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THE INDUCED HOMOLOGY AND HOMOTOPY FUNCTORS ON THE COARSE SHAPE CATEGORY

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Abstract. In this paper we consider some algebraic invariants of the coarse shape. We introduce functors pro^*-H_n and pro^*-π_n relating the (pointed) coarse shape category $(Sh^*_\star)$ Sh^* to the category pro^*-Grp. The category $(Sh^*_\star)$ Sh^*, which is recently constructed, is the supercategory of the (pointed) shape category $(Sh_\star)$ Sh^*, having all (pointed) topological spaces as objects. The category pro^*-Grp is the supercategory of the category of *pro*-groups *pro*-*Grp*, both having the same object class. The functors pro^*-H_n and pro^*-π_n extend standard functors *pro*-H_n and *pro*-π_n which operate on $(Sh_\star)$ Sh^*. The full analogue of the well known Hurewicz theorem holds also in $Sh^*_\star$. We proved that the *pro*-homology (homotopy) sequence of every pair (X, A) of topological spaces, where A is normally embedded in X, is also exact in pro^*-Grp. Regarding this matter the following general result is obtained: for every category $\mathcal{C}$ with zero-objects and kernels, the category *pro*-$\mathcal{C}$ is also a category with zero-objects and kernels, while morphisms of pro^*-$\mathcal{C}$ generally don't have kernels.

1. Introduction

In the last few years several articles have been published in which are introduced some new classifications of metric compacta (see [11,12]) and even classifications of all topological spaces (see [5, 13]) which are coarser than the shape type classification. These classifications are playing an important role because they provide better information about spaces having different shape types than the standard shape classification. For instance, recall that all fibres of a shape fibration over an arbitrary metric continuum, generally,

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don't have the same shape type, but there is a certain equivalence relation, called S-equivalence, such that all fibres of a shape fibration are mutually S-equivalent (see [4]). Although, K. Borsuk in 1976, for the very first time, has presented some relations between metric compacta coarser than shape, we may say that Mardešić's S-equivalence, introduced in [9], was an origin of most later ideas about relations coarser than shape. Namely, it has been proven in [3] that Borsuk's relation of quasi-equivalence, introduced in [1], is not transitive. Therefore the interest for this relation has decreased while a studying of the S-equivalence was carried on and recently intensified. Since the question whether the S-equivalence admits characterization in terms of category isomorphisms is still an open issue, Mardešić and Uglešić in [11] have described an S^*-equivalence relation which is an "uniformization" of the S-equivalence. They have proven that the S^*-equivalence admits a characterization via isomorphisms of a category constructed over metric compacta as objects. Although the S^*-equivalence is finer than S-equivalence (but still strictly coarser than shape) it has kept all known S-invariants and all nice properties which S-equivalence has. Recently, Uglešić and the author in [5] have extended the S^*-equivalence to all topological spaces. They have constructed a coarse shape category Sh^* whose objects are all topological spaces such that its isomorphisms classify topological spaces strictly coarser than the shape. The coarse shape classification restricted to the class of metric compacta is exactly the S^*-equivalence. On the other hand, since the shape category Sh can be considered as the subcategory of Sh^* one may say that the coarse shape generalizes the shape theory. The coarse shape preserves some important topological or shape invariants (see [8]) as connectedness, movability, strong movability, n-movability, shape dimension and stability.

In the present paper we consider some algebraic invariants of the coarse shape. Homology (homotopy) *pro*-groups of (pointed) topological spaces can be considered as objects of the category *pro**-*Grp* which is a supercategory of the *pro*-*Grp* (the class of objects in *pro*-*Grp* and *pro**-*Grp* coincide, but sets of morphisms between *pro*-groups are essentially larger in *pro**-*Grp* than in *pro*-*Grp*). The isomorphisms of *pro**-*Grp* induce an equivalence relation among homology (homotopy) *pro*-groups which is an invariant of a (pointed) coarse shape type. An application of this technique in the studying of shape and coarse shape theory can be readily seen in Example 3.5 (first homology *pro*-groups of two spaces having different shape and the same coarse shape type are considered) and in Example 3.6 (0-dimensional homotopy "*pro*-groups" of two pointed spaces having different pointed coarse shape type are considered). However, this approach provides not only some useful coarse shape invariants but also induces functors *pro**-H_n (*pro**-π_n) relating Sh^* ($Sh^*_\star$-the pointed coarse shape category) and the category *pro**-*Grp*. We introduce these functors in Section 3. Since it is proven that zero-objects in the categories *pro**-*Grp* and *pro*-*Grp* coincide, the notions of n-shape connectedness and of n-coarse shape connectedness are mutually equivalent (Corollary 4.7). In Section 4 we have established the full analogue of the well known Hurewicz theorem in $Sh^*_\star$ (Theorem 4.11). By considering the relative homology (homotopy) groups in *pro**-*Grp* it is proven that the pro-homology (homotopy) sequence of every pair (X, A) of topological spaces, where A is normally embedded in X, is also exact in *pro**-*Grp* (Theorem 5.10). Prior to this, in Section 5, an arbitrary category $\mathcal{C}$ with zero-objects and kernels is considered, and several general results has been obtained. It is proven that the category *pro*-$\mathcal{C}$ is also a category with zero-objects and kernels (Theorem 5.4), while morphisms in *pro**-$\mathcal{C}$ generally don't have kernels (Example 5.6). However, the induced morphisms of *pro**-$\mathcal{C}$ have kernels (Theorem 5.7).

2. Preliminaries

We begin by recalling some main notions concerning the coarse shape and *pro**-category (see [5]). Let $\mathcal{C}$ be a category and let $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be two inverse systems in $\mathcal{C}$. An *S^*-morphism of inverse systems*, $(f, f_\mu^n) : \boldsymbol{X} \to \boldsymbol{Y}$, consists of a function $f : M \to \Lambda$, and of a set of $\mathcal{C}$-morphisms $f_\mu^n : X_{f(\mu)} \to Y_\mu$, $n \in \mathbb{N}$, $\mu \in M$, such that, for every related pair $\mu \leq \mu'$ in M, there exists a $\lambda \in \Lambda$, $\lambda \geqslant f(\mu), f(\mu')$, and there exists an $n \in \mathbb{N}$ so that, for every $n' \geqslant n$,

$$f_\mu^{n'} p_{f(\mu)\lambda} = q_{\mu\mu'} f_{\mu'}^{n'} p_{f(\mu')\lambda}.$$

If the index function f is increasing and, for every pair $\mu \leq \mu'$, one may put $\lambda = f(\mu')$, then (f, f_μ^n) is said to be a *simple* S^*-morphism. If, in addition, $M = \Lambda$ and $f = 1_\Lambda$, then $(1_\Lambda, f_\lambda^n)$ is said to be a *level* S^*-morphism.

The *composition* of S^*-morphisms $(f, f_\mu^n) : \boldsymbol{X} \to \boldsymbol{Y}$ and $(g, g_\nu^n) : \boldsymbol{Y} \to \boldsymbol{Z}$ is an S^*-morphism $(h, h_\nu^n) = (g, g_\nu^n)(f, f_\mu^n) : \boldsymbol{X} \to \boldsymbol{Z}$, where $h = fg$ and $h_\nu^n = g_\nu^n f_{g(\nu)}^n$. The *identity* S^*-morphism on $\boldsymbol{X}$ is an S^*-morphism $(1_\Lambda, 1_{X_\lambda}^n) : \boldsymbol{X} \to \boldsymbol{X}$, where 1_Λ is the identity function and $1_{X_\lambda}^n = 1_{X_\lambda}$ is the identity morphisms in $\mathcal{C}$, for all $n \in \mathbb{N}$ and $\lambda \in \Lambda$.

An S^*-morphism $(f, f_\mu^n) : \boldsymbol{X} \to \boldsymbol{Y}$ of inverse systems in $\mathcal{C}$ is said to be *equivalent to* an S^*-morphism $(f', f_\mu'^n) : \boldsymbol{X} \to \boldsymbol{Y}$, denoted by $(f, f_\mu^n) \sim (f', f_\mu'^n)$, provided every $\mu \in M$ admits a $\lambda \in \Lambda$, $\lambda \geqslant f(\mu), f'(\mu)$, and an $n \in \mathbb{N}$, such that, for every $n' \geqslant n$,

$$f_\mu^{n'} p_{f(\mu)\lambda} = f_\mu'^{n'} p_{f'(\mu)\lambda}.$$

The relation $\sim$ is an equivalence relation among S^*-morphisms of inverse systems in $\mathcal{C}$. The equivalence class $[(f, f_\mu^n)]$ of an S^*-morphism $(f, f_\mu^n) : \boldsymbol{X} \to \boldsymbol{Y}$ is briefly denoted by $\boldsymbol{f}^*$.

The category *pro**-$\mathcal{C}$ has as objects all inverse systems $\boldsymbol{X}$ in $\mathcal{C}$ and as morphisms all equivalence classes $\boldsymbol{f}^* = [(f, f_\mu^n)]$ of S^*-morphisms (f, f_μ^n).

The composition in pro^*-$\mathcal{C}$ is well defined by putting

$$\boldsymbol{g}^*\boldsymbol{f}^* = \boldsymbol{h}^* \equiv [(h, h^n_\nu)],$$

where $(h, h^n_\nu) = (g, g^n_\nu)(f, f^n_\mu) = (fg, g^n_\nu f^n_{g(\nu)})$. For every inverse system $\boldsymbol{X}$ in $\mathcal{C}$, the identity morphism in pro^*-$\mathcal{C}$ is $\boldsymbol{1}^*_{\boldsymbol{X}} = [(1_\Lambda, 1_{X_\lambda})]$.

A functor $\underline{J} \equiv \underline{J}_{\mathcal{C}} : pro\text{-}\mathcal{C} \to pro^*\text{-}\mathcal{C}$ is defined as follows. It keeps objects fixed, i.e., $\underline{J}(\boldsymbol{X}) = \boldsymbol{X}$, for every inverse system $\boldsymbol{X}$ in $\mathcal{C}$. If $\boldsymbol{f} \in pro\text{-}\mathcal{C}(\boldsymbol{X}, \boldsymbol{Y})$ and if (f, f_μ) is any representative of $\boldsymbol{f}$, then a morphism $\underline{J}(\boldsymbol{f}) = \boldsymbol{f}^* = [(f, f^n_\mu)] \in pro^*\text{-}\mathcal{C}(\boldsymbol{X}, \boldsymbol{Y})$ is represented by the S^*-morphism (f, f^n_μ), where $f^n_\mu = f_\mu$ for all $\mu \in M$ and $n \in \mathbb{N}$. The morphism $\boldsymbol{f}^*$ is said to be *induced* by $\boldsymbol{f}$. Since the functor $\underline{J}$ is faithful, we may consider the category pro-$\mathcal{C}$ as a subcategory of pro^*-$\mathcal{C}$. Thus, every morphism $\boldsymbol{f}$ in pro-$\mathcal{C}$ can be considered as a morphism of the category pro^*-$\mathcal{C}$, too.

Let $\mathcal{D}$ be a full and pro-reflective (or dense) subcategory of $\mathcal{C}$ (see [10]). Let $\boldsymbol{p} : X \to \boldsymbol{X}$ and $\boldsymbol{p'} : X \to \boldsymbol{X'}$ be $\mathcal{D}$-expansions of the same object X of $\mathcal{C}$, and let $\boldsymbol{q} : Y \to \boldsymbol{Y}$ and $\boldsymbol{q'} : Y \to \boldsymbol{Y'}$ be $\mathcal{D}$-expansions of the same object Y of $\mathcal{C}$. Then there exist two natural (unique) isomorphisms $\boldsymbol{i} : \boldsymbol{X} \to \boldsymbol{X'}$ and $\boldsymbol{j} : \boldsymbol{Y} \to \boldsymbol{Y'}$ in pro-$\mathcal{D}$. Consequently, $\boldsymbol{i}^* \equiv \underline{J}(\boldsymbol{i}) : \boldsymbol{X} \to \boldsymbol{X'}$ and $\boldsymbol{j}^* \equiv \underline{J}(\boldsymbol{j}) : \boldsymbol{Y} \to \boldsymbol{Y'}$ are isomorphisms in pro^*-$\mathcal{D}$. A morphism $\boldsymbol{f}^* : \boldsymbol{X} \to \boldsymbol{Y}$ is said to be *pro^*-$\mathcal{D}$ equivalent to* a morphism $\boldsymbol{f}'^* : \boldsymbol{X'} \to \boldsymbol{Y'}$, denoted by $\boldsymbol{f}^* \sim \boldsymbol{f}'^*$, provided the following diagram in pro^*-$\mathcal{D}$ commutes:

$$\begin{array}{ccc} \boldsymbol{X} & \xrightarrow{\boldsymbol{i}^*} & \boldsymbol{X'} \\ \boldsymbol{f}^* \downarrow & & \downarrow \boldsymbol{f}'^* \\ \boldsymbol{Y} & \xrightarrow{\boldsymbol{j}^*} & \boldsymbol{Y'} \end{array}.$$

Hereby is defined an equivalence relation on the appropriate subclass of $Mor(pro^*\text{-}\mathcal{D})$, such that $\boldsymbol{f}^* \sim \boldsymbol{f}'^*$ and $\boldsymbol{g}^* \sim \boldsymbol{g}'^*$ imply $\boldsymbol{g}^*\boldsymbol{f}^* \sim \boldsymbol{g}'^*\boldsymbol{f}'^*$ whenever it is defined. The equivalence class of an $\boldsymbol{f}^*$ is denoted by $\langle \boldsymbol{f}^* \rangle$.

We define the (*abstract*) *coarse shape category $Sh^*_{(\mathcal{C},\mathcal{D})}$ for* $(\mathcal{C}, \mathcal{D})$ as follows. The objects of $Sh^*_{(\mathcal{C},\mathcal{D})}$ are all the objects of $\mathcal{C}$. A morphism $F^* \in Sh^*_{(\mathcal{C},\mathcal{D})}(X, Y)$ is a pro^*-$\mathcal{D}$ equivalence class $\langle \boldsymbol{f}^* \rangle$ of a morphism $\boldsymbol{f}^* : \boldsymbol{X} \to \boldsymbol{Y}$, with respect to any choice of a pair of $\mathcal{D}$-expansions $\boldsymbol{p} : X \to \boldsymbol{X}$, $\boldsymbol{q} : Y \to \boldsymbol{Y}$. The *composition* of an $F^* : X \to Y$, $F^* = \langle \boldsymbol{f}^* \rangle$ and a $G^* : Y \to Z$, $G^* = \langle \boldsymbol{g}^* \rangle$, is defined by the representatives, i.e., $G^*F^* : X \to Z$, $G^*F^* = \langle \boldsymbol{g}^*\boldsymbol{f}^* \rangle$. The *identity coarse shape morphism* on an object X, $1^*_X : X \to X$, is the pro^*-$\mathcal{D}$ equivalence class $\langle \boldsymbol{1}^*_{\boldsymbol{X}} \rangle$ of the identity morphism $\boldsymbol{1}^*_{\boldsymbol{X}}$ in pro^*-$\mathcal{D}$. Since

$$Sh^*_{(\mathcal{C},\mathcal{D})}(X, Y) \approx pro^*\text{-}\mathcal{D}(\boldsymbol{X}, \boldsymbol{Y}),$$

one may say that pro^*-$\mathcal{D}$ is the realizing category for the coarse shape category $Sh^*_{(\mathcal{C},\mathcal{D})}$ in the same way as it is pro-$\mathcal{D}$ for the shape category $Sh_{(\mathcal{C},\mathcal{D})}$. If X and Y are isomorphic objects of $Sh^*_{(\mathcal{C},\mathcal{D})}$, then we say that they have the same *coarse shape type*, and we write $sh^*(X) = sh^*(Y)$. The abstract coarse shape type classification on $\mathcal{D}$ coincides with the abstract shape type classification. We denote by $J \equiv J_{(\mathcal{C},\mathcal{D})} : Sh_{(\mathcal{C},\mathcal{D})} \to Sh^{*}_{(\mathcal{C},\mathcal{D})}$ a faithful functor keeping the objects fixed whose acting on the sets of shape morphisms is induced by the "inclusion" functor $\underline{J}$: *pro*-$\mathcal{D} \to$ *pro**-$\mathcal{D}$. A functor $S^* \equiv S^{*}_{(\mathcal{C},\mathcal{D})} : \mathcal{C} \to$: $Sh^{*}_{(\mathcal{C},\mathcal{D})}$ which factorizes as $S^* = J_{(\mathcal{C},\mathcal{D})}S$, where $S : \mathcal{C} \to Sh_{(\mathcal{C},\mathcal{D})}$ is the shape functor, we call *the coarse shape functor.*

As in the case of the abstract shape, the most interesting example of the above construction is $\mathcal{C} = HTop$ - the homotopy category of topological spaces and $\mathcal{D} = HPol$ - the homotopy category of spaces having the homotopy type of polyhedra. In this case, one speaks about the (ordinary) *coarse shape category* $Sh^{*}_{(HTop,HPol)} \equiv Sh^{*}$ of topological spaces. The realizing category for Sh^* is the category *pro**-$HPol$. On locally nice spaces (polyhedra, CW-complexes, ANR's...) the coarse shape type classification coincides with the shape type classification and, consequently, with the homotopy type classification, but coarse shape differ from shape on the class of all topological spaces. Since the pointed homotopy category of polyhedra $HPol_\star$ is pro-reflective in the pointed homotopy category $HTop_\star$ ([10, Theorem 7 in I.4.3]) the *pointed coarse shape category* $Sh^{*}_{(HTop_\star,HPol_\star)} \equiv Sh^{*}_{\star}$ is well defined. For an inverse system $((X_\lambda,\star),[p_{\lambda\lambda'}],\Lambda)$ in $HPol_\star$-expansion $\boldsymbol{p} : (X,\star) \to ((X_\lambda,\star),[p_{\lambda\lambda'}],\Lambda)$ of pointed space $(X,\star)$ we will use abbreviation $(\boldsymbol{X},\star)$. One can also define the *coarse shape category of pairs* $Sh^{*}_{(HTop^2,HPol^2)} \equiv Sh^{*2}$, where $HTop^2$ is the homotopy category of pairs and $HPol^2$ (the homotopy category of polyhedral pairs) is its pro-reflective subcategory. One also introduces the *pointed coarse shape category of pairs* $Sh^{*}_{(HTop^2_\star,HPol^2_\star)} \equiv Sh^{*2}_{\star}$ via the pointed homotopy category of pairs $HTop^2_\star$ and its pro-reflective subcategory $HPol^2_\star$ (the pointed homotopy category of polyhedral pairs).

3. The induced *pro**-functors on the coarse shape categories

Let $U : \mathcal{D} \to \mathcal{K}$ be a covariant functor. Let us use the abbreviated notations $X_\#$ for the $\mathcal{K}$-object $U(X)$, $X \in Ob(\mathcal{D})$, and $f_\#$ for a $\mathcal{K}$-morphism $U(f) : U(X) \to U(Y)$, $f \in \mathcal{D}(X,Y)$. The functor U induces the rule which associates with every inverse system $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ in $\mathcal{D}$ an inverse system

$$\boldsymbol{X}_\# = (X_{\lambda\#}, p_{\lambda\lambda'\#}, \Lambda) \text{ in } \mathcal{K},$$

and with every morphism $\boldsymbol{f} = [(f, f_\mu)] : \boldsymbol{X} \to \boldsymbol{Y}$ in *pro*-$\mathcal{D}$ a morphism

$$\boldsymbol{f}_\# = [(f, f_{\mu\#})] : \boldsymbol{X}_\# \to \boldsymbol{Y}_\# \text{ in } pro\text{-}\mathcal{K}.$$

It is trivial to check that these rules induce a functor

$$\boldsymbol{U} : pro\text{-}\mathcal{D} \to pro\text{-}\mathcal{K},$$

that assigns to each inverse system $\boldsymbol{X}$ in $\mathcal{D}$ an inverse system $\boldsymbol{U}(\boldsymbol{X}) = \boldsymbol{X}_{\#}$ in $\mathcal{K}$ and to each morphism $\boldsymbol{f}$ of *pro*-$\mathcal{D}$ a morphism $\boldsymbol{U}(\boldsymbol{f}) = \boldsymbol{f}_{\#} : \boldsymbol{X}_{\#} \to \boldsymbol{Y}_{\#}$ of *pro*-$\mathcal{K}$.

Similarly as above, the functor U induces the rule which associates with every morphism $\boldsymbol{f}^* = \left[\left(f, f_\mu^n\right)\right] : \boldsymbol{X} \to \boldsymbol{Y}$ in *pro**-$\mathcal{D}$ a morphism

$$\boldsymbol{f}^*_{\#} = [(f, f^n_{\mu\#})] : \boldsymbol{X}_{\#} \to \boldsymbol{Y}_{\#} \text{ in } pro^*\text{-}\mathcal{K}.$$

In order to prove that $\boldsymbol{f}^*_{\#}$ is well defined, notice first that $(f, f^n_{\mu\#}) : \boldsymbol{X}_{\#} \to \boldsymbol{Y}_{\#}$ is an S^* morphism. Indeed, since for every related pair $\mu \leq \mu'$, there exist a $\lambda \in \Lambda$ and an $n \in \mathbb{N}$ such that $f_\mu^{n'} p_{f(\mu)\lambda} = q_{\mu\mu'} f_{\mu'}^{n'} p_{f(\mu')\lambda}$, for every $n' \geq n$, by properties of functor U it follows $f^{n'}_{\mu\#} p_{f(\mu)\lambda\#} = q_{\mu\mu'\#} f^{n'}_{\mu'\#} p_{f(\mu')\lambda\#}$, for every $n' \geq n$. Moreover, using similar arguments one can show that $\left(f', f_\mu^n\right) \sim \left(f, f_\mu^n\right)$ implies $(f', f^n_{\mu\#}) \sim (f, f^n_{\mu\#})$.

It is readily seen that

$$(\boldsymbol{g}^*\boldsymbol{f}^*)_{\#} = \boldsymbol{g}^*_{\#}\boldsymbol{f}^*_{\#}$$

holds for every pair $\boldsymbol{f}^* : \boldsymbol{X} \to \boldsymbol{Y}$, $\boldsymbol{g}^* : \boldsymbol{Y} \to \boldsymbol{Z}$ of morphisms of *pro**-$\mathcal{D}$. Further, a morphism $(\boldsymbol{1}^*_{\boldsymbol{X}})_{\#} = [(1_\Lambda, 1_{X_\lambda \#})]$ in *pro**-$\mathcal{K}$ is obviously identity on $\boldsymbol{X}_{\#}$, for every inverse system $\boldsymbol{X}$ in $\mathcal{D}$. Therefore, the correspondence $\boldsymbol{X} \to \boldsymbol{U}^*(\boldsymbol{X}) = \boldsymbol{X}_{\#}$ and $[\boldsymbol{f}^* : \boldsymbol{X} \to \boldsymbol{Y}] \to [\boldsymbol{U}^*(\boldsymbol{f}^*) : \boldsymbol{X}_{\#} \to \boldsymbol{Y}_{\#}]$ induces the functor

$$\boldsymbol{U}^* : pro^*\text{-}\mathcal{D} \to pro^*\text{-}\mathcal{K}.$$

Proposition 3.1. *For every covariant functor $U : \mathcal{D} \to \mathcal{K}$, the following diagram commutes*

$$\begin{array}{ccc} pro\text{-}\mathcal{D} & \xrightarrow{\underline{J}} & pro^*\text{-}\mathcal{D} \\ \downarrow \boldsymbol{U} & & \downarrow \boldsymbol{U}^* \\ pro\text{-}\mathcal{K} & \xrightarrow{\underline{J}} & pro^*\text{-}\mathcal{K} \end{array} .$$

Proof. For every inverse system $\boldsymbol{X}$ in $\mathcal{D}$ it holds

$$\boldsymbol{U}^*\underline{J}(\boldsymbol{X}) = \boldsymbol{U}^*(\boldsymbol{X}) = \boldsymbol{X}_{\#} = \underline{J}(\boldsymbol{X}_{\#}) = \underline{J}\boldsymbol{U}(\boldsymbol{X}).$$

Let $\boldsymbol{f} = [(f, f_\mu)] : \boldsymbol{X} \to \boldsymbol{Y}$ be an arbitrary morphism in *pro*-$\mathcal{D}$. Since $\underline{J}(\boldsymbol{f}) = \left[\left(f, f_\mu^n\right)\right]$, $f_\mu^n = f_\mu$, for all μ and n, it follows
$\boldsymbol{U}^*\underline{J}(\boldsymbol{f}) = \boldsymbol{U}^*\left(\left[\left(f, f_\mu^n\right)\right]\right) = \left[\left(f, f^n_{\mu\#}\right)\right] = \underline{J}([(f, f_{\mu\#})]) = \underline{J}\boldsymbol{U}([(f, f_\mu)])$
$= \underline{J}\boldsymbol{U}(\boldsymbol{f})$, which completes the proof. ∎

Since the functor $\underline{J}$ is faithful keeping the objects fixed, we may consider it as the inclusion functor. Therefore, the previous proposition allows us to treat the functor $\boldsymbol{U}$ as a restriction of $\boldsymbol{U}^*$ to the subcategory *pro*-$\mathcal{D}$.

Let a category $\mathcal{D}$ be a pro-reflective subcategory of a category $\mathcal{C}$. In order to define an induced functor $Sh_{(\mathcal{C},\mathcal{D})} \to pro\text{-}\mathcal{K}$, via functor $\boldsymbol{U} : pro\text{-}\mathcal{D} \to pro\text{-}\mathcal{K}$, let us apply the axiom of choice and choose, in advance, for every object

$X \in Ob(\mathcal{C})$, a fixed $\mathcal{D}$-expansion $\boldsymbol{p} : X \to \boldsymbol{X}$ (see also Remark 3.2 below). Further, for every morphism $F : X \to Y$ in $Sh_{(\mathcal{C},\mathcal{D})}$, let us consider its unique representative $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in *pro*-$\mathcal{D}$, where $\boldsymbol{p} : X \to \boldsymbol{X}$ and $\boldsymbol{q} : Y \to \boldsymbol{Y}$ are chosen $\mathcal{D}$-expansions of objects X and Y respectively. Now, one can associate with an object X an inverse system $\boldsymbol{U}(\boldsymbol{X})$ denoted by *pro*-$U(X)$. Also, one can associate with a shape morphism $F : X \to Y$ a morphism $\boldsymbol{U}(\boldsymbol{f})$ in *pro*-$\mathcal{K}$ denoted by *pro*-$U(F)$. In such a manner we have constructed a functor

$$pro\text{-}U : Sh_{(\mathcal{C},\mathcal{D})} \to pro\text{-}\mathcal{K}$$

which is said to be the *functor* (*on the shape category*) *induced by the functor* U. By using properties of the functor $\boldsymbol{U}$, it is trivial to show that the *pro*-U is a functor indeed.

In the same way we define a functor

$$pro^*\text{-}U : Sh^*_{(\mathcal{C},\mathcal{D})} \to pro^*\text{-}\mathcal{K}$$

relating the coarse shape category and *pro**-$\mathcal{K}$, and it is said to be the *functor* (*on the coarse shape category*) *induced by the functor* U. More precisely, it is defined via functor $\boldsymbol{U}^* : pro^*\text{-}\mathcal{D} \to pro^*\text{-}\mathcal{K}$ as follows: for every $X \in Ob(\mathcal{C}) = Ob\left(Sh^*_{(\mathcal{C},\mathcal{D})}\right)$ we put $pro^*\text{-}U(X) = \boldsymbol{U}^*(\boldsymbol{X})$, and for every coarse shape morphism $F^* : X \to Y$ we put $pro^*\text{-}U(F) = \boldsymbol{U}^*(\boldsymbol{f}^*)$, where $\boldsymbol{p} : X \to \boldsymbol{X}$ and $\boldsymbol{q} : Y \to \boldsymbol{Y}$ are chosen $\mathcal{D}$-expansions of objects X and Y, respectively, and $\boldsymbol{f}^* : \boldsymbol{X} \to \boldsymbol{Y}$ is the unique morphism of *pro**-$\mathcal{D}$ which represents the coarse shape morphism F^*.

REMARK 3.2. Formally, every particular choice of $\mathcal{D}$-expansions of $\mathcal{C}$-objects yields a different induced functor *pro**-U (*pro*-U), but all those functors induced by U associate with the same object mutually isomorphic inverse systems in *pro**-$\mathcal{K}$ and in *pro*-$\mathcal{K}$. Namely, for every pair $\boldsymbol{p} : X \to \boldsymbol{X}$ and $\boldsymbol{p}' : X \to \boldsymbol{X}'$ of $\mathcal{D}$-expansions of the same $\mathcal{C}$-object X, there exists a natural isomorphism $\boldsymbol{i}^* : \boldsymbol{X} \to \boldsymbol{X}'$ in *pro**-$\mathcal{D}$ induced by a natural isomorphism $\boldsymbol{i} : \boldsymbol{X} \to \boldsymbol{X}'$ in *pro*-$\mathcal{D}$. Therefore, $\boldsymbol{U}^*(\boldsymbol{i}^*) : \boldsymbol{U}^*(\boldsymbol{X}) \to \boldsymbol{U}^*\left(\boldsymbol{X}'\right)$ and $\boldsymbol{U}(\boldsymbol{i}) : \boldsymbol{U}(\boldsymbol{X}) \to \boldsymbol{U}\left(\boldsymbol{X}'\right)$ are also isomorphisms of *pro**-$\mathcal{K}$ and *pro*-$\mathcal{K}$, respectively. Since the inverse systems $\boldsymbol{U}^*(\boldsymbol{X})$ and $\boldsymbol{U}^*\left(\boldsymbol{X}'\right)$ ($\boldsymbol{U}(\boldsymbol{X})$ and $\boldsymbol{U}\left(\boldsymbol{X}'\right)$) are equal in *pro**-$\mathcal{K}$ (in *pro*-$\mathcal{K}$) up to isomorphism, we may both inverse systems denote by *pro**-$U(X)$ (*pro*-$U(X)$). From that point of view, we will consider that functors *pro**-U (*pro*-U) induced by U do not depend on particular choice of $\mathcal{D}$-expansions of $\mathcal{C}$-objects and, in the sequel, we will treat all of them as the unique functor $pro^*\text{-}U : Sh^*_{(\mathcal{C},\mathcal{D})} \to pro^*\text{-}\mathcal{K}$ ($pro\text{-}U : Sh_{(\mathcal{C},\mathcal{D})} \to pro\text{-}\mathcal{K}$).

An immediate consequence of Proposition 3.1 is the following corollary.

COROLLARY 3.3. *For every covariant functor $U : \mathcal{D} \to \mathcal{K}$ the following diagram of functors*

$$\begin{array}{ccc} Sh_{(\mathcal{C},\mathcal{D})} & \xrightarrow{J} & Sh^*_{(\mathcal{C},\mathcal{D})} \\ \downarrow pro\text{-}U & & \downarrow pro^*\text{-}U \\ pro\text{-}\mathcal{K} & \xrightarrow{\underline{J}} & pro^*\text{-}\mathcal{K} \end{array},$$

commutes.

REMARK 3.4. As we mentioned above, the functors J and $\underline{J}$ are faithful keeping the objects fixed and therefore we may consider the categories *pro*-$\mathcal{D}$ and $Sh_{(\mathcal{C},\mathcal{D})}$ to be the subcategories of the pro^*-$\mathcal{D}$ and $Sh^*_{(\mathcal{C},\mathcal{D})}$, respectively. Now, in the light of the previous corollary, one may say that the functor pro^*-U induced by U, is an extension of the functor *pro*-U over the category $Sh^*_{(\mathcal{C},\mathcal{D})}$.

From the topological point of view the most interesting applications of the coarse shape theory involve the categories Sh^*, $Sh^*_\star$, Sh^{*2}, $Sh^{*2}_\star$. We are going to consider functors on these categories induced by homology and homotopy functors.

Let $\mathcal{C} = HTop$, $\mathcal{D} = HPol$, $\mathcal{K} = Ab$, the category of Abelian groups, and let U be the k-th homology functor $H_k(\cdot; G) : HPol \to Ab$ (for an Abelian group G and $k \in \mathbb{N}_0$,), which assigns the k-th singular homology group $H_k(P; G)$ with coefficients in G to every polyhedron P (for $G = \mathbb{Z}$ we will use the standard abbreviation $H_k(\cdot)$). The functor

$$pro^*\text{-}H_k(\cdot; G) : Sh^* \to pro^*\text{-}Ab$$

on the coarse shape category induced by U is defined in the way we described above. The functor pro^*-$H_k(\cdot; G)$ is said to be *the induced k-th homology functor on the coarse shape category with coefficients in G*. According to Remark 3.4, the functor pro^*-$H_k(\cdot; G)$ is an extension of the well known functor

$$pro\text{-}H_k(\cdot; G) : Sh \to pro\text{-}Ab$$

which operates on the shape category. Moreover, the both functors associate, with a topological space X, the same object the inverse system

$$pro^*\text{-}H_k(X; G) = \boldsymbol{H}^*_k(\boldsymbol{X}; G) = \boldsymbol{H}_k(\boldsymbol{X}; G) = pro\text{-}H_k(X; G),$$

which is a standard k-th homology *pro*-group (see [10]). Now, one might thought that, beside relating the coarse shape category and pro^*-Ab , the functor pro^*-$H_k(\cdot; G)$ doesn't give any new algebraic informations on the coarse shape. Quite contrary, the homology *pro*-groups of topological spaces viewed as objects of the category *pro*-Ab are shape invariants, which is a well known fact in the shape theory. On the other hand, the same *pro*-groups considered as objects in the category pro^*-Ab are coarse shape invariants. Therefore, depending on the context, the k-th homology *pro*-group with coefficients in G of a space X, will be denoted by $pro\text{-}H_k(X;G)$ or by $pro^*\text{-}H_k(X;G)$. The full sense of considering homology *pro*-groups $pro^*\text{-}H_k(X;G)$ and $pro^*\text{-}H_k(Y;G)$ of spaces X and Y as objects of the category $pro^*\text{-}Ab$ gives a case when k-th homology *pro*-groups of this spaces are not isomorphic in $pro\text{-}Ab$. We shall illustrate this approach by the next example.

EXAMPLE 3.5. Let X and Y be inverse limits of inverse sequences $\boldsymbol{X} = (X_i, [p_{ii+1}])$ and $\boldsymbol{Y} = (Y_j, [q_{jj+1}])$, respectively, where each term is equal to a 2-torus T, i.e., $X_i = Y_j = T$, for all $i, j \in \mathbb{N}$, and the bonding homotopy classes $[p_{ii+1}], [q_{jj+1}] : T \to T$ are represented by continuous homomorphisms given by integral matrices

$$P_{ii+1} = \begin{bmatrix} 1 & 0 \\ 0 & -2^{4i} \end{bmatrix} \text{ and } Q_{jj+1} = \begin{bmatrix} -1 & 0 \\ 2^{2j} & -2^{4j} \end{bmatrix},$$

respectively, for all $i, j \in \mathbb{N}$. The first homology *pro*-groups with coefficients in $\mathbb{Z}$ of metric compacta X and Y are inverse sequences

$$pro\text{-}H_1(X) = pro^*\text{-}H_1(X) = (X_{i\#}, p_{ii+1\#})$$

and

$$pro\text{-}H_1(Y) = pro^*\text{-}H_1(Y) = (Y_{j\#}, q_{jj+1\#})$$

of groups $X_{i\#} = Y_{j\#} = \mathbb{Z}^2$, for all $i, j \in \mathbb{N}$, and homomorphisms $p_{ii+1\#}, q_{jj+1\#} : \mathbb{Z}^2 \to \mathbb{Z}^2$ given by matrices P_{ii+1} and Q_{jj+1}, respectively, for all $i, j \in \mathbb{N}$. Refer to Example 7.1 of [5], these *pro*-groups are not isomorphic in $pro\text{-}Ab$. Therefore we may write $pro\text{-}H_1(X) \not\cong pro\text{-}H_1(Y)$ and infer that $sh(X) \neq sh(Y)$. On the other hand, these *pro*-groups are isomorphic in $pro^*\text{-}Ab$ and we may write $pro^*\text{-}H_1(X) \cong pro^*\text{-}H_1(Y)$. Generally, this gives no information about the coarse shape types of spaces X and Y. However, in this case one can easily prove that $sh^*(X) = sh^*(Y)$.

The homotopy functors $\pi_0 : HPol_\star \to Set_\star$ ($Set_\star$ denotes the category of pointed sets) and $\pi_k : HPol_\star \to Grp$ (Grp denotes the category of groups), for $k \in \mathbb{N}$, which assigns the homotopy groups $\pi_0(P, \star)$ and $\pi_k(P, \star)$ to every pointed polyhedron $(P, \star)$, induce the functors

$$pro^*\text{-}\pi_0 : Sh^*_\star \to pro^*\text{-}Set_\star$$

and

$$pro^*\text{-}\pi_k : Sh^*_\star \to pro^*\text{-}Grp, \text{ for every } k \in \mathbb{N},$$

which are said to be *the induced k-th homotopy functors on the pointed coarse shape category.* According to Remark 3.4, the functor $pro^*\text{-}\pi_k$ is an extension of the well known functor

$$pro\text{-}\pi_k : Sh_\star \to pro\text{-}Grp$$

which operates on the pointed shape category.

In order to compare (pointed) spaces $(X,\star)$ and $(Y,\star)$ having different (pointed) homotopy types, powerful tools are the functors *pro*-H_k and *pro*-π_k. If the appropriate k-th homology or k-th homotopy *pro*-groups of these spaces are not isomorphic in the *pro*-*Ab* (*pro*-*Grp*, *pro*-$Set_\star$) at least for one k, we infer that $sh\,(X,\star) \neq sh\,(Y,\star)$. Then we pass to the "coarser" level by using the functors pro^*-H_k and pro^*-π_k, i.e., we compare the homology or the homotopy *pro*-groups of these spaces in the category pro^*-*Ab* (pro^*-*Grp*, pro^*-$Set_\star$). An application of this technique can be seen in the following example.

EXAMPLE 3.6. Let (X,x_0) and (Y,y_0) be a pair of pointed 0-dimensional metric compacta having exactly 1 and 2 accumulation points respectively and both having for a base point an accumulation point x_0 and y_0 respectively. We will prove that the homology *pro*-groups *pro*-$\pi_0\,(X,x_0)$ and *pro*-$\pi_0\,(Y,y_0)$ of this pointed spaces are not isomorphic in *pro*-Set_* nor in pro^*-Set_*. As we know, the study of shape types of 0-dimensional metric compacta reduces to the study of their topological types. Therefore, pointed spaces (X,x_0) and (Y,y_0) don't belong to the same pointed shape type. This is, for this particular case, equivalent to *pro*-$\pi_0\,(X,x_0) \ncong$ *pro*-$\pi_0\,(Y,y_0)$. Let us prove that these homotopy "*pro*-groups" are not isomorphic in pro^*-Set_*, i.e., pro^*-$\pi_0\,(X,x_0) \ncong pro^*$-$\pi_0\,(Y,y_0)$. By this we will infer that $sh^*\,(X,x_0) \neq sh^*\,(Y,y_0)$. Since every 0-dimensional metric compact can be embedded in the Euclidian space $\mathbb{R}$, there is no loss of generality in assuming that $X = \{1\} \cup \left\{1+\frac{1}{n} \mid n \in \mathbb{N}\right\} \subseteq \mathbb{R}$, $x_0 = 1$, and $Y = X \cup (-X)$, $y_0 = 1$ ($-X$ denotes a set $\{-x \mid x \in X\}$). Now, one can compute

$$pro^*\text{-}\pi_0\,(X,x_0) = ((X_i,x_0)\,, p_{ii+1}, \mathbb{N})\,,$$

where $X_i = \{1\} \cup \left\{1+\frac{1}{k} \mid 1 < k \leq i\right\}$ and the bonding pointed functions $p_{ii+1} : (X_{i+1},x_0) \to (X_i,x_0)$ are given by

$$p_{ii+1}\,(x) = \begin{cases} x, \ 1+\frac{1}{i} \leq x \\ 1, \ 1+\frac{1}{i} \nleq x \end{cases}, \text{ for every } i \in \mathbb{N}.$$

A similar computation yields

$$pro^*\text{-}\pi_0\,(Y,y_0) = ((Y_j,y_0)\,, q_{jj+1}, \mathbb{N})\,,$$

where $Y_j = X_j \cup (-X_j)$ and the bonding pointed functions $q_{jj+1} : (Y_{j+1},y_0) \to (Y_j,y_0)$ are given by

$$q_{jj+1}\,(x) = \begin{cases} p_{jj+1}\,(x)\,, \ x \in X_j \\ -p_{jj+1}\,(-x)\,, \ x \in -X_j \end{cases}, \text{ for every } j \in \mathbb{N}.$$

Let us assume that $\boldsymbol{f}^* : pro^*\text{-}\pi_0\,(X,x_0) \to pro^*\text{-}\pi_0\,(Y,y_0)$ is an isomorphism in pro^*-$Set_\star$. According to Lemma 3.23 of [5] $\boldsymbol{f}^*$ admits a simple representative $\left(f, f_j^n\right) : ((X_i,x_0)\,, p_{ii+1}, \mathbb{N}) \to ((Y_j,y_0)\,, q_{jj+1}, \mathbb{N})$ and we may assume that $f\,(1) \geq 2$. Now, by restricting the inverse sequence

pro^*-$\pi_0(X,x_0)$ to the isomorphic subsequence $\left(\left(X'_j,x_0\right),p'_{jj+1},\mathbb{N}\right)$, where $X'_j = X_{f(j)}$ and $p'_{jj+1} = p_{f(j)f(j+1)}$, one obtains a level S^*-morphisms $\left(1_{\mathbb{N}},f_j^n\right) : \left(\left(X'_j,x_0\right),p'_{jj+1},\mathbb{N}\right) \rightarrow \left(\left(Y_j,y_0\right),q_{jj+1},\mathbb{N}\right)$ which represents an isomorphism in pro^*-$Set_\star$. Therefore, by Theorem 6.1 of [5], there exist an index $k \geq 1$, an index $l \geq k$, an integer n, and pointed functions $h_1^n : (Y_k,y_0) \rightarrow (X'_1,x_0)$ and $h_k^n : (Y_l,y_0) \rightarrow (X'_k,x_0)$ such that the following diagram in $Set_\star$ commutes:

$$\begin{array}{ccccccc} (X'_1,x_0) & \overset{p'_{1k}}{\longleftarrow} & (X'_k,x_0) & \overset{p'_{kl}}{\longleftarrow} & (X'_l,x_0) \\ f_1^n\downarrow & h_1^n\nwarrow & f_k^n\downarrow & h_k^n\nwarrow & \downarrow f_l^n \\ (Y_1,y_0) & \underset{q_{1k}}{\longleftarrow} & (Y_k,y_0) & \underset{q_{kl}}{\longleftarrow} & (Y_l,y_0) \end{array}. \tag{1}$$

Notice that we can always achieve an index k to be sufficiently large, particularly, to be $k \geq f(1)-1$. Since all bonding functions are surjective, it follows that f_1^n, h_1^n and f_k^n are surjective too. Let $C = Y_1\backslash\{y_0\}$, $A = (f_1^n)^{-1}(C)$, $B = (p'_{1k})^{-1}(A)$, and $D = (q_{1k})^{-1}(C)$. Since $x_0 \notin A\,(\neq \emptyset)$, it follows that A and B have the same cardinality α, and it holds that

$$\alpha \leq f(1)-1 \leq k. \tag{2}$$

Obviously, C and D have cardinality $|C| = 3$ and $|D| = k+1$ respectively. Let us prove that

$$f_k^n(B) = D. \tag{3}$$

First, notice that surjectivity of p'_{1k} and f_1^n implies

$$f_1^n p'_{1k}(B) = f_1^n p'_{1k}\left((p'_{1k})^{-1}\left((f_1^n)^{-1}(C)\right)\right) = C. \tag{4}$$

Now, since diagram (1) commutes, it follows by (4) that $q_{1k}f_k^n(B) = C$, and we infer that

$$D = (q_{1k})^{-1}(C) \supseteq f_k^n(B). \tag{5}$$

For an arbitrary $y \in D$, by (1), it holds $f_1^n h_1^n(y) = q_{1k}(y) \in C$, and consequently, we have

$$h_1^n(y) \in (f_1^n)^{-1}(C) = A. \tag{6}$$

Since f_k^n is a surjective function, there exists an $x \in X'_k$ such that

$$f_k^n(x) = y. \tag{7}$$

By (1), (6) and (7), it follows that

$$p'_{1k}(x) = h_1^n f_k^n(x) = h_1^n(y) \in A,$$

which implies that

$$x \in (p'_{1k})^{-1}(A) = B. \tag{8}$$

Now, by (7) and (8), it follows that $y \in f_k^n(B)$, which proves that

$$D \subseteq f_k^n(B)\,. \tag{9}$$

Finally, (5) and (9) imply (3). By using (3) and (2) one obtains

$$k \geq f(1) - 1 \geq \alpha = |B| \geq |D| = k + 1,$$

which is a contradiction.

REMARK 3.7. Recall that, for every $k \in \mathbb{N}_0$ and every Abelian group G, the functor $H_k(\cdot;G)$ extends to the functor $H_k(\cdot,\cdot;G) : HPol^2 \to Ab$ which assigns the relative k-th singular homology group $H_k(P,P_0;G)$ with coefficients in G to every pair (P,P_0) of polyhedra. The functor $H_k(\cdot,\cdot;G)$ induces the functor

$$pro^*\text{-}H_k(\cdot,\cdot;G) : Sh^{*2} \to pro^*\text{-}Ab$$

which is called *the induced k-th homology functor on the coarse shape category of pairs with coefficients in G.*

Similarly, the relative homotopy functors $\pi_1 : HPol_\star^2 \to Set_*$ and $\pi_k : HPol_\star^2 \to Grp$, for all integer $k \geq 2$, induce the functors

$$pro^*\text{-}\pi_1(\cdot,\cdot) : Sh_\star^{*2} \to pro^*\text{-}Set_*$$

and

$$pro^*\text{-}\pi_k(\cdot,\cdot) : Sh_\star^{*2} \to pro^*\text{-}Grp,\ k \geq 2,$$

which are called *the induced homotopy functors on the pointed coarse shape category of pairs.*

4. The n-coarse shape connectedness and Hurewicz theorem in the coarse shape theory

Let us briefly recall some general relevant facts. An object 0 of a category $\mathcal{C}$ is said to be a *zero-object* of $\mathcal{C}$ if, for every object X of $\mathcal{C}$, the sets of morphisms $\mathcal{C}(0,X)$ and $\mathcal{C}(X,0)$ are singletons. Any two zero-objects of $\mathcal{C}$ are isomorphic and any object isomorphic to a zero-object is itself a zero-object. We say that $\mathcal{C}$ is a *category with zero-objects* if there exists at least one zero-object of $\mathcal{C}$. For instance, the categories Grp and Set_* are the categories with zero-objects, which are the singletons. A morphism $o : X \to Y$ of a category $\mathcal{C}$ with zero-objects is said to be a *zero-morphism* if it factorizes through a zero-object, i.e., if there exist morphisms $f \in \mathcal{C}(X,0)$ and $g \in \mathcal{C}(0,Y)$ such that $o = gf$. Clearly, for every pair of objects X and Y there exists a unique zero-morphism $o : X \to Y$.

PROPOSITION 4.1. *Let $\mathcal{C}$ be a category with a zero-object* 0. *Then the rudimentary inverse system $\mathbf{0} = (0)$ is a zero-object of pro^*-$\mathcal{C}$.*

Proof. Let $\boldsymbol{X}$ denote an inverse system $(X_\lambda, p_{\lambda\lambda'}, \Lambda)$ in $\mathcal{C}$. For every $n \in \mathbb{N}$ and for any $\lambda \in \Lambda$, let $o^n : X_\lambda \to 0$ denote the zero-morphism. Then the morphism $[(o^n)] : \boldsymbol{X} \to \mathbf{0}$ is the unique element of $(pro^*\text{-}\mathcal{C})(\boldsymbol{X}, \mathbf{0})$. Indeed, for every S^*-morphism $(f^n) : \boldsymbol{X} \to \mathbf{0}$ it holds $(f^n) \sim (o^n)$. Namely, an S^*-morphism $(f^n) : \boldsymbol{X} \to \mathbf{0}$ is uniquely determined by a $\lambda' \in \Lambda$ and by a sequence (f^n) of $\mathcal{C}$-morphism $f^n : X_{\lambda'} \to 0$. Since 0 is a zero-object of the category $\mathcal{C}$, for every $\lambda_0 \geq \lambda, \lambda'$, the set $\mathcal{C}(X_{\lambda_0}, 0)$ is a singleton. Now it follows that

$$o^n p_{\lambda\lambda_0} = f^n_{\lambda'} p_{\lambda'\lambda_0} \in \mathcal{C}(X_{\lambda_0}, 0),$$

for every $n \in \mathbb{N}$. Similarly, one can prove that a morphism $[(o^n_\lambda)] : \mathbf{0} \to \boldsymbol{X}$, represented by an S^*-morphism (o^n_λ), where $o^n_\lambda : X_\lambda \to 0$ denotes the zero-morphism for all $\lambda \in \Lambda$ and $n \in \mathbb{N}$, is the unique element of $(pro^*\text{-}\mathcal{C})(\mathbf{0}, \boldsymbol{X})$. $\square$

Corollary 4.2. *The categories pro***-Grp and pro***-Set*$_\star$ *are the categories with zero-objects.*

The previous corollary allows us to define the notion of an n-coarse shape connectedness as the full analogue of n-(shape) connectedness.

Definition 4.3. *A pointed topological space* $(X, \star)$ *is said to be* n*-coarse shape connected if pro**-$\pi_k(X, \star)$ *is a zero-object of pro***-Grp (pro***-Set*$_\star$, $k = 0$*) for every* $k \in \mathbb{N}_0$, $k \leq n$.

Recall that a pointed space $(X, \star)$ is called n-shape connected provided *pro*-$\pi_k(X, \star)$ is a zero-object of *pro-Grp* (*pro-Set*$_\star$) for every $k \in \mathbb{N}_0$, $k \leq n$. One of the main goals of this section is to show that the n-shape connectedness is equivalent to n-coarse shape connectedness. In order to do it, we will prove that a zero-object is characterized in the same way in *pro*-$\mathcal{C}$ as in *pro**-$\mathcal{C}$, whenever $\mathcal{C}$ is a category with zero-objects.

Theorem 4.4. *Let* $\mathcal{C}$ *be a category with zero-objects. An inverse system* $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ *in* $\mathcal{C}$ *is a zero-object of pro**-$\mathcal{C}$ *if and only if, for every* $\lambda \in \Lambda$, *there exists a* $\lambda' \in \Lambda$, $\lambda' \geq \lambda$, *such that* $p_{\lambda\lambda'}$ *is a zero-morphism of* $\mathcal{C}$, *i.e.,* $p_{\lambda\lambda'} = o$.

Proof. Let $\boldsymbol{X}$ be a zero-object of the category *pro**-$\mathcal{C}$. Then, by Proposition 4.1, $\boldsymbol{X}$ is isomorphic to the rudimentary system $\mathbf{0} = (0)$. Thus, there exists an isomorphism $\boldsymbol{f}^* = [(f^n_\lambda)] : \mathbf{0} \to \boldsymbol{X}$. For all $\lambda, \lambda' \in \Lambda$ and $n \in \mathbb{N}$, let $0_\lambda = 0$ be a zero-object of $\mathcal{C}$ and let $o_{\lambda\lambda'} = 1_0 : 0_{\lambda'} \to 0_\lambda$ be the identity (zero-)morphism. It follows that $(1_\Lambda, f^n_\lambda,) : (0_\lambda, o_{\lambda\lambda'}, \Lambda) \to \boldsymbol{X}$ is a level S^*-morphism which represents an isomorphism in *pro**-$\mathcal{C}$. Now, by Theorem 6.1 of [5], for every $\lambda \in \Lambda$, there exist a $\lambda' \geq \lambda$ and an $n \in \mathbb{N}$ such that, for every $n' \geq n$, there exists a morphism $h^{n'}_\lambda : X_{\lambda'} \to 0_\lambda$ of $\mathcal{C}$, such that

$$f^{n'}_\lambda h^{n'}_\lambda = p_{\lambda\lambda'}.$$

It implies that $p_{\lambda\lambda'}$ factorizes through the zero-object $0_\lambda = 0$, which means that $p_{\lambda\lambda'} = o$. Conversely, let $\boldsymbol{X}$ satisfy the assumption of the theorem. Let $\boldsymbol{f}^*$ and $\boldsymbol{g}^*$ be the unique morphisms of $(pro^*\text{-}\mathcal{C})(\mathbf{0}, \boldsymbol{X})$ and of $(pro^*\text{-}\mathcal{C})(\boldsymbol{X}, \mathbf{0})$, respectively. Let an S^*-morphism $(f_\lambda^n) : \mathbf{0} \to \boldsymbol{X}$ be a representative of $\boldsymbol{f}^*$. Let $\boldsymbol{g}^*$ be represented by an S^*-morphism $(g^n) : \boldsymbol{X} \to \mathbf{0}$ which is determined by a $\lambda_0 \in \Lambda$ and by $\mathcal{C}$-morphisms $g^n : X_{\lambda_0} \to 0$, $n \in \mathbb{N}$. By assumption, for an arbitrary $\lambda \in \Lambda$, there exists a $\lambda' \geq \lambda$ such that $p_{\lambda\lambda'} = o$. Therefore, for any $\lambda'' \geq \lambda', \lambda_0$, $p_{\lambda\lambda''} = o \circ p_{\lambda'\lambda''}$ is a zero-morphism. On the other hand, since the composite morphism

$$f_\lambda^n g^n p_{\lambda_0\lambda''} : X_{\lambda''} \overset{g^n p_{\lambda_0\lambda''}}{\to} 0 \overset{f_\lambda^n}{\to} X_\lambda$$

factorizes through 0, for every $n \in \mathbb{N}$, it follows that

$$f_\lambda^n g^n p_{\lambda_0\lambda''} = o = p_{\lambda\lambda''}.$$

This proves that $(f_\lambda^n)(g^n) \sim (1_\Lambda, 1_{X_\lambda})$, i.e., $\boldsymbol{f}^*\boldsymbol{g}^* = \mathbf{1}_{\boldsymbol{X}}^*$. Further, since 1_0 is the unique morphism of the set $\mathcal{C}(0,0)$, it follows that $g^n f_{\lambda_0}^n = 1_0$, for every $n \in \mathbb{N}$. Therefore, one infers $\boldsymbol{g}^*\boldsymbol{f}^* = \mathbf{1}_{\mathbf{0}}^*$, which implies that $\boldsymbol{X}$ is isomorphic to $\mathbf{0}$ and, consequently, $\boldsymbol{X}$ is a zero-object of $pro^*\text{-}\mathcal{C}$. ∎

Notice that the previous theorem also holds if one puts $pro\text{-}\mathcal{C}$ instead of $pro^*\text{-}\mathcal{C}$. That is because in [10, Theorem 7., II.2.3], instead of *Grp*, we may put, more generally, a category $\mathcal{C}$ with zero-objects. In such a manner one obtains the following corollary.

COROLLARY 4.5. *Let $\mathcal{C}$ be a category with zero-objects. An inverse system $\boldsymbol{X}$ is a zero-object of pro-$\mathcal{C}$ if and only if, $\boldsymbol{X}$ is a zero-object of pro*-$\mathcal{C}$.*

We need, especially,

COROLLARY 4.6. *An inverse system $\boldsymbol{X}$ is a zero-object of pro*-Grp (pro*-Set$_\star$) if and only if, $\boldsymbol{X}$ is a zero-object of pro-Grp (pro-Set$_\star$).*

Now, by using Corollary 4.6 and taking into account that, for every $k \in \mathbb{N}_0$, the inverse systems $pro\text{-}\pi_k(X, \star)$ and $pro^*\text{-}\pi_k(X, \star)$ are the same objects considering in two different categories, it follows that the following holds:

COROLLARY 4.7. *A pointed topological space $(X, \star)$ is n-shape connected if and only if, it is n-coarse shape connected.*

COROLLARY 4.8. *A pointed topological space $(X, \star)$ is 0-coarse shape connected if and only if, the space X is connected.*

Let us recall that, for every pointed topological space $(X, \star)$ and for every $k \in \mathbb{N}$, there exists the homomorphism $h_k \equiv h_k(X, \star) : \pi_k(X, \star) \to H_k(X)$ (called the Hurewicz homomorphism, see [2]) such that, for every homotopy class $[f] : (X, \star) \to (Y, *)$, the following diagram in *Grp*

$$\begin{array}{ccc} \pi_k(X,\star) & \overset{h_k(X,\star)}{\longrightarrow} & H_k(X) \\ \downarrow \pi_k(f) & & \downarrow H_k(f) \\ \pi_k(Y,\star) & \overset{h_k(Y,\star)}{\longrightarrow} & H_k(Y) \end{array} \quad . \tag{10}$$

commutes. Further, for a pointed topological space $(X, \star)$ and for a $k \in \mathbb{N}$, the Hurewicz morphism in *pro-Grp* (see [10]) is the morphism

$$\boldsymbol{\varphi}_k \equiv \boldsymbol{\varphi}_k(X,\star) : pro\text{-}\pi_k(X,\star) \to pro\text{-}H_k(X)$$

in *pro-Grp* represented by a level morphism

$$(1_\Lambda, \varphi_\lambda) : (\pi_k(X_\lambda,\star), \pi_k(p_{\lambda\lambda'}), \Lambda) \to (H_k(X_\lambda), H_k(p_{\lambda\lambda'}), \Lambda),$$

where $\boldsymbol{p} : (X,\star) \to ((X_\lambda,\star), [p_{\lambda\lambda'}], \Lambda)$ is an $HPol_\star$-expansion of $(X,\star)$ and, for every λ, $\varphi_\lambda = h_k(X_\lambda,\star) : \pi_k(X_\lambda,\star) \to H_k(X_\lambda)$ is the Hurewicz homomorphism. The following theorem in the shape theory (Hurewicz isomorphism theorem in $Sh_\star$, [10, Theorem 1., II 4.1]) is the well known analogue of the classical Hurewicz isomorphism theorem ([2, Theorem 4.37]).

THEOREM 4.9. *Let $(X,\star)$ be a pointed topological space, which is $(n-1)$-shape connected. Then*

- (H_1) *pro-$H_k(X)$ is a zero-object of pro-Grp, $1 \le k \le n-1$, for $n \ge 2$;*
- (H_2) *the Hurewicz morphism $\boldsymbol{\varphi}_n$: pro-$\pi_n(X,\star) \to$ pro-$H_n(X)$ is an isomorphism of pro-Grp, for $n \ge 2$;*
- (H_3) *the Hurewicz morphism $\boldsymbol{\varphi}_{n+1}$: pro-$\pi_{n+1}(X,\star) \to$ pro-$H_{n+1}(X)$ is an epimorphism of pro-Grp, for $n \ge 2$;*
- (H_4) *the Hurewicz morphism $\boldsymbol{\varphi}_1$: pro-$\pi_1(X,\star) \to$ pro-$H_1(X)$ is an epimorphism of pro-Grp, for $n = 1$.*

Let a pointed space $(X,\star)$ and its $HPol_*$-expansion $\boldsymbol{p} : (X,\star) \to ((X_\lambda,\star), [p_{\lambda\lambda'}], \Lambda)$ be given. For $(X,\star)$ and $k \in \mathbb{N}$, we define the *Hurewicz morphism in pro*-Grp* to be the morphism

$$\boldsymbol{\varphi}_k^* \equiv \boldsymbol{\varphi}_k^*(X,\star) : pro^*\text{-}\pi_k(X,\star) \to pro^*\text{-}H_k(X)$$

in *pro*-Grp* induced by $\boldsymbol{\varphi}_k(X,\star)$, i.e., $\boldsymbol{\varphi}_k^* = \underline{J}(\boldsymbol{\varphi}_k)$. The morphism $\boldsymbol{\varphi}_k^*$ is represented by a level S^*-morphism

$$(1_\Lambda, \varphi_\lambda^n) : (\pi_k(X_\lambda,\star), \pi_k(p_{\lambda\lambda'}), \Lambda) \to (H_k(X_\lambda), H_k(p_{\lambda\lambda'}), \Lambda),$$

where $\varphi_\lambda^n = \varphi_\lambda = h_k(X_\lambda,\star) : \pi_k(X_\lambda,\star) \to H_k(X_\lambda)$ is the Hurewicz homomorphism, for all $\lambda \in \Lambda$, $n \in \mathbb{N}$. Naturality remains an elementary property of the Hurewicz (homo)morphisms also in *pro*-Grp*, i.e., the following theorem holds.

Theorem 4.10. *Let $F^*:(X,\star)\to(Y,\star)$ be a pointed coarse shape morphism and let $k\in\mathbb{N}$. Then the following diagram in pro*-Grp*

$$\begin{array}{ccc} pro^*\text{-}\pi_k(X,\star) & \xrightarrow{\boldsymbol{\varphi}_k^*(X,\star)} & pro^*\text{-}H_k(X) \\ \downarrow pro^*\text{-}\pi_k(F^*) & & \downarrow pro^*\text{-}H_k(F^*) \\ pro^*\text{-}\pi_k(Y,\star) & \xrightarrow{\boldsymbol{\varphi}_k^*(Y,\star)} & pro^*\text{-}H_k(Y) \end{array}$$

commutes.

Proof. Let a morphism $\boldsymbol{f}^*=[(f,[f_\lambda^n])]:(\boldsymbol{X},\star)\to(\boldsymbol{Y},\star)$ in pro^*-$HPol_\star$ represents a pointed coarse shape morphism F^*. Then, the morphism pro^*-$\pi_k(F^*)$ is represented by the S^*-morphism

$$(f,\pi_k(f_\lambda^n)):(\pi_k(X_\lambda,\star),\pi_k(p_{\lambda\lambda'}),\Lambda)\to(\pi_k(Y_\mu,\star),\pi_k(q_{\mu\mu'}),M),$$

and the morphism pro^*-$H_k(F^*)$ is represented by the S^*-morphism

$$(f,H_k(f_\lambda^n)):(H_k(X_\lambda),H_k(p_{\lambda\lambda'}),\Lambda)\to(H_k(Y_\mu),H_k(q_{\mu\mu'}),M).$$

Notice that the Hurewicz morphisms $\boldsymbol{\varphi}_k^*(X,\star)$ and $\boldsymbol{\varphi}_k^*(Y,\star)$ are represented by the S^*-morphisms $(1_\Lambda,\varphi_\lambda^n)$ and $(1_M,\varphi_\mu^n)$ respectively. For an arbitrary $\mu\in M$, by using naturality of the Hurewicz homomorphism and applying diagram (10) for homotopy class $[f_\mu^n]:X_{f(\mu)}\to Y_\mu$, one infers that the following diagram in *Grp*

$$\begin{array}{ccc} \pi_k(X_{f(\mu)},\star) & \xrightarrow{\varphi_{f(\mu)}^n} & H_k(X_{f(\mu)}) \\ \downarrow \pi_k(f_\mu^n) & & \downarrow H_k(f_\mu^n) \\ \pi_k(Y_\mu,\star) & \xrightarrow{\varphi_\mu^n} & H_k(Y_\mu) \end{array},$$

commutes for every $n\in\mathbb{N}$. Hence,

$$(1_M,\varphi_\mu^n)(f,\pi_k(f_\lambda^n))=(f,H_k(f_\lambda^n))(1_\Lambda,\varphi_\lambda^n),$$

which completes the proof. ∎

The following Hurewicz isomorphism theorem in $Sh_\star^*$ is the full analogue of Theorem 4.9.

Theorem 4.11. *Let $(X,\star)$ be a pointed topological space, which is $(n-1)$-coarse shape connected. Then*

- (H_1^*) *pro^*-$H_k(X)$ is a zero-object of pro*-Grp, $1\le k\le n-1$, for $n\ge 2$;*
- (H_2^*) *the Hurewicz morphism $\boldsymbol{\varphi}_n^*:pro^*$-$\pi_n(X,\star)\to pro^*$-$H_n(X)$ is an isomorphism of pro*-Grp, for $n\ge 2$;*
- (H_3^*) *the Hurewicz morphism $\boldsymbol{\varphi}_{n+1}^*:pro^*$-$\pi_{n+1}(X,\star)\to pro^*$-$H_{n+1}(X)$ is an epimorphism of pro*-Grp, for $n\ge 2$;*
- (H_4^*) *the Hurewicz morphism $\boldsymbol{\varphi}_1^*:pro^*$-$\pi_1(X,\star)\to pro^*$-$H_1(X)$ is an epimorphism of pro*-Grp, for $n=1$.*

Proof. If $(X,*)$ is $(n-1)$-coarse shape connected then, by Corollary 4.7 and Theorem 4.9, properties $(H_1)-(H_4)$ hold. By referring to Corollary 4.6, properties (H_1) and (H_1^*) are equivalent. Since the Hurewicz morphism $\boldsymbol{\varphi}_k^*$ is the morphism induced by $\boldsymbol{\varphi}_k$, by Theorem 3.2 of [7], it follows that (H_2) and (H_2^*) are equivalent. Finally, since the category *Grp* admits products, Theorem 3 of [6] implies that (H_3) and (H_3^*) , as well as (H_4) and (H_4^*) are equivalent. □

Remark 4.12. One can also define the Hurewicz morphisms in pro^*-*Grp* for every pointed pair of topological spaces. By using the same technique as in the absolute case, it can be seen that Theorems 4.10 and 4.11 also hold in the relative case, i.e., that an analogue of the relative Hurewicz theorem holds in $Sh_\star^{*2}$.

5. Exact sequence of homology and homotopy pro^*-groups

Let $\mathcal{C}$ be a category with zero-objects. A morphism $k : N \to X$ is said to be a *kernel* of a morphism $f : X \to Y$ of $\mathcal{C}$ provided that $fk = o$ and, for every morphism $g : Z \to X$ with $fg = o$, there exists a unique morphism $h : Z \to N$ such that $g = kh$. If we put the additional condition that k is a monomorphism, then we may omit a condition that h is a unique morphism with $g = kh$. Actually, these conditions are equivalent. Kernel of f is unique up to a unique isomorphism. Notice that the composition $ik : N' \to X$ of a kernel k and any isomorphism $i : N' \to N$ in $\mathcal{C}$ is also a kernel of f. If in a category $\mathcal{C}$ with zero-objects, every morphism has a kernel, we say that $\mathcal{C}$ is a *category with kernels.* For instance, the categories *Grp* and $Set_\star$ are the categories with kernels (which are the inclusions of inverse images of $\star$). The following fact is readily seen.

Lemma 5.1. *Let $\mathcal{C}$ be a category with zero-objects and let $i : X \to X'$ and $j : Y \to Y'$ be isomorphisms of $\mathcal{C}$. If a morphism $k : N \to X$ is a kernel of a morphism $f : X \to Y$ in $\mathcal{C}$, then $k' = ik$ is a kernel of morphism $f' = jfi^{-1} : X' \to Y'$.*

It has been proven that *pro-Grp* is the category with kernels ([10, Remark 2 in II.2.3]). Moreover, this statement remains true if we put, instead of *Grp*, any category with kernels and zero objects. But first, let us give the following useful characterization of zero-morphisms in *pro-*$\mathcal{C}$ and pro^*-$\mathcal{C}$ over a category $\mathcal{C}$ having zero-objects.

Theorem 5.2. *Let $\mathcal{C}$ be a category with zero-objects. A morphism $\boldsymbol{f}$: $(X_\lambda, p_{\lambda\lambda'}, \Lambda) \to (Y_\mu, q_{\mu\mu'}, M)$ is a zero-morphism of pro-$\mathcal{C}$ if and only if, for every index function $f : M \to \Lambda$, the morphism (f, o_μ) represents $\boldsymbol{f}$, where $o_\mu : X_{f(\mu)} \to Y_\mu$ is a zero-morphism of $\mathcal{C}$. Further, a morphism $\boldsymbol{f}^*$ is a zero-morphism of pro^*-$\mathcal{C}$ if and only if, it is induced by a zero-morphism of pro-$\mathcal{C}$.*

Proof. Let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ be a morphism of *pro*-$\mathcal{C}$. If $\boldsymbol{f}$ is a zero-morphism, then there exist morphisms $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{0} = (0)$ and $\boldsymbol{g}' : \boldsymbol{0} \to \boldsymbol{Y}$ such that $\boldsymbol{f} = \boldsymbol{g}'\boldsymbol{g}$. It follows that $\boldsymbol{g}$ is represented by a morphism $(g) : \boldsymbol{X} \to \boldsymbol{0}$, for some $\mathcal{C}$-morphism $g : X_{\lambda_0} \to 0$, and $\boldsymbol{g}'$ is represented by some morphism $\left(g'_\mu\right) : \boldsymbol{0} \to \boldsymbol{Y}$, $g'_\mu : 0 \to Y_\mu$, $\mu \in M$. Therefore, the morphism $(c, f_\mu) = \left(g'_\mu\right)(g)$ represents $\boldsymbol{f}$, where $c : M \to \Lambda$, $c(\mu) = \lambda_0$, is the constant index function and $f_\mu = g'_\mu g : X_{\lambda_0} \to Y_\mu$ is a zero-morphism of $\mathcal{C}$ for each $\mu \in M$. Let $f : M \to \Lambda$ be an arbitrary index function and let $o_\mu : X_{f(\mu)} \to Y_\mu$ be a zero-morphism, for every $\mu \in M$. Obviously, $(f, o_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ is a morphism such that, for every $\mu \in M$ and $\lambda \in \Lambda$, $\lambda \geq f(\mu), \lambda_0$, it holds $f_\mu p_{\lambda_0 \lambda} = o = o_\mu p_{f(\mu)\lambda}$. It follows that $(f, o_\mu) \sim (c, f_\mu)$ and therefore (f, o_μ) is a representative of $\boldsymbol{f}$. Conversely, for every index function $f : M \to \Lambda$, let $(f, o_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ be a representative of the morphism $\boldsymbol{f}$, where $o_\mu : X_{f(\mu)} \to Y_\mu$ is a zero-morphism of $\mathcal{C}$. Particularly, for a constant function $c : M \to \Lambda$, the morphism (c, o_μ) represents $\boldsymbol{f}$. Given any $\mu \in M$, for the zero-morphism $o_\mu : X_{c(\mu)} \to Y_\mu$, there exist morphisms $g : X_{c(\mu)} \to 0$ and $g'_\mu : 0 \to Y_\mu$ such that $o_\mu = g'_\mu g$. Therefore, $(g) : \boldsymbol{X} \to (0)$ and $\left(g'_\mu\right) : (0) \to \boldsymbol{Y}$ are morphisms such that $(c, o_\mu) = \left(g'_\mu\right)(g)$. It follows that $\boldsymbol{f} = [(c, o_\mu)]$ is a zero-morphism of *pro*-$\mathcal{C}$ as we asserted. Since *pro*-$\mathcal{C}$ is a subcategory of the category *pro**-$\mathcal{C}$, both having the same object class and the same zero-objects (Corollary 4.5), by the uniqueness of a zero-morphism between any pair of objects, it follows second statement. □

By using the definition of a monomorphism, one can easily verify that in Corollary 5 of [6] we may omit the condition that the index set is cofinite. Along with the fact that the category *pro*-$\mathcal{C}$ can be considered as a subcategory of *pro**-$\mathcal{C}$, it yields the following lemma.

Lemma 5.3. *Let $\boldsymbol{f}^* : \boldsymbol{X} \to (Y_\lambda, q_{\lambda\lambda'}, \Lambda)$ be a morphism of pro*-$\mathcal{C}$ having a level representative $(1_\Lambda, f^n_\lambda)$. If for every $\lambda \in \Lambda$, f^n_λ is a monomorphism of $\mathcal{C}$ for almost all $n \in \mathbb{N}$, then $\boldsymbol{f}^*$ is a monomorphism of pro*-$\mathcal{C}$. Especially, every morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ of pro-$\mathcal{C}$ having a level representative $(1_\Lambda, f_\lambda)$ and consisting of monomorphisms f_λ of $\mathcal{C}$ is a monomorphism.*

Theorem 5.4. *Let $\mathcal{C}$ be a category with kernels and zero-objects. Then pro-$\mathcal{C}$ is a category with kernels.*

Proof. Let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ be a morphism of *pro*-$\mathcal{C}$. By the "reindexing theorem" ([10, Theorem 3 in I.1.3]) and Lemma 5.1, there is no loss of generality in assuming that the both $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\boldsymbol{Y} = (Y_\lambda, q_{\lambda\lambda'}, \Lambda)$ have the same index set Λ and that $\boldsymbol{f}$ admits a level representative $(1_\Lambda, f_\lambda)$. Let $k_\lambda : N_\lambda \to X_\lambda$ be a kernel of $f_\lambda : X_\lambda \to Y_\lambda$ in $\mathcal{C}$, for every $\lambda \in \Lambda$. Since, for every pair $\lambda \leq \lambda'$, it holds

$$f_\lambda (p_{\lambda\lambda'} k_{\lambda'}) = q_{\lambda\lambda'} f_{\lambda'} k_{\lambda'} = q_{\lambda\lambda'} o = o,$$

by the property of a kernel, there exists a unique morphism $t_{\lambda\lambda'} : N_{\lambda'} \to N_\lambda$ such that $k_\lambda t_{\lambda\lambda'} = p_{\lambda\lambda'} k_{\lambda'}$ (see (15)). Obviously, for all $\lambda, \lambda', \lambda'' \in \Lambda$, $\lambda \leq \lambda' \leq \lambda''$, it holds

$$k_\lambda t_{\lambda\lambda'} t_{\lambda'\lambda''} = p_{\lambda\lambda'} k_{\lambda'} t_{\lambda'\lambda''} = p_{\lambda\lambda'} p_{\lambda'\lambda''} k_{\lambda''} = p_{\lambda\lambda''} k_{\lambda''}.$$

Therefore, by the uniqueness, it follows that $t_{\lambda\lambda''} = t_{\lambda\lambda'} t_{\lambda'\lambda''}$. Hence, $\boldsymbol{N} = (N_\lambda, t_{\lambda\lambda'}, \Lambda)$ is an inverse system in $\mathcal{C}$ and $(1_\Lambda, k_\lambda) : \boldsymbol{N} \to \boldsymbol{X}$ is a level morphism. We will prove that $\boldsymbol{k} = [(1_\Lambda, k_\lambda)] : \boldsymbol{N} \to \boldsymbol{X}$ is a kernel of $\boldsymbol{f}$ in *pro*-$\mathcal{C}$. Since $f_\lambda k_\lambda = o$ holds for every $\lambda \in \Lambda$, by using Theorem 5.2 one infers that $\boldsymbol{fk} = \boldsymbol{o}$. Further, since k_λ is a monomorphism of $\mathcal{C}$, for every $\lambda \in \Lambda$, it follows by Lemma 5.3 that $\boldsymbol{k}$ is a monomorphism of *pro*-$\mathcal{C}$. In order to verify that $\boldsymbol{k}$ is a kernel of $\boldsymbol{f}$, it is sufficient to prove that, for every inverse system $\boldsymbol{Z} = (Z_\mu, r_{\mu\mu'}, M)$ in $\mathcal{C}$ and every morphism $\boldsymbol{g} : \boldsymbol{Z} \to \boldsymbol{X}$ of *pro*-$\mathcal{C}$ satisfying

$$\boldsymbol{fg} = \boldsymbol{o}, \tag{11}$$

there exists a morphism $\boldsymbol{h} : \boldsymbol{Z} \to \boldsymbol{N}$ of *pro*-$\mathcal{C}$ such that

$$\boldsymbol{kh} = \boldsymbol{g}. \tag{12}$$

Let (g, g_λ) be a representative of $\boldsymbol{g}$. By combining (11) and Theorem 5.2, we conclude that, for every $\lambda \in \Lambda$, there exists a $\mu \geq g(\lambda)$ such that

$$o = f_\lambda g_\lambda r_{g(\lambda)\mu}.$$

Thus, there is no loss of generality in assuming that

$$o = f_\lambda g_\lambda : Z_{g(\lambda)} \to Y_\lambda. \tag{13}$$

Therefore, by the property of the kernel k_λ of f_λ in $\mathcal{C}$, for every $\lambda \in \Lambda$ there exists a unique $\mathcal{C}$-morphisms $h_\lambda : Z_{g(\lambda)} \to N_\lambda$ such that

$$k_\lambda h_\lambda = g_\lambda. \tag{14}$$

Let us prove that $(g, h_\lambda) : \boldsymbol{Z} \to \boldsymbol{N}$ is a morphism of *inv*-$\mathcal{C}$. First, notice that, for every pair $\lambda \leq \lambda'$, there exists an index $\mu \in M$ such that

$$p_{\lambda\lambda'} g_{\lambda'} r_{g(\lambda')\mu} = g_\lambda r_{g(\lambda)\mu}. \tag{15}$$

Consider the morphism $g_\lambda r_{g(\lambda)\mu} : Z_\mu \to X_\lambda$ of $\mathcal{C}$. Since,

$$f_\lambda \left(g_\lambda r_{g(\lambda)\mu} \right) \overset{(13)}{=} o,$$

by the property of the kernel k_λ, there exists a unique morphism $l : Z_\mu \to N_\lambda$ such that $k_\lambda l = g_\lambda r_{g(\lambda)\mu}$. Since we have established that

$$k_\lambda \left(h_\lambda r_{g(\lambda)\mu} \right) \overset{(14)}{=} g_\lambda r_{g(\lambda)\mu},$$

it follows that $l = h_\lambda r_{g(\lambda)\mu}$. On the other hand, by chasing the diagram below,

$$
\begin{array}{cccccccc}
 & N_\lambda & & \longrightarrow & & X_\lambda & \longrightarrow & Y_\lambda \\
 & & \nwarrow & & \nearrow & & & \\
 & \uparrow & & Z_{g(\lambda)} & & \uparrow & & \uparrow \\
 & & & \uparrow & & & & \\
(16) & N_{\lambda'} & & \longrightarrow & & X_{\lambda'} & \longrightarrow & Y_{\lambda'}\ , \\
 & \nwarrow & & | & \nearrow & & & \\
 & & Z_{g(\lambda')} & & & & & \\
 & & \uparrow & | & & & & \\
 & & & Z_\mu & & & &
\end{array}
$$

we infer that

$$k_\lambda \left(t_{\lambda\lambda'} h_{\lambda'} r_{g(\lambda')\mu}\right) = p_{\lambda\lambda'} k_{\lambda'} h_{\lambda'} r_{g(\lambda')\mu} \overset{(14)}{=} p_{\lambda\lambda'} g_{\lambda'} r_{g(\lambda')\mu} \overset{(15)}{=} g_\lambda r_{g(\lambda)\mu}.$$

Therefore, it holds that

$$t_{\lambda\lambda'} h_{\lambda'} r_{g(\lambda')\mu} = l = h_\lambda r_{g(\lambda)\mu}.$$

This means that (g, h_λ) is a morphism of *inv*-$\mathcal{C}$. Thus, it represents a morphisms $\boldsymbol{h} : \boldsymbol{Z} \to \boldsymbol{N}$ of *pro*-$\mathcal{C}$, and because of (14), holds (12), which completes the proof. ∎

THEOREM 5.5. *Let $\mathcal{C}$ be a category with zero-objects such that pro-$\mathcal{C}$ is the category with kernels. Then $\mathcal{C}$ is the category with kernels if and only if, for every morphism $f : X \to Y$ of $\mathcal{C}$, the rudimentary morphism $(f) : (X) \to (Y)$ of pro-$\mathcal{C}$ has a kernel $\boldsymbol{k} : \boldsymbol{N} \to (X)$ such that the inverse system $\boldsymbol{N}$ is stable.*

PROOF. Suppose $\mathcal{C}$ is a category with kernels. By the proof of Theorem 5.4, a kernel of a rudimentary morphism $(f) : (X) \to (Y)$ of *pro*-$\mathcal{C}$, where $f : X \to Y$ is a morphism in $\mathcal{C}$, is a morphism $(n) : (E) \to (X)$ of *pro*-$\mathcal{C}$, where $n : E \to X$ is a kernel of f in $\mathcal{C}$. Now, for every kernel $\boldsymbol{k} : \boldsymbol{N} \to (X)$ of (f) in *pro*-$\mathcal{C}$, it follows that $\boldsymbol{N}$ and (E) are mutually isomorphic in *pro*-$\mathcal{C}$, which implies that $\boldsymbol{N}$ is stable. Conversely, let $f : X \to Y$ be a morphism of $\mathcal{C}$, and let a morphism $\boldsymbol{k} : \boldsymbol{N} \to (X)$ be a kernel of the rudimentary morphism $(f) : (X) \to (Y)$ of *pro*-$\mathcal{C}$, such that $\boldsymbol{N}$ is stable. Then, there exist an object E of $\mathcal{C}$ and an isomorphism $\boldsymbol{i} : (E) \to \boldsymbol{N}$ in *pro*-$\mathcal{C}$. Now, $\boldsymbol{ki} : (E) \to (X)$ is a kernel of (f) in *pro*-$\mathcal{C}$. Let $n : E \to X$ be a representative of the morphism $\boldsymbol{ki}$. Then, it is trivial to check that n is a kernel of f in $\mathcal{C}$. ∎

The next example shows that an analogue of Theorem 5.4, generally, does not hold for *pro**-$\mathcal{C}$.

EXAMPLE 5.6. Let $\boldsymbol{X} = \boldsymbol{Y} = \left(S^1\right)$ be the rudimentary systems in *Grp*, where S^1 is the multiplicative group of complex numbers $z = e^{i\varphi}$, $\varphi \in [0, 2\pi)$. Consider an S^*-morphism $(f^n) : \boldsymbol{X} \to \boldsymbol{Y}$ given by the homomorphisms $f^n : S^1 \to S^1$ $f^n(z) = z^n$, $n \in \mathbb{N}$. Suppose that $\boldsymbol{k}^* : \boldsymbol{E} \to \boldsymbol{X}$ is a kernel of $\boldsymbol{f}^* =$

$[(f^n)] : \boldsymbol{X} \to \boldsymbol{Y}$ in *pro**-*Grp*. Then, $\boldsymbol{k}^*$ is represented by an S^*-morphism $(e^n) : (E_\lambda, p_{\lambda\lambda'}, \Lambda) \to \boldsymbol{X}$, which is defined by homomorphisms $e^n : E_{\lambda_0} \to S^1$. By Corollary 3.26 of [5], we may assume that Λ is cofinite. Since $\boldsymbol{f}^*\boldsymbol{k}^* = \boldsymbol{o}^*$, there exists, by Theorem 5.2, a $\lambda \geq \lambda_0$ such that $f^n e^n p_{\lambda_0\lambda} = o$ for almost all $n \in \mathbb{N}$. There is no loss of generality in assuming that

$$f^n e^n = o, \text{ for almost all } n \in \mathbb{N} \tag{17}$$

(otherwise, one can take $(e^n p_{\lambda_0\lambda})$ to be a representative of $\boldsymbol{k}^*$). Let $\boldsymbol{E}' = (E_\lambda, p_{\lambda\lambda'}, \Lambda')$ denote a subsystem of $\boldsymbol{E}$ indexed by $\Lambda' = \{\lambda \in \Lambda \mid \lambda \geq \lambda_0\}$ and let $\boldsymbol{i} = [(i, i_\lambda)] : \boldsymbol{E} \to \boldsymbol{E}' = (E_\lambda, p_{\lambda\lambda'}, \Lambda')$ be the restriction morphism in *pro-Grp* ($i : \Lambda' \to \Lambda$ is the inclusion and i_λ is the identity on E_λ, for every $\lambda \in \Lambda'$). Now, for a morphism $\boldsymbol{k}'^* = [(e^n)] : \boldsymbol{E}' \to \boldsymbol{X}$ in *pro**-*Grp*, it holds $\boldsymbol{k}^* = \boldsymbol{k}'^* \underline{J}(\boldsymbol{i})$. Since $\boldsymbol{k}^*$ is a monomorphism (being the kernel) and since $\boldsymbol{i}$ is an isomorphism, it follows that $\boldsymbol{k}'^*$ is also a monomorphism. Let $\boldsymbol{X}' = (X'_\lambda, i_{\lambda\lambda'}, \Lambda')$ denote the inverse system, where $X'_\lambda = S^1$ and $i_{\lambda\lambda'}$ is the identity for all $\lambda \leq \lambda'$ in Λ'. Obviously, the morphism $\boldsymbol{j} : \boldsymbol{X} \to \boldsymbol{X}' = (X'_\lambda, id_{\lambda\lambda'}, \Lambda')$ of *pro-Grp*, which is represented by (j_λ) consisting of the identities $j_\lambda : S^1 \to S^1$, is an isomorphism. Hence, $\underline{J}(\boldsymbol{j})\boldsymbol{k}'^* : \boldsymbol{E}' \to \boldsymbol{X}'$ is a monomorphism of *pro**-*Grp* and it is represented by the level S^*-morphism $(1_{\Lambda'}, e^n p_{\lambda_0\lambda})$. Now, by using [7, Example 4.5 and Proposition 4.4], one infers that, for every $\lambda \in \Lambda'$, there exist a $\lambda' \geq \lambda$ and $n_0 \in \mathbb{N}$ such that $(e^n p_{\lambda_0\lambda'})^{-1}(1) \subseteq (p_{\lambda\lambda'})^{-1}(1)$, for every $n \geq n_0$ (hereby we denote the unit element of every group by 1). It follows that

$$((e^n p_{\lambda_0\lambda})^{-1}(1)) \cap p_{\lambda\lambda'}(E_{\lambda'}) = \{1\}, n \geq n_0,$$

which means that the homomorphism $e^n p_{\lambda_0\lambda}$, restricted to the subgroup $p_{\lambda\lambda'}(E_{\lambda'})$ of E_λ, is a monomorphism for every $n \geq n_0$. Now, (17) implies that $e^n p_{\lambda_0\lambda}(p_{\lambda\lambda'}(E_{\lambda'}))$ is the subgroup of $(f^n)^{-1}(1)$ (the subgroup of the n-th root of unity) for almost all $n \in \mathbb{N}$. Consequently, the order of the group $(f^n)^{-1}(1)$, which is exactly n, is divisible by the order of the group $p_{\lambda\lambda'}(E_{\lambda'})$, for almost all $n \in \mathbb{N}$. Therefore $p_{\lambda\lambda'}(E_{\lambda'})$ is the trivial group and the homomorphism $p_{\lambda\lambda'}$ is the zero-morphism. According to Theorem 4.4, the inverse systems $\boldsymbol{E}'$, and consequently $\boldsymbol{E}$, are zero-objects. By using Proposition 4.1 and denoting by E the trivial group $\{1\}$, one concludes that the kernel of $\boldsymbol{f}^*$ is the only morphism of *pro**-*Grp* of (E) to (S^1), i.e., the morphism $\boldsymbol{n}^* : (E) \to (S^1)$ represented by the S^*-morphisms consisting of all the trivial homomorphisms. Let us denote by Z the group $\bigoplus_{n\in\mathbb{N}} Z_n$, where $Z_n = (f^n)^{-1}(1)$, for every $n \in \mathbb{N}$, and by $\iota^n : Z_n \to Z$ the canonical injection corresponding to the summand Z_n. According to the universal property of a direct sum, for every $n \in \mathbb{N}$, there exists a unique homomorphism $g^n : Z \to X$ such that

$$g^n \iota^n(z) = z, \; z \in Z_n \text{ and}$$

$$g^n \iota^m (z) = 1, \ z \in Z_m, \ m \neq n.$$

A straightforward verification shows that $f^n g^n = o$, for every $n \in \mathbb{N}$. Therefore, $\boldsymbol{f}^* \boldsymbol{g}^* = \boldsymbol{o}^*$, where $\boldsymbol{g}^* = [(g^n)] : (Z) \to \boldsymbol{X}$. Since, the only S^*-morphism $(h^n) : (Z) \to (E)$ consists of the constant homomorphisms $h^n : Z \to E$, $n \in \mathbb{N}$, the only morphism of *pro**-*Grp* of (Z) to (E) is the morphism $\boldsymbol{h}^* = [(h^n)]$. Obviously, $\boldsymbol{n}^* \boldsymbol{h}^* \neq \boldsymbol{g}^*$ which contradicts to the property of a kernel.

Although, for a category $\mathcal{C}$ with kernels, the category *pro**-$\mathcal{C}$ need not to be a category with kernels, an important class of morphisms of *pro**-$\mathcal{C}$ still have kernels.

THEOREM 5.7. *Let $\mathcal{C}$ be a category with kernels and zero-objects. Then every induced morphism of pro*-$\mathcal{C}$ has a kernel. More precisely, if $\boldsymbol{k} : \boldsymbol{N} \to \boldsymbol{X}$ is a kernel of a morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in pro-$\mathcal{C}$, then $\boldsymbol{k}^* = \underline{J}(\boldsymbol{k}) : \boldsymbol{N} \to \boldsymbol{X}$ is a kernel of the induced morphism $\boldsymbol{f}^* = \underline{J}(\boldsymbol{f}) : \boldsymbol{X} \to \boldsymbol{Y}$ in pro*-$\mathcal{C}$.*

PROOF. As in the proof of Theorem 5.4, we may assume that the both $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\boldsymbol{Y} = (Y_\lambda, q_{\lambda\lambda'}, \Lambda)$ have the same index set Λ and that $\boldsymbol{f}^*$ admits a level representative $(1_\Lambda, f_\lambda^n)$, $f_\lambda^n = f_\lambda$, for every $n \in \mathbb{N}$. Since a kernel is unique up to an isomorphism, we may assume that a kernel $\boldsymbol{k} : \boldsymbol{N} \to \boldsymbol{X}$ of $\boldsymbol{f}$ in *pro*-$\mathcal{C}$ is the morphism constructed in the proof of Theorem 5.4, represented by the morphism (k_λ), where $k_\lambda : N_\lambda \to X_\lambda$ is the kernel of f_λ. Notice that, according to Lemma 5.3, a morphism $\boldsymbol{k}^* = [(k_\lambda^n)] : \boldsymbol{N} \to \boldsymbol{X}$, $k_\lambda^n = k_\lambda$, $\lambda \in \Lambda$, $n \in \mathbb{N}$, is a monomorphism of *pro**-$\mathcal{C}$. Thus, it is sufficient to prove that, for every morphism $\boldsymbol{g}^* : \boldsymbol{Z} \to \boldsymbol{X}$ of *pro**-$\mathcal{C}$, $\boldsymbol{Z} = (Z_\mu, r_{\mu\mu'}, M)$, satisfying

$$\boldsymbol{f}^* \boldsymbol{g}^* = \boldsymbol{o}^*, \tag{18}$$

there exists a morphism $\boldsymbol{h}^* : \boldsymbol{Z} \to \boldsymbol{N}$ such that

$$\boldsymbol{k}^* \boldsymbol{h}^* = \boldsymbol{g}^*. \tag{19}$$

Let (g, g_λ^n) be a representative of $\boldsymbol{g}^*$. By using (18) and Theorem 5.2, one infers that, for every $\lambda \in \Lambda$, there exist an $n_\lambda \in \mathbb{N}$ and a $\mu \geq g(\lambda)$ such that

$$o = f_\lambda^n g_\lambda^n r_{g(\lambda)\mu}, \text{ for every } n \geq n_\lambda.$$

Thus, there is no loss of generality in assuming that

$$o = f_\lambda^n g_\lambda^n : Z_{g(\lambda)} \to Y_\lambda, \ n \geq n_\lambda.$$

Therefore, by the property of the kernel k_λ^n of f_λ^n in $\mathcal{C}$, for every $\lambda \in \Lambda$ and every $n \geq n_\lambda$, there exists a unique $\mathcal{C}$-morphisms $h_\lambda^n : Z_{g(\lambda)} \to N_\lambda$ such that

$$k_\lambda^n h_\lambda^n = g_\lambda^n. \tag{20}$$

By performing the obvious changes in the proof of Theorem 5.4, it follows that, for every pair $\lambda \leq \lambda'$, there exists an index $\mu \in M$ such that

$$t_{\lambda\lambda'} h_{\lambda'}^n r_{g(\lambda')\mu} = h_\lambda^n r_{g(\lambda)\mu},$$

for every $n \geq \max\{n_\lambda, n_{\lambda'}\}$. Therefore, letting $h_\lambda^n = h_\lambda^{n_\lambda}$, for every $\lambda \in \Lambda$ and $n < n_\lambda$, we can define an S^*-morphism $(g, h_\lambda^n) : \boldsymbol{Z} \to \boldsymbol{N}$. Now, for $\boldsymbol{h}^* = [(g, h_\lambda^n)]$, because of (20), it holds (19), as asserted. ◻

Recall, that a sequence

$$\cdots \to X' \xrightarrow{f'} X \xrightarrow{f} X'' \to \cdots$$

of morphisms in a category with zero-objects is said to be *exact at* X provided the following holds: $ff' = o$, f has a kernel and, for a kernel $k : N \to X$ of the morphism $f : X \to X''$, there exists a unique epimorphism $h : X' \to N$ satisfying $f' = kh$. A sequence of morphisms is *exact* if it is exact at each of its terms.

THEOREM 5.8. *Let $\mathcal{C}$ be a category with zero-objects and kernels, and let $\boldsymbol{Z}$, $\boldsymbol{X}$ and $\boldsymbol{Y}$ be inverse systems in $\mathcal{C}$ having the same index set Λ. Let morphisms $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{g} : \boldsymbol{Z} \to \boldsymbol{X}$ of pro-$\mathcal{C}$ admit level representatives $(1_\Lambda, f_\lambda)$ and $(1_\Lambda, g_\lambda)$, respectively. If the sequence $Z_\lambda \xrightarrow{g_\lambda} X_\lambda \xrightarrow{f_\lambda} Y_\lambda$ of morphisms of $\mathcal{C}$ is exact for every $\lambda \in \Lambda$, then the sequence $\boldsymbol{Z} \xrightarrow{\boldsymbol{g}} \boldsymbol{X} \xrightarrow{\boldsymbol{f}} \boldsymbol{Y}$ of morphisms of pro-$\mathcal{C}$ is also exact.*

PROOF. By applying Theorem 5.2, one obtains that $\boldsymbol{fg} = \boldsymbol{o}$. Let $\boldsymbol{k} : \boldsymbol{N} \to \boldsymbol{X}$ be the kernel of the morphism $\boldsymbol{f}$ constructed in the proof of Theorem 5.4. Repeating the same procedure from that proof for $M = \Lambda$ and $g = 1_\Lambda$, it is readily seen that the unique morphism $\boldsymbol{h} : \boldsymbol{Z} \to \boldsymbol{N}$ of *pro*-$\mathcal{C}$ satisfying (12) is represented by $(1_\Lambda, h_\lambda)$, where h_λ is a unique morphism such that (14) holds, for every $\lambda \in \Lambda$. Since the sequence $Z_\lambda \xrightarrow{g_\lambda} X_\lambda \xrightarrow{f_\lambda} Y_\lambda$ is exact, it follows that h_λ is an epimorphism of $\mathcal{C}$ for every $\lambda \in \Lambda$. Consequently, by using [6, Corollary 2 and Proposition 1], one infers that $\boldsymbol{h}$ is an epimorphism of *pro*-$\mathcal{C}$. ◻

THEOREM 5.9. *Let $\mathcal{C}$ be a category with zero-objects and kernels and let $\boldsymbol{Z}$, $\boldsymbol{X}$ and $\boldsymbol{Y}$ be inverse systems in $\mathcal{C}$ having the same index set Λ. Let a morphism $\boldsymbol{f}^* : \boldsymbol{X} \to \boldsymbol{Y}$ of pro*-$\mathcal{C}$ be induced by a morphism $\boldsymbol{f}$ of pro-$\mathcal{C}$ admitting a level representative $(1_\Lambda, f_\lambda)$, and let $\boldsymbol{g}^* : \boldsymbol{Z} \to \boldsymbol{X}$ of pro*-$\mathcal{C}$ admit a level representative $(1_\Lambda, g_\lambda^n)$. If, for every $\lambda \in \Lambda$, there exists an $n_\lambda \in \mathbb{N}$ such that the sequence $Z_\lambda \xrightarrow{g_\lambda^n} X_\lambda \xrightarrow{f_\lambda} Y_\lambda$ of morphisms of $\mathcal{C}$ is exact, for every $n \geq n_\lambda$, then the sequence $\boldsymbol{Z} \xrightarrow{\boldsymbol{g}^*} \boldsymbol{X} \xrightarrow{\boldsymbol{f}^*} \boldsymbol{Y}$ of morphisms of pro*-$\mathcal{C}$ is also exact.*

PROOF. By analogy with the proofs of Theorem 5.8 and Theorem 5.7, one obtains that $\boldsymbol{f}^*\boldsymbol{g}^* = \boldsymbol{o}^*$, and that there exists a unique morphism $\boldsymbol{h}^*$ of *pro**-$\mathcal{C}$ satisfying (17), which is represented by an S^*-morphism $(1_\Lambda, h_\lambda^n)$, where h_λ^n is a unique epimorphism of $\mathcal{C}$ such that (19) holds, for every $\lambda \in \Lambda$ and every $n \geq n_\lambda$. Therefore, by Corollary 2 of [6], it follows that $\boldsymbol{h}^*$ is an epimorphism, which completes the proof. ◻

Recall that, for every $k \in \mathbb{N}$, there exists a natural transformation $\partial_k : U_k \rightsquigarrow V_k$ of the functor $U_k : HTop^2 \to Ab$ to the functor $V_k : HTop^2 \to Ab$, where $U_k(X,A) = H_k(X,A)$ and $V_k(X,A) = H_{k-1}(A)$, for every topological pair (X,A). That means that, for every topological pair (X,A), there exists *the homology boundary homomorphism* $\partial_k \equiv \partial_k(X,A) : H_k(X,A) \to H_{k-1}(A)$ and, for every homotopy class $[f] : (X,A) \to (Y,B)$ the following diagram in Ab :

$$
\begin{array}{lcl}
H_k(X,A) & \xrightarrow{\partial_k} & H_{k-1}(A) \\
\downarrow H_k(f) & & \downarrow H_{k-1}(f|A) \\
H_k(Y,B) & \xrightarrow{\partial_k} & H_{k-1}(B)
\end{array}
\tag{21}
$$

commutes. Further, letting $i : A \to X$ and $j : (X,\emptyset) \to (X,A)$ to be the inclusions, the following sequence of homomorphisms in Ab :

$$\cdots \overset{\partial_{k+1}}{\to} H_k(A) \overset{H_k(i)}{\to} H_k(X) \overset{H_k(j)}{\to} H_k(X,A) \overset{\partial_k}{\to} H_{k-1}(A) \overset{H_{k-1}(i)}{\to} \cdots$$

$$\cdots \overset{\partial_1}{\to} H_0(A) \overset{H_0(i)}{\to} H_0(X) \overset{H_0(j)}{\to} H_0(X,A) \to 0 \tag{22}$$

is exact.

For a topological pair (X,A), where subspace A is normally embedded in X, there exists an $HPol^2$-expansion $\boldsymbol{p} = ([p_\lambda]) : (X,A) \to (\boldsymbol{X},\boldsymbol{A}) = ((X_\lambda, A_\lambda), [p_{\lambda\lambda'}], \Lambda)$ such that $\boldsymbol{p} : X \to \boldsymbol{X}$ and $\boldsymbol{p}|A = ([p_\lambda|A]) : A \to \boldsymbol{A} = (A_\lambda, [p_{\lambda\lambda'}|A_{\lambda'}], \Lambda)$ are $HPol$-expansions (see [10, II. 3.1]). We will say that this expansion is a *normal* $HPol^2$-expansion (of a pair). Therefore, if A and B are normally embedded in X and Y, respectively, then for every coarse shape morphism $F^* : (X,A) \to (Y,B)$ in Sh^{*2} which is represented by $\boldsymbol{f}^* = [(f,[f_\mu^n])] : (\boldsymbol{X},\boldsymbol{A}) \to (\boldsymbol{Y},\boldsymbol{B})$, there exists the "restricted" coarse shape morphism $F^*|A : A \to B$ in Sh^* which is defined via a representative $\boldsymbol{f}^*|\boldsymbol{A} = [(f,[f_\mu^n|A_{f(\mu)}])] : \boldsymbol{A} \to \boldsymbol{B}$, where $\boldsymbol{p} : (X,A) \to (\boldsymbol{X},\boldsymbol{A})$ and $\boldsymbol{q} : (Y,B) \to (\boldsymbol{Y},\boldsymbol{B})$ are the normal $HPol^2$-expansions.

For every topological pair (X,A), where A is normally embedded in X, and every $k \in \mathbb{N}$, we are now able to define *the homology boundary morphism in pro***-Ab* to be the morphism

$$\boldsymbol{\partial}_k^* : pro^*\text{-}H_k(X,A) \to pro^*\text{-}H_{k-1}(A)$$

in pro^*-Ab represented by the S^*-morphism

$$\left(1_\Lambda, \partial_{\lambda,k}^n\right) : \left(H_k(X_\lambda,A_\lambda), H_k(p_{\lambda\lambda'}), \Lambda\right) \to \left(H_{k-1}(A_\lambda), H_{k-1}(p_{\lambda\lambda'}|A_\lambda), \Lambda\right),$$

where $\boldsymbol{p} = ([p_\lambda]) : (X,A) \to ((X_\lambda,A_\lambda),[p_{\lambda\lambda'}],\Lambda)$ is the normal $HPol^2$-expansion of (X,A) and $\partial_{\lambda,k} = \partial_{\lambda,k}^n = \partial_k(X_\lambda,A_\lambda) : H_k(X_\lambda,A_\lambda) \to H_{k-1}(A_\lambda)$ is the boundary homology homomorphism, for all $n \in \mathbb{N}$ and $\lambda \in \Lambda$.

Obviously, (21) assures commutativity of the following diagram in Ab

$$\begin{array}{ccc} H_k\left(X_{\lambda'}, A_{\lambda'}\right) & \overset{\partial^n_{\lambda',k}}{\longrightarrow} & H_{k-1}\left(A_{\lambda'}\right) \\ \downarrow H_k(p_{\lambda\lambda'}) & & \downarrow H_{k-1}\left(p_{\lambda\lambda'}|A_{\lambda'}\right) \\ H_k\left(X_{\lambda}, A_{\lambda}\right) & \overset{\partial^n_{\lambda,k}}{\longrightarrow} & H_{k-1}\left(A_{\lambda}\right) \end{array},$$

which shows that $\left(1_\Lambda, \partial^n_{\lambda,k}\right)$ is a level S^*-morphism indeed. Notice that $\boldsymbol{\partial}^*_k$ is induced by the morphism $[(1_\Lambda, \partial_{\lambda,k})]$ of *pro-Ab*.

THEOREM 5.10. *Let A be a subspace normally embedded in a space X. If $\boldsymbol{i}^*_k = pro^*\text{-}H_k\left(S^*(i)\right)$ and $\boldsymbol{j}^*_k = pro^*\text{-}H_k\left(S^*(j)\right)$ are morphisms of pro^*-Ab, where $i : A \to X$ and $j : (X, \emptyset) \to (X, A)$ are the inclusions, then the following sequence of morphisms of pro^*-Ab*

$$\ldots \overset{\boldsymbol{\partial}^*_{k+1}}{\to} pro^*\text{-}H_k(A) \overset{\boldsymbol{i}^*_k}{\to} pro^*\text{-}H_k(X) \overset{\boldsymbol{j}^*_k}{\to}$$

$$pro^*\text{-}H_k(X, A) \overset{\boldsymbol{\partial}^*_k}{\to} pro^*\text{-}H_{k-1}(A) \overset{\boldsymbol{i}^*_{k-1}}{\to} \cdots$$

$$\cdots \overset{\boldsymbol{\partial}^*_1}{\to} pro^*\text{-}H_0(A) \overset{\boldsymbol{i}^*_0}{\to} pro^*\text{-}H_0(X) \overset{\boldsymbol{j}^*_0}{\to} H_0(X, A) \overset{\boldsymbol{o}^*}{\to} \mathbf{0}$$

is exact.

PROOF. Let $\boldsymbol{p} : (X, A) \to (\boldsymbol{X}, \boldsymbol{A}) = ((X_\lambda, A_\lambda), [p_{\lambda\lambda'}], \Lambda)$ be a normal $HPol^2$-expansion. Then, the coarse shape morphisms $S^*(i) : A \to X$ and $S^*(j) : (X, \emptyset) \to (X, A)$ are represented by morphisms $[(1_\Lambda, [i^n_\lambda])] : \boldsymbol{A} \to \boldsymbol{X}$ of pro^*-$HPol$ and $[(1_\Lambda, [j^n_\lambda])] : (\boldsymbol{X}, \boldsymbol{\emptyset}) \to (\boldsymbol{X}, \boldsymbol{A})$ of pro^*-$HPol^2$, respectively, where $i_\lambda = i^n_\lambda : A_\lambda \hookrightarrow X_\lambda$ and $j_\lambda = j^n_\lambda : (X_\lambda, \emptyset) \hookrightarrow (X_\lambda, A_\lambda)$ are the inclusions, for all $\lambda \in \Lambda$ and $n \in \mathbb{N}$, and $\boldsymbol{\emptyset}$ denotes the inverse system having all terms to be the empty space. Notice that, by Theorems 4.4 and 5.2, we may assume that a zero-object is $\mathbf{0} = (0_\lambda, o_{\lambda\lambda'}, \Lambda)$ and that a zero-morphism $\boldsymbol{o}^*$ admits a level representative $(1_\Lambda, o^n_\lambda)$, where 0_λ is the trivial group and o^n_λ is the constant homomorphism, for all $\lambda \in \Lambda$ and $n \in \mathbb{N}$. Notice that the morphisms $\boldsymbol{i}^*_k$, $\boldsymbol{j}^*_k$ and $\boldsymbol{\partial}^*_k$ are induced by morphisms $[(1_\Lambda, H_k(i_\lambda))]$, $[(1_\Lambda, H_k(j_\lambda))]$ and $[(1_\Lambda, \partial_{\lambda,k})]$ of *pro-Ab*, respectively. Now, since the sequence (22) is exact, it follows that the following sequence

$$\ldots \overset{\partial^n_{\lambda,k+1}}{\to} H_k(A_\lambda) \overset{H_k(i^n_\lambda)}{\to} H_k(X_\lambda) \overset{H_k(j^n_\lambda)}{\to}$$

$$H_k(X_\lambda, A_\lambda) \overset{\partial^n_{\lambda,k}}{\to} H_{k-1}(A_\lambda) \overset{H_{k-1}(i^n_\lambda)}{\to} \ldots$$

$$\ldots \overset{\partial^n_{\lambda,1}}{\to} H_0(A_\lambda) \overset{H_0(i^n_\lambda)}{\to} H_0(X_\lambda) \overset{H_0(j^n_\lambda)}{\to} H_0(X_\lambda, A_\lambda) \overset{o^n_\lambda}{\to} 0_\lambda,$$

is exact, for all $\lambda \in \Lambda$ and $n \in \mathbb{N}$. Therefore, we may apply Theorem 5.9 to complete the proof. ∎

Theorem 5.11. *Let A be normally embedded in a space X and let B be normally embedded in a space Y. If $F^{*}:(X,A)\rightarrow(Y,B)$ is a coarse shape morphism of Sh^{*2}, then the following diagram commutes in the pro*-Ab, for every $k\in\mathbb{N}$,*

$$\begin{array}{ccc} pro^{*}\text{-}H_{k}\left(X,A\right) & \xrightarrow{\boldsymbol{\partial}_{k}^{*}} & pro^{*}\text{-}H_{k-1}\left(A\right) \\ \downarrow pro^{*}\text{-}H_{k}\left(F^{*}\right) & & \downarrow pro^{*}\text{-}H_{k}(F^{*}|A) \\ pro^{*}\text{-}H_{k}\left(Y,B\right) & \xrightarrow{\boldsymbol{\partial}_{k}^{*}} & pro^{*}\text{-}H_{k-1}\left(B\right) \end{array}.$$

Proof. Let $k\in\mathbb{N}$, and let $\boldsymbol{p}:(X,A)\rightarrow(\boldsymbol{X},\boldsymbol{A})$ and $\boldsymbol{q}:(Y,B)\rightarrow(\boldsymbol{Y},\boldsymbol{B})$ be the normal $HPol^{2}$-expansions of (X,A) and (Y,B) respectively. If a coarse shape morphism $F^{*}:(X,A)\rightarrow(Y,B)$ is represented by a morphism $\boldsymbol{f}^{*}=[(f,[f_{\mu}^{n}])]:(\boldsymbol{X},\boldsymbol{A})\rightarrow(\boldsymbol{Y},\boldsymbol{B})$ of pro^{*}-$HPol^{2}$, then the restricted coarse shape morphism $F^{*}|A:A\rightarrow B$ is represented by a morphism $\boldsymbol{f}^{*}|\boldsymbol{A}=\left(f,\left[f_{\mu}^{n}|A_{f(\mu)}\right]\right):\boldsymbol{A}\rightarrow\boldsymbol{B}$ of pro^{*}-$HPol$. Therefore, in order to prove that

$$pro^{*}\text{-}H_{k}(F^{*}|A)\circ\boldsymbol{\partial}_{k}^{*}=\boldsymbol{\partial}_{k}^{*}\circ pro^{*}\text{-}H_{k}\left(F^{*}\right),$$

it is sufficient to verify that

$$\left(f,H_{k}(f_{\mu}^{n}|A_{f(\mu)})\right)\left(1_{\Lambda},\partial_{\lambda,k}^{n}\right)\sim\left(1_{M},\partial_{\mu,k}^{n}\right)\left(f,H_{k}(f_{\mu}^{n})\right).$$

However, this is a consequence of the commutativity of the following diagram

$$\begin{array}{ccc} H_{k}\left(X_{f(\mu)},A_{f(\mu)}\right) & \xrightarrow{\partial_{f(\mu),k}^{n}} & H_{k-1}\left(A_{f(\mu)}\right) \\ \downarrow H_{k}(f_{\mu}^{n}) & & \downarrow H_{k-1}\left(f_{\mu}^{n}|A_{f(\mu)}\right) \\ H_{k}\left(Y_{\mu},B_{\mu}\right) & \xrightarrow{\partial_{\mu,k}^{n}} & H_{k-1}\left(B_{\mu}\right) \end{array},$$

which holds since (21) commutes. □

Remark 5.12. If we put any Abelian group G instead of $\mathbb{Z}$, i.e., if we consider the functors $H_{k}\left(\cdot;G\right)$, pro^{*}-$H_{k}\left(\cdot,\cdot;G\right)$ and pro^{*}-$H_{k}\left(\cdot;G\right)$, then all the above (particularly Theorems 5.10 and 5.11) also holds.

By taking the functors $\pi_{k}\left(\cdot,\cdot\right)$, pro^{*}-$\pi_{k}\left(\cdot,\cdot\right)$ and pro^{*}-$\pi_{k}\left(\cdot\right)$ into consideration, we can obtain as above, *the homotopy boundary morphism* $\boldsymbol{\partial}_{k}^{*}$ of pro^{*}-Grp, for $k\geq2$, and $\boldsymbol{\partial}_{1}^{*}$ of pro^{*}-$Set_{\star}$. One can easily verify that the analogous theorems to Theorems 5.10 and 5.11 hold for the induced homotopy functors on the pointed coarse shape category (of pairs) and for the homotopy boundary morphisms as well.

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