GLASNIK MATEMATIČKI
Vol. 45(65)(2010), 525 – 530

ON INVERSE LIMITS OF COMPACT SPACES. CORRECTION OF A PROOF

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Abstract. For a compact Hausdorff space X and an ANR for metrizable spaces M, one considers the space M^X of all mappings from X to M, endowed with the compact-open topology. Since a mapping $f\colon X' \to X$ induces a natural mapping $M^f\colon M^X \to M^{X'}$, an inverse system of compact Hausdorff spaces $\boldsymbol{X}$ determines a direct system $M^{\boldsymbol{X}}$ of spaces as well as the corresponding direct system of singular homology groups $H_n(M^{\boldsymbol{X}}; G)$. There is a natural isomorphism between the direct limit $\operatorname{dir\,lim} H_n(M^{\boldsymbol{X}}; G)$ and the singular homology group $H_n(M^X; G)$, where $X = \operatorname{inv\,lim} \boldsymbol{X}$. This continuity theorem, used by some authors, was published more than 50 years ago. Unfortunately, the author discovered a serious error in the proofs of two lemmas on which the result depended. The present paper gives new correct proofs of these lemmas.

1. Introduction

A compact Hausdorff space X and a metrizable space M determine the space M^X of all (continuous) mappings $\phi\colon X \to M$, endowed with the compact-open topology. Every mapping $f\colon X' \to X$ between compact Hausdorf spaces induces a mapping $M^f\colon M^X \to M^{X'}$, which assigns to $\phi \in M^X$ the composition $M^f(\phi) = \phi f$. Therefore, if one has an inverse system $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ of compact Hausdorff spaces and a metrizable space M, the connecting mappings $p_{\lambda\lambda'}\colon X_{\lambda'} \to X_\lambda$, $\lambda \leq \lambda'$, induce mappings $P_{\lambda\lambda'} = M^f(p_{\lambda\lambda'})\colon M^{X_\lambda} \to M^{X_{\lambda'}}$ and one obtains a direct system $M^{\boldsymbol{X}} = (M^{X_\lambda}, P_{\lambda\lambda'}, \Lambda)$. If X is the limit of $\boldsymbol{X}$ and $p_\lambda\colon X \to X_\lambda, \lambda \in \Lambda$, are the corresponding canonical projections, then they induce mappings $P^\lambda =$

2010 *Mathematics Subject Classification.* 54C35, 54B35, 54C55, 55N10.
Key words and phrases. Homology of spaces of mappings, compact Hausdorff space, absolute neighborhood retract, absolute neighborhood extensor, direct system, inverse system.

$M^{p_\lambda}\colon M^{X_\lambda} \to M^X$ such that $P_{\lambda'} P_{\lambda\lambda'} = P_\lambda$. Denote by $H_n(.;G)$ the functor of singular n-dimensional homology group with coefficients in an abelian group G. Denote by $P_{\lambda\lambda' n}\colon H_n(M^{X_\lambda};G) \to H_n(M^{X_{\lambda'}};G)$ the homomorphism induced by $P_{\lambda\lambda'}\colon M^{X_\lambda} \to M^{X_{\lambda'}}$. Then, $M^{\boldsymbol{X}}$ determines the direct system $H_n(M^{\boldsymbol{X}};G) = (H_n(M^{X_\lambda};G), P_{\lambda\lambda' n}, \Lambda)$ of homology groups. Moreover, the mappings P^λ induce homomorphisms $P^{\lambda n}\colon H_n(M^{X_\lambda};G) \to H_n(M^X;G)$ such that $P_{\lambda' n} P_{\lambda\lambda' n} = P_{\lambda n}$. Clearly, these homomorphisms induce a homomorphism P_n from the direct limit $\operatorname{dir\,lim} H_n(M^{\boldsymbol{X}};G)$ to the homology group $H_n(M^X;G)$.

A straightforward verification shows that $P_n\colon \operatorname{dir\,lim} H_n(M^{\boldsymbol{X}};G) \to H_n(M^X;G)$ is a natural homomorphism, i.e., if $\boldsymbol{X}' = ({X'}_\lambda, {p'}_{\lambda\lambda'}, \Lambda)$ is another system of compact Hausdorff spaces and $\boldsymbol{g} = (g_\lambda, \Lambda)\colon \boldsymbol{X}' \to \boldsymbol{X}$ is a level-preserving mapping of inverse systems, then the following diagram commutes.

$$\begin{array}{ccc} \operatorname{dir\,lim} H_n(M^{\boldsymbol{X}};G) & \xrightarrow{P_n} & H_n(M^X;G) \\ {\scriptstyle \operatorname{dir\,lim} H_n(M^{\boldsymbol{g}};G)}\downarrow & & \downarrow{\scriptstyle H_n(M^g;G)} \\ \operatorname{dir\,lim} H_n(M^{\boldsymbol{X}'};G) & \xrightarrow[{P'}_n]{} & H_n(M^{X'};G), \end{array}$$

where $H_n(M^g;G)\colon H_n(M^X;G) \to H_n(M^{X'};G)$ is the homomorphism induced by $g = \lim \boldsymbol{g}\colon X' \to X$ and $\operatorname{dir\,lim} H_n(M^{\boldsymbol{g}};G)\colon \operatorname{dir\,lim} H_n(M^{\boldsymbol{X}};G) \to \operatorname{dir\,lim} H_n(M^{\boldsymbol{X}'};G)$ is the homomorphism induced by $M^{\boldsymbol{g}}\colon M^{\boldsymbol{X}} \to M^{\boldsymbol{X}'}$.

The main result of the present paper is the following theorem.

THEOREM 1.1. *If $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ is an inverse system of compact Hausdorff spaces with limit $X = \lim \boldsymbol{X}$ and M is an ANR for metrizable spaces, then the natural homomorphism $P_n\colon \operatorname{dir\,lim} H_n(M^{\boldsymbol{X}};G) \to H_n(M^X;G)$ is an isomomorphism.*

The special case when $\boldsymbol{X}$ is an inverse sequence of metrizable compacta and M is a compact ANR was proved already in [11] as Theorem 13 on page 200. The generalization to inverse systems of compact Hausdorff spaces and arbitrary ANRs appears in [12] as Theorem 6 on page 254. It follows the proof of the theorem in the special case, given in [11]. However, two lemmas used in that proof, i.e., a lemma due to M. Abe and Lemma 8, stated on page 199 of [12], had to be generalized as follows.

LEMMA 1.2. *Let $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be an inverse system of compact Hausdorff spaces with limit $X = \lim \boldsymbol{X}$ and canonical projections $p_\lambda\colon X \to X_\lambda$, $\lambda \in \Lambda$. If M is an ANR for metrizable spaces, then every mapping $f\colon X \to M$ admits a $\lambda \in \Lambda$ such that, for every $\mu \geq \lambda$, there exists a mapping $f_\mu\colon X \to M$, having the property that $f_\mu p_\mu \simeq f$. Moreover, if $\nu \geq \mu \geq \lambda$, then $f_\mu p_{\mu\nu} = f_\nu$.*

LEMMA 1.3. *Let* $(\boldsymbol{Y}, \boldsymbol{X}) = ((Y_\lambda, X_\lambda), q_{\lambda\lambda'}, \Lambda)$ *be an inverse system of compact Hausdorff pairs,* $X_\lambda \subseteq Y_\lambda$*, with limit* $(Y, X) = \lim(\boldsymbol{Y}, \boldsymbol{X})$ *and canonical projections* $q_\lambda \colon (Y, X) \to (Y_\lambda, X_\lambda)$*,* $\lambda \in \Lambda$*. Let* $p_{\lambda\lambda'} \colon X_{\lambda'} \to X_\lambda$ *and* $p_\lambda \colon X \to X_\lambda$ *be the restrictions of the mappings* $q_{\lambda\lambda'}$ *and* q_λ *to* $X_{\lambda'}$ *and* X*, respectively. If* M *is an ANR for metrizable spaces and for a given* $\lambda \in \Lambda$*,* $f_\lambda \colon X_\lambda \to M$ *is a mapping such that* $f_\lambda p_\lambda \colon X \to M$ *admits an extension* $g \colon Y \to M$ *to all of* Y*, then there is a* $\mu \geq \lambda$ *such that* $f_\lambda p_{\lambda\mu} \colon X_\mu \to M$ *admits an extension* $g_\mu \colon Y_\mu \to M$ *to all of* Y_μ*.*

By an extension we always mean a continuous extension. Lemmas 1.2 and 1.3 appear in [12] as Theorems 4 and 5. Unfortunately, the proofs given in that paper are not correct. Indeed, they are based on the following theorem of R. Arens (see [3, Theorem 4.1]).

Let C *be a closed convex subset of a Banach space* L*. Every mapping* $f \colon A \to C$ *of a closed subset* A *of a (Hausdorff) paracompact space* X *to* C *admits an extension* $g \colon X \to C$ *to the whole space* X*.*

The assumption that C is closed in L is not fulfilled in the application made in [12]. In fact, there the Arens' theorem was misstated, because "closed convex" was replaced by "convex".

In the present paper we give correct proofs of Lemmas 1.2 and 1.3 and thus, we obtain a correct proof of Theorem 1.1. The new proof of Lemma 1.3 uses the following proposition.

PROPOSITION 1.4. *Every ANR for metrizable spaces is an ANE for compact Hausdorff spaces.*

Yu.T. Lisica proved the following theorem (see [10, Theorem 1]).

Every convex subset of a Banach space is an AE for paracompact p*-spaces.*

By the well-known Kuratowski-Wojdisławski embedding theorem, every metrizable space embeds in a Banach space as a closed subset of its convex hull ([17], also see [14, I.3.1, Theorem 2]). Therefore, the following proposition holds.

PROPOSITION 1.5. *Every ANR for metrizable spaces is an ANE for paracompact* p*-spaces.*

p-spaces were introduced by A.V. Arhangelskiĭ ([4]), who proved that paracompact p-spaces coincide with spaces which admit perfect mappings to metrizable spaces ([4, Theorem 16]). It is clear from this characterization, that compact Hausdorff spaces are paracompact p-spaces. Consequently, Proposition 1.4 is an immediate consequence of Proposition 1.5. In fact, Proposition 1.4 is an immediate consequence of the Kuratowski-Wojdisławski embedding theorem and the following special case of the theorem of Lisica.

Proposition 1.6. *Every convex subset C of a Banach space is an AE for compact Hausdorff spaces.*

Although the paper [12] has been published more than 50 years ago, to the author it appears justified to correct the faulty proofs, because the paper has been cited at least in the papers [1,2,5,8,9,15] and [16] and Theorem 1.1 was used in an essential way in [15] and [16].

2. Proofs of Lemmas 1.2 and 1.3

In the proofs we will use the fact that the limit $\boldsymbol{p} = (p_\lambda, \Lambda)\colon X \to \boldsymbol{X}$ of an inverse system of compact Hausdorff spaces $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ is a resolution ([13, Theorem 6.20]) and a homotopy expansion ([13, Corollary 7.8]). This allows us to use property (R1) (see [13, page 104 and Lemma 6.3]), property (B2′) (see [13, page 107, Theorem 6.7 and Remark 6.13]) and Morita's property (M1) ([13, page 129]).

Proof of Lemma 1.2. Since $\boldsymbol{p}\colon X \to \boldsymbol{X}$ is a homotopy expansion, it has Morita's property (M1). Therefore, for a mapping $f\colon X \to M$ into an ANR for metrizable spaces M, there exist a $\lambda \in \Lambda$ and a mapping $f_\lambda\colon X_\lambda \to M$ such that $f_\lambda p_\lambda \simeq f$. For $\mu \geq \lambda$, put $f_\mu = f_\lambda p_{\lambda\mu}$. Note that $f_\mu p_\mu = f_\lambda p_{\lambda\mu} p_\mu = f_\lambda p_\lambda \simeq f$. If $\nu \geq \mu$, then $\nu \geq \lambda$ and thus, $f_\nu = f_\lambda p_{\lambda\nu} = f_\lambda p_{\lambda\mu} p_{\mu\nu} = f_\mu p_{\mu\nu}$. ∎

Proof of Lemma 1.3. For $\lambda \in \Lambda$, let $f_\lambda\colon X_\lambda \to M$ and $g\colon Y \to M$ be mappings such that $g|X = f_\lambda p_\lambda$. Since M is an ANR for metrizable spaces and X_λ is a closed subset of the compact Hausdorff space Y_λ, Proposition 1.4 yields an open neighborhood U_λ of X_λ in Y_λ and an extension $\overline{f}_\lambda\colon U_\lambda \to M$ of f_λ. Since M is an ANR for metrizable spaces, there exists an open covering $\mathcal{V}$ of M such that any two $\mathcal{V}$-near mappings into M are homotopic ([14, I.3.2, Corollary 1]). Since $\boldsymbol{q}$ is a resolution, by property (R1) for $\boldsymbol{q}$, there exist a $\lambda' \in \Lambda$ and a mapping $g_{\lambda'}\colon Y_{\lambda'} \to M$ such that the mappings $g_{\lambda'} q_{\lambda'}, g\colon Y \to M$ are $\mathcal{V}$-near. There is no loss of generality in assuming that $\lambda' \geq \lambda$.

Let us show that $g_{\lambda'}|p_{\lambda'}(X)$ and $f_\lambda p_{\lambda\lambda'}|p_{\lambda'}(X)$ are $\mathcal{V}$-near mappings. Indeed, if $x_{\lambda'} \in p_{\lambda'}(X)$, there is a point $x \in X$ such that $x_{\lambda'} = p_{\lambda'}(x)$. Note that $g_{\lambda'}(x_{\lambda'}) = g_{\lambda'} p_{\lambda'}(x) = g_{\lambda'} q_{\lambda'}(x)$. Therefore, there is a $V \in \mathcal{V}$ such that $g_{\lambda'}(x_{\lambda'}), g(x) \in V$. The assertion follows, because $g(x) = f_\lambda p_\lambda(x) = f_\lambda p_{\lambda\lambda'} p_{\lambda'}(x) = f_\lambda p_{\lambda\lambda'}(x_{\lambda'})$ and thus, $g_{\lambda'}(x_{\lambda'}), f_\lambda p_{\lambda\lambda'}(x_{\lambda'}) \in V$.

Now put $U_{\lambda'} = (q_{\lambda\lambda'})^{-1}(U_\lambda) \subseteq Y_{\lambda'}$ and note that $p_{\lambda'}(X) \subseteq U_{\lambda'}$, because $q_{\lambda\lambda'}(p_{\lambda'}(X)) = p_{\lambda\lambda'} p_{\lambda'}(X) = p_\lambda(X) \subseteq X_\lambda \subseteq U_\lambda$. Also note that for $y_{\lambda'} \in U_{\lambda'}$, one has $q_{\lambda\lambda'}(y_{\lambda'}) \in U_\lambda$ and therefore, $\overline{f}_\lambda q_{\lambda\lambda'}\colon U_{\lambda'} \to M$ is well defined. Moreover, since the restrictions $g_{\lambda'}|p_{\lambda'}(X)$ and $\overline{f}_\lambda q_{\lambda\lambda'}|p_{\lambda'}(X) = \overline{f}_\lambda p_{\lambda\lambda'}|p_{\lambda'}(X) = f_\lambda p_{\lambda\lambda'}|p_{\lambda'}(X)$ are $\mathcal{V}$-near mappings, there exists an open neighborhood $U'_{\lambda'} \subseteq U_{\lambda'}$ of $p_{\lambda'}(X)$ in $Y_{\lambda'}$ such that the mappings $g_{\lambda'}|U'_{\lambda'}$ and $\overline{f}_\lambda q_{\lambda\lambda'}|U'_{\lambda'}$ are also $\mathcal{V}$-near mappings.

Since $U'_{\lambda'} \cap X_{\lambda'}$ is an open neighborhood of $p_{\lambda'}(X)$ in $X_{\lambda'}$, property (B2)$'$ of the inverse limit $\boldsymbol{p} = (p_\lambda, \Lambda)\colon X \to \boldsymbol{X}$ yields a $\mu \geq \lambda'$ such that $p_{\lambda'\mu}(X_\mu) \subseteq U'_{\lambda'} \cap X_{\lambda'} \subseteq U'_{\lambda'}$. We now define a mapping $g_\mu\colon Y_\mu \to M$ by the formula $g_\mu = g_{\lambda'} q_{\lambda'\mu}$. Let us show that $g_\mu | X_\mu$ and $f_\lambda p_{\lambda\mu}$ are $\mathcal{V}$-near mappings and therefore,

$$g_\mu | X_\mu \simeq f_\lambda p_{\lambda\mu}.$$

Indeed, if $x_\mu \in X_\mu$, then $p_{\lambda'\mu}(x_\mu) \in U'_{\lambda'}$ and therefore, there is a $V \in \mathcal{V}$ such that the points $g_{\lambda'} p_{\lambda'\mu}(x_\mu), \overline{f}_{\lambda'} q_{\lambda\lambda'} p_{\lambda'\mu}(x_\mu) \in V$. However, $g_{\lambda'} p_{\lambda'\mu}(x_\mu) = g_\mu(x_\mu)$ and $\overline{f}_\lambda q_{\lambda\lambda'} p_{\lambda'\mu}(x_\mu) = \overline{f}_\lambda p_{\lambda\mu}(x_\mu) = f_\lambda p_{\lambda\mu}(x_\mu)$, because $p_{\lambda'\mu}(x_\mu) \in X_{\lambda'}$ and thus, $q_{\lambda\lambda'} p_{\lambda'\mu}(x_\mu) = p_{\lambda\lambda'} p_{\lambda'\mu}(x_\mu) = p_{\lambda\mu}(x_\mu)$.

Clearly, the mapping $g_\mu | X_\mu \colon X_\mu \to M$ admits an extension to all of Y_μ, because $g_\mu\colon Y_\mu \to M$ is such a mapping. Now the homotopy extension property (HEP) shows that the mapping $f_\lambda p_{\lambda\mu}$, being homotopic to $g_\mu | X_\mu$, also admits an extension to all of Y_μ. □

Remark 2.1. Recall that a pair of spaces (Z, B), where B is a closed subset of Z, is said to have the HEP with respect to a space M, provided every mapping $F\colon (B \times I) \cup (Z \times 0) \to M$ admits an extension $H\colon (Z \times I) \to M$. The well-known Dowker lemma asserts that this is the case, whenever Z is normal and there is a neighborhood U of B in Z such that F admits an extension $G\colon (U \times I) \cup (Z \times 0) \to M$ ([7, Lemma IV.2.1]). Consequently, if $\mathcal{C}$ is a class of normal spaces, having the property that $Z \in \mathcal{C}$ implies $Z \times I \in \mathcal{C}$, it follows that every pair (Z, B), where $Z \in \mathcal{C}$, has the HEP with respect to any space M, which is an absolute neighborhood extensor for the class $\mathcal{C}$. In particular, Proposition 1.4 shows that this is the case when $\mathcal{C}$ is the class of compact Hausdorff spaces and M is an ANR for metrizable spaces.

3. A more elementary proof of Proposition 1.6

Proof. Let (X, A) be a pair of compact Hausdorff spaces and let $f\colon A \to C$ be a mapping to a convex subset C of a Banach space L. We must exhibit an extension $g\colon X \to C$ of f to all of X. First note that there is an extension $\overline{f}\colon X \to L$ of $f\colon A \to C \subseteq L$. It suffices to apply the Arens theorem to L, viewed as a closed convex subset of L. Now note that $(L, f(A))$ is a pair of metrizable spaces; $f(A)$ is closed in L, because it is compact. Consider the inclusion mapping $i\colon f(A) \to C$. Applying the well-known Dugundji extension theorem ([6], also see [14, I.3.1, Theorem 3]) to $(L, f(A))$ and $C \subseteq L$, one obtains a mapping $h\colon L \to C$ such that $h | f(A) = i$. Clearly, $g = h\overline{f}\colon X \to C$ is a mapping which extends g, because $g(a) = h\overline{f}(a) = hf(a) = f(a)$, for $a \in A$. □

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Received: 3.9.2009.
Revised: 19.10.2009.