p spaces,* Trans. Amer. Math. Soc. **259** (1980), 235–254.

[17] X. Li and D. Yang, *Boundedness of some sublinear operators on Herz spaces,* Illinois J. Math. **40** (1996), 484–501.

[18] S. Lu, K. Yabuta and D. Yang, *Boundedness of some sublinear operators in weighted Herz-type spaces,* Kodai Math. J. **23** (2000), 391–410.

[19] S. Lu and D. Yang, *The decomposition of weighted Herz spaces on $\mathbb{R}^n$ and its application,* Sci. China Ser. A **38** (1995), 147–158.

[20] S. Lu and D. Yang, *Herz-type Sobolev and Bessel potential spaces and their applications,* Sci. China Ser. A **40** (1997), 113–129.

[21] Y. Meyer, Wavelets and Operators, Cambridge University Press, Cambridge, 1992.

[22] E. Nakai, N. Tomita and K. Yabuta, *Density of the set of all infinitely differentiable functions with compact support in weighted Sobolev spaces*, Sci. Math. Jpn. **60** (2004), 121–127.

[23] A. Nekvinda, *Hardy-Littlewood maximal operator on $L^{p(x)}(\mathbb{R}^n)$,* Math. Inequal. Appl. **7** (2004), 255–265.

[24] E. M. Stein, Singular integrals and differentiability properties of functions, Princeton University Press, Princeton, 1970.

[25] L. Tang and D. Yang, *Boundedness of vector-valued operators on weighted Herz spaces,* Approx. Theory Appl. (N.S.) **16** (2000), 58–70.

[26] H. Triebel, Theory of Function Spaces, Birkhäuser, Basel, 1983.

[27] J. Xu and D. Yang, *Applications of Herz-type Triebel-Lizorkin spaces*, Acta Math. Sci. Ser. B Engl. Ed. **23** (2003), 328–338.

[28] J. Xu and D. Yang, *Herz-type Triebel-Lizorkin spaces. I,* Acta Math. Sci. (Engl. Ser.) **21** (2005), 643–654.

M. Izuki
Department of Mathematics
Faculty of Science
Hokkaido University
Kita 10 Nishi 8, Kita-ku, Sapporo, Hokkaido 060-0810
Japan
E-mail: `mitsuo@math.sci.hokudai.ac.jp`

Received: 1.7.2009.
Revised: 19.1.2010.