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A PROPERTY OF GROUPS OF ORDER $\leq p^{p(e+1)}$ AND EXPONENT p^e

Yakov Berkovich
University of Haifa, Israel

To Professor Noboru Ito on the occasion of his 80th anniversary

Abstract. Let G be a p-group of exponent p^e and order p^m, where $m < p(e+1)$ if $p > 2$ and $m \leq 2(e+1)$ if $p = 2$. Then, if $\mho^{e-1}(G)$ is irregular, then $p = 2$, $e = 2$ and $\mho^{e-1}(G) \cong \mathrm{D}_8 \times \mathrm{C}_2$, where $|\mathrm{C}_2| = 2$ and D_8 is dihedral of order 8.

It is proved in [B4, Lemma 4.1] that if $G > \{1\}$ is a group of order p^m and exponent p^e, where $m \leq pe$, then $\mho_{e-1}(G)$ is of order $\leq p^p$ and exponent p. In this note we improve this result essentially.

We use the standard notation as in [B2, B4]. In what follows, G is a p-group, where p is a prime. For $n \in \mathbb{N} \cup \{0\}$, we set $\mho_n(G) = \langle x^{p^n} \mid x \in G\rangle$ and $\Omega_n(G) = \langle x \in G \mid x^{p^n} = 1\rangle$ so that $\mho_0(G) = G$ and $\Omega_0(G) = \{1\}$. The characteristic subgroups $\mho_n(G)$ and $\Omega_n(G)$, introduced by Philip Hall, determine the power structure of G. If G is of exponent p^e, then $\mho_{e-1}(G) \leq \Omega_1(G)$ (the strong inequality is possible as the group $G = \mathrm{D}_8$ shows); moreover, $\mho_{e-1}(G)$ is generated by elements of order p so, if $\exp(\mho_{e-1}(G)) > p$, then $\mho_{e-1}(G)$ is irregular (see Lemma 2(f)). A p-group G satisfying $|G : \mho_1(G)| < p^p$, is called, according to Blackburn, *absolutely regular*. By the Hall regularity criterion, absolutely regular p-groups are regular. It is easy to show that subgroups and epimorphic images of absolutely regular p-groups are absolutely regular. All these assertions are used freely in what follows.

Let $\mathrm{d}(G)$ stand for the minimal number of generators of G. We have $p^{\mathrm{d}(G)} = |G : \Phi(G)|$, where $\Phi(G)$ is the Frattini subgroup of G. Next, D_{2^n} and

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Q_{2^n} are dihedral and generalized quaternion groups of order 2^n, respectively. The abelian group of type $(2,2)$ is denoted by E_4.

We define the series $G = \mho^0(G) > \mho^1(G) > \mho^2(G) > \dots$ of characteristic subgroups inductively, setting

$$\mho^0(G) = G,\ \mho^1(G) = \mho_1(G),\ \mho^{i+1}(G) = \mho_1(\mho^i(G)).$$

Since $\exp(G/\mho^i(G)) \le p^i$, we get $\mho_i(G) \le \mho^i(G)$ for all $i \in \mathbb{N}$. For $\exp(G) = p^e > p$, the inequality $\mho^e(G) > \{1\}$ is possible as the following remark shows.

Remark 1. We will show that the strong inequality $\mho_2(G) < \mho^2(G)$ is possible. Let G be the two-generator group of exponent 4 of maximal order; then $|G : \mho_1(G)| = |G : \Phi(G)| = 4$. It is known (Burnside) that $|G| = 2^{12}$. By Schreier's Theorem on the number of generators of a subgroup, $d(\mho_1(G)) \le 1+(d(G)-1)|G : \mho_1(G)| = 5$. Therefore, in view of $|\mho_1(G)| = 2^{10}$, the subgroup $\mho_1(G)$ is not elementary abelian so $\mho^2(G) = \mho_1(\mho_1(G)) > \{1\} = \mho_2(G)$, as was to be shown.

However, if G is regular, then $\mho^i(G) = \mho_i(G)$ for all i. Moreover, the last equality holds if $\mho_i(H) = \{x^{p^i} \mid x \in H\}$ for all $i \in \mathbb{N}$ and all sections H of G (or, what is the same, if G is a $\mathcal{P}_1$-group [Man]). Indeed, suppose we have proved that $\mho^{i-1}(G) = \mho_{i-1}(G)$. Take $x \in \mho^i(G)$. Then $x = y^p$ for some $y \in \mho^{i-1}(G) = \mho_{i-1}(G) = \{z^{p^{i-1}} \mid z \in G\}$. It follows that, for some $z \in G$, we get $y = z^{p^{i-1}}$ so $x = z^{p^i} \in \mho_i(G)$. Thus, $\mho^i(G) \le \mho_i(G)$, and we are done, since the reverse inclusion is true. Note that $\mathcal{P}_1$-groups are not necessary regular (all 2-groups of maximal class are irregular $\mathcal{P}_1$-groups).

Recall [B3] that a p-group G is said to be *pyramidal* if $|\Omega_i(G) : \Omega_{i-1}(G)| \ge |\Omega_{i+1}(G) : \Omega_i(G)|$ and $|\mho_{i-1}(G) : \mho_i(G)| \ge |\mho_i(G) : \mho_{i+1}(G)|$ for all $i = 1, 2, \dots$.

In this note we prove the following

Main Theorem *Let G be a p-group of order p^m and exponent p^e, where $m < p(e+1)$ if $p > 2$ and $m \le 2(e+1)$ if $p = 2$. Then*

(a) *$|\mho_{e-1}(G)| < p^{2p}$ if $p > 2$ and $|\mho_{e-1}(G)| \le 2^4$ if $p = 2$.*
(b) *If $\mho^{e-1}(G)$ is irregular, then $p = 2$, $e = 2$, $|G| = 2^6$ and $\mho^1(G) = C \times M$, where $|C| = 2$ and $M \cong D_8$.*

Part (a) of the Main Theorem is trivial for pyramidal p-groups.

Some known results are collected in the following

Lemma 2. *Let G be a p-group.*

(a) *(Hall) If G is regular, it is pyramidal and $|\Omega_1(G)| = |G/\mho_1(G)|$.*
(b) *Suppose that G is a noncyclic group of order p^3 and P a Sylow p-subgroup of $Aut(G)$. Then P is nonabelian of order p^3 (moreover, if $p > 2$, then $\exp(P) = p$ and if $p = 2$, then P is dihedral).*
(c) *(Hall) If G is of class $< p$, it is regular. Next, G is regular if $p > 2$ and G' is cyclic.*

(d) *(Burnside) Let $R < \Phi(G)$ be normal in G. If $Z(R)$ is cyclic so is R.*
(e) [B2, Theorem 4.1] *If $R \leq \Phi(G)$ is normal in G and R is generated by two elements, it is metacyclic.*
(f) *(Hall) If G is regular, then $\exp(\Omega_i(G)) \leq p^i$ for all $i \in \mathbb{N}$.*
(g) [B1, Lemma 1.4] *Let $N \trianglelefteq G$. If N has no G-invariant abelian subgroup of type (p,p), it is either cyclic or a 2-group of maximal class.*
(h) [B2, §7, Remark 2] *Suppose that a p-group G is neither absolutely regular nor of maximal class. Then the number of subgroups of order p^p and exponent p in G is $\equiv 1 \pmod p$.*

Let us show that if G is noncyclic of order p^3, $p > 2$, then $\exp(P) = p$, where $P \in \mathrm{Syl}_p(\mathrm{Aut}(G))$. Take $\mu \in P^{\#}$. In that case, G has a μ-invariant subgroup F of type (p,p) and F has a μ-invariant subgroup $H \triangleleft G$ of order p. Given $g \in G$, we have $g^{\mu} = gf$ for some $f \in F$, and let $f^{\mu} = fh$ for some $h \in H$. Then, by induction, we get $g^{\mu^n} = gf^n h^{(n-1)n/2}$. Taking, in the last formula, $n = p$, we get $g^{\mu^p} = gf^p h^{(p-1)p/2} = g$ in view of $\exp(F) = \exp(H) = p$ and $p > 2$. Thus, $\mu^p = \mathrm{id}_G$ for all $\mu \in P^{\#}$ so $\exp(P) = p$. It is easy to check that a Sylow p-subgroup of the automorphism group of a noncyclic abelian p-group is nonabelian.

If, in the Main Theorem, $p > 2$, then $\mho_{e-1}(G)$ is of exponent p. Indeed, $\mho_{e-1}(G)$ is generated by elements of order p and, by the Main Theorem, regular so the assertion follows from Lemma 2(f).

Remark 3 ([J]). There exists a group G of order 2^6 with $\exp(G) = 4$ and $\mho_1(G)(= \Phi(G)) = \mathrm{D}_8 \times \mathrm{C}_2$. Indeed, let

$$G = \langle x, s \mid s^2 = t,\ x^s = y,\ y^s = xu,\ u^s = uz,$$
$$u^2 = z^2 = t^2 = s^4 = [x,y] = [x,u] = [x,z] = [y,u] = [y,z] =$$
$$[u,z] = [u,t] = [z,t] = [z,x] = 1,\ x^2 = u,\ y^2 = uz\rangle.$$

Here $|G| = 2^6$, $\Phi(G) = \langle t, z, u, xy\rangle = \mathrm{D}_8 \times \mathrm{C}_2$, $G' = \langle xy, u\rangle$ and $M = \langle x, y\rangle = \mathrm{C}_4 \times \mathrm{C}_4$ is a normal (even characteristic) abelian subgroup of G and the involution t inverts M, $\mathrm{Z}(G) = \langle z\rangle$ is of order 2 and $\exp(G) = 4$. Let us check these assertions. It follows from $\Phi(G) = \mho_1(G)G'$ that elements $t = s^2$, $u = x^2$, $z = [u,s]$, $xy = u[x,s]$ are contained in $\Phi(G)$. Since $G = \langle x, y\rangle$ and $\langle t, xy\rangle \times \langle u\rangle \cong \mathrm{D}_8 \times \mathrm{C}_2$, we get $\Phi(G) \cong \mathrm{D}_8 \times \mathrm{C}_2$. We compute

$$x^t = (x^s)^s = y^s = xu = xx^2 = x^3 = x^{-1},$$
$$y^t = (y^s)^s = (xu)^s = yuz = yy^2 = y^3 = y^{-1}$$

so t inverts $M = \langle x, y\rangle \cong \mathrm{C}_4 \times \mathrm{C}_4$. We have

$$[x,t] = x^{-1}x^t = x^2 = u,\ [y,t] = y^{-1}y^t = y^2.$$
$$[y,s] = y^{-1}xu = x^3y^3 = xuyuz = xyz,\ x^2y^2 = z.$$

Since $|G : G'| > 4$ (Taussky's Theorem), we get $G' = \langle xy, u\rangle \cong \mathrm{C}_4 \times \mathrm{C}_2$.

Clearly, $G = \langle s\rangle \cdot M$ is a semidirect product with kernel M so $|G| = 2^2|M| = 2^6$. Thus, M is a maximal abelian subgroup of G so $\mathrm{Z}(G) < M$ and $\mathrm{Z}(G) \le \Omega_1(M)$. Since $(x^2)^s = y^2 \neq x^2$ and $(y^2)^s = (xu)^2 = x^2 \neq y^2$, it follows that $\mathrm{Z}(G) = \langle z\rangle$, in view of $\Omega_1(M) = \{1, x^2, y^2, x^2y^2 = z\}$. If $a \in M^\#$, then $h = (sa)^2 = ta^sa \in \langle t, M\rangle - M$ is an involution so $(sa)^4 = h^2 = 1$, and we conclude that $\exp(G) = 4$.

We find this group G as a subgroup of the symmetric group S_8, if we set:

$$s = (1,5)(2,8,4,7)(3,6),\ x = (5,7,6,8)$$

and check that these permutations satisfy all above relations, and since

$$z = uu^s = (1,3)(2,4)(5,6)(7,8)$$

is a nontrivial permutation (where $\langle z\rangle = \mathrm{Z}(G)$), we have obtained a faithful permutation representation of our group G and so G exists.

Our theorem follows easily from Lemmas 4–7.

Lemma 4. *Suppose that G is a p-group of order p^{s+pe} and exponent p^e, $s \ge 0$, $e > 1$. If $|\mho^{e-1}(G)| > p^{s+p}$, then there is $i \le e-2$ such that $\mho^i(G)$ is absolutely regular; in particular, in that case we have $|\mho_{e-1}(G)| < p^p$.*

Proof. We have

$$p^{s+pe} = |G| = (\prod_{i=0}^{e-2} |\mho^i(G) : \mho^{i+1}(G)|)|\mho^{e-1}(G)| \tag{1}$$

so that, by hypothesis,

$$\prod_{i=0}^{e-2} |\mho^i(G) : \mho^{i+1}(G)| < p^{p(e-1)}. \tag{2}$$

It follows from (2) that $|\mho^i(G) : \mho^{i+1}(G)| < p^p$ for some $i \le e-2$, whence $\mho^i(G)$ is absolutely regular so regular, and we have $|\Omega_1(\mho^i(G))| < p^p$. Then $\mho_{e-1}(G) = \Omega_1(\mho_{e-1}(G)) \le \Omega_1(\mho^i(G))$ (Lemma 2(a,f)), and the proof is complete. □

Corollary 5. *Suppose that G is a p-group of order p^{s+pe} and exponent p^e, $s \ge 0$, $e > 1$. Then $|\mho_{e-1}(G)| \le p^{s+p}$.*

Proof. Assume that $|\mho_{e-1}(G)| > p^{s+p}$; then $|\mho^{e-1}(G)| > p^{s+p}$. In that case, $\mho_{e-1}(G)$ is absolutely regular (Lemma 4) so $|\mho_{e-1}(G)| = |\Omega_1(\mho_{e-1}(G))| < p^p < p^{s+p}$ (Lemma 2(a,f)), contrary to the assumption. □

Lemma 6. *Suppose that G is a group of order p^{s+pe} and exponent p^e, $s \ge 0$, $e > 1$. If $|\mho_{e-1}(G)| = p^{s+p}$, then $|\mho^i(G) : \mho^{i+1}(G)| = p^p$ for $i = 1, \dots, e-2$ so $\mho^{e-1}(G) = \mho_{e-1}(G)$.*

PROOF. Since the subgroup $\mho_{e-1}(G)$ of order $p^{s+p} > p^{p-1}$ is generated by elements of order p, it is not absolutely regular so is $\mho^{e-1}(G)$ as it contains $\mho_{e-1}(G)$. It follows from Lemma 4 that $|\mho^{e-1}(G)| \le p^{s+p} = |\mho_{e-1}(G)|$ so that $\mho_{e-1}(G) = \mho^{e-1}(G)$. In particular, $|\mho^{i-1}(G) : \mho^{i}(G)| \ge p^p$ for $i \le e-1$ since, for these i, $\mho^{i-1}(G)$ is not absolutely regular. Then, by (1),

$$p^{s+pe} = (\prod_{i=0}^{e-2} |\mho^i(G) : \mho^{i+1}(G)|)p^{s+p} \ge p^{p(e-1)}p^{s+p} = p^{s+pe}$$

so $|\mho^i(G) : \mho^{i+1}(G)| = p^p$ for $i = 0, 1, \dots, e-2$. □

In the sequel we use freely the following known fact. If $K < L \le \Phi(G)$ are normal subgroups of a p-group G and $|L/K| = p^2$, then $L/K \le \mathrm{Z}(\Phi(G/K))$.

LEMMA 7. *Let $N \le \Phi(G)$ be a normal irregular subgroup of order p^n in a p-group G, where $n < 2p$ if $p > 2$ and $n \le 2p$ if $p = 2$. Then $p = 2$ and $N = C \times M$, where $|C| = 2$ and M is nonabelian of order* 8.

PROOF. Assume that $\mathrm{d}(N) = 2$. Then N, by Lemma 2(e), is metacyclic. By hypothesis, N is irregular so $p = 2$ (Lemma 2(c)) and $|N| \le 2^4$. In that case, by Lemma 2(d), the center of N is noncyclic so $|N| = 2^4$ and $\exp(N) = 4$. It is known that there exists only one nonabelian metacyclic group of order 2^4 and exponent 4 and so $N = \langle a, b \mid a^4 = b^4 = 1, a^b = a^{-1}\rangle$. It is easy to see that N contains exactly three involutions, namely, a^2, b^2 and a^2b^2, and all of them lie in $\mathrm{Z}(N)$. Since $N/\langle a^2b^2\rangle \cong \mathrm{Q}_8$ and N has no cyclic subgroup of index 2, we conclude that a^2b^2 is the unique involution of N which is not a square in N. It follows that $L = \langle a^2b^2\rangle$ is a characteristic subgroup of N so L is normal in G. However, the nonabelian subgroup N/L with cyclic center is normal in G/N and contained in $\Phi(G/L)$, contrary to Lemma 2(d). Thus, $\mathrm{d}(N) > 2$.

Assume that $p > 2$. Then, by the previous paragraph, $|N'| \le p^{n-3} \le p^{2p-4}$ so the class of N does not exceed $1 + \frac{2p-4}{2} = p - 1$ (see the paragraph, preceding the lemma). Then N is regular (Lemma 2(c)), a contradiction.

Thus, $p = 2$ so $|N| = 2^4$. In view of $\mathrm{d}(N) > 2$, N is neither of maximal class nor minimal nonabelian. In that case, N has a G-invariant abelian subgroup R of type $(2, 2)$ (Lemma 2(g)); then $R \le \mathrm{Z}(N)$ in view of $R \le \Phi(G)$. Let M be a minimal nonabelian subgroup of N; then $|M| = 8$. Let $C < R$ be such that $C \not\le M$ (C exists since $\mathrm{Z}(M)$ has order 2). Then $|C| = 2$ and $N = C \times M$, completing the proof. □

Now we are ready to prove our main result.

PROOF OF THE MAIN THEOREM. (a) follows from Corollary 5 since $|G| = p^{s+pe}$, where $s \le p$ and $s < p$ if $p > 2$.

(b) Since $\mho^{e-1}(G)$ is irregular, its order is at least p^{2p} (Lemma 7). On the other hand, $|\mho^{e-1}(G)| \le p^{2p}$ (Lemma 4). It follows that $|\mho^{e-1}(G)| = p^{2p}$, and then, by Lemma 7, we must have $p = 2$. In that case, $|\mho^{e-1}(G)| = 2^4$, and we have $N = C \times M$, where $|C| = 2$ and M is nonabelian of order 8 (Lemma 7).

It remains to prove that $e = 2$ and $M \cong \mathrm{D}_8$.

Assume that $e > 2$. By Lemma 6, we have $|\mho^1(G) : \mho^2(G)| = p^p = 2^2$ so $\mathrm{d}(\mho^1(G)) = 2$. It follows that $\Phi(G) = \mho^1(G)$ is metacyclic (Lemma 2(e)), a contradiction since it contains, by what has been proved already, a nonmetacyclic subgroup $\mho^{e-1}(G)$. Thus, $e = 2$ so that $\exp(G) = 2^e = 4$.

Now assume that $M \not\cong \mathrm{D}_8$; then $M \cong \mathrm{Q}_8$. Since the subgroup $\mho_1(G) = \mho^1(G) = \mathrm{C}_2 \times \mathrm{Q}_8$ is not generated by involutions, it contains an element x of order 4 which is a square in G, i.e., $x = y^2$ for some $y \in G$. It follows that $\exp(G) \geq o(y) = 2^3 > 2^2 = 2^e$, a final contradiction. The proof is complete. (In Remark 3 the group, satisfying part (b) of the theorem, is presented.) □

Corollary 8. *Suppose that G is a p-group of order $p^{p(e+t+1)}$ and exponent $p^e, e > 1$, $t \geq 0$. If $\exp(\mho_{e-1}(G)) > p^{t+1}$, then $|\mho_{e-1}(G)| = p^{p(t+2)}$.*

Proof. We proceed by induction on t. Set $H = \mho_{e-1}(G)$.

We have $|G| = p^{p(e+t+1)} = p^{p(t+1)+pe}$ so, by Corollary 5, $|H| \leq p^{p(t+1)+p} = p^{p(t+2)}$. By hypothesis, $\exp(H) > p^{t+1} \geq p$ so, since H is generated by elements of order p, it is irregular (Lemma 2(f)). Since H is not of maximal class (Lemma 2(d)), it has a G-invariant subgroup R of order p^p and exponent p (Lemma 2(h)).

(i) Let $t = 0$. Since H is irregular, then, by Lemma 7, $|H| \geq p^{2p}$ so, by the previous paragraph (take there $t = 0$), we get $|H| = p^{2p} = p^{p(t+2)}$. Thus, the corollary is true for $t = 0$.

(ii) Now let $t > 0$. In that case, we have $|G/R| = p^{p[e+(t-1)+1]}$. The subgroup $H/R = \mho_{e-1}(G/R)$ has exponent $\geq \frac{\exp(G)}{\exp(R)} > \frac{p^{t+1}}{p} = p^{(t-1)+1}$. Therefore, by induction, $|H/R| = p^{p[(t-1)+2]} = p^{p(t+1)}$ so $|H| = |H/R||R| = p^{p(t+2)}$.

□

Remark 9. Let N be a normal subgroup of order p^3 in a p-group G. We claim that if $N \leq \Phi(\Phi(G))$, then $N \leq \mathrm{Z}(\Phi(\Phi(G))$. Set $N_1 = N \cap \mathrm{Z}(\Phi(G))$ and $C = \mathrm{C}_{\Phi(G)}(N)$; then C is normal in G and $|N_1| \geq p^2$ since each G-invariant subgroup of order p^2 in $\Phi(G)$ is contained in the center of $\Phi(G)$ (see the paragraph preceding Lemma 7); in particular, N is abelian. We have to prove that $\Phi(\Phi(G)) \leq C$. (i) Let N be cyclic; then $\mathrm{Aut}(N)$ is abelian so $\Phi(G)/C$ is. If $p = 2$, then $G/\mathrm{C}_G(N)$ is isomorphic to a subgroup of $\mathrm{Aut}(N) \cong \mathrm{E}_4$. It follows that $\Phi(G)$ centralizes N; in particular, $N \leq \mathrm{Z}(\Phi(G))$ so $N \leq \mathrm{Z}(\Phi(\Phi(G))$. Now let $p > 2$. Set $N = \langle x \rangle$ and take $y \in \Phi(G)$; then $x^y = x^r$ for some $r \in \mathbb{N}$. Since $x^p \in N_1 \leq \mathrm{Z}(\Phi(G))$, we get $x^{pr} = (x^p)^y = x^p$ so $x^{p(r-1)} = 1$. It follows that $r = 1 + kp^2$ for some nonnegative integer k since

$o(x) = p^3$. In that case, $(1+kp^2)^p \equiv 1 \pmod{o(x)}$ so $x^{y^p} = x^{(1+kp^2)^p} = x$ and $y^p \in C$, and we conclude that $\Phi(G)/C$ is abelian of exponent p. It follows that $\Phi(\Phi(G)) \leq C$, completing the case where N is cyclic. (ii) Now let N be noncyclic. Then N_1 must be noncyclic since N has a noncyclic G-invariant subgroup of order p^2, and the last one is contained in $\mathrm{Z}(\Phi(G))$. Then $\Phi(G)/C$ is isomorphic to a subgroup of a nonabelian group of order p^3 (Lemma 2(b)). By Lemma 2(d), however, $\Phi(G)/C$ must be abelian. If $\exp(\Phi(G)/C) = p$, then $\Phi(\Phi(G)) \leq C$, and we are done. Now assume that $\exp(\Phi(G)/C) > p$. Then $p = 2$ and $\Phi(G)/C$ is cyclic of order 4 (Lemma 2(b)). Let $n \in N$ and let $g \in \Phi(G)$ be such that $g^2 \not\in C$. Then $n^g = nn_1$ for some $n_1 \in N_1$ since $N/N_1 \in \mathrm{Z}(\Phi(G)/N_1)$. Next, $n^{g^2} = nn_1^2 = n$ since $N_1 \leq \mathrm{Z}(\Phi(G))$ and $\exp(N_1) = 2$. It follows that g^2 centralizes N so $g^2 \in C$, contrary to the choice of g. Thus, $\exp(\Phi(G)/C) \leq 2$ so $\Phi(\Phi(G)) \leq C$, completing the proof. It follows from what has just been proved that if $M \leq \Phi(\Phi(G))$ is a normal subgroup of order p^4 in a p-group G, then M must be abelian.

It follows from Remark 9 that $e = 2$ in part (b) of the Main Theorem. Indeed, otherwise $\mho^{e-1}(G)$, as a G-invariant subgroup of order 2^4 in $\Phi(\Phi(G))$, must be abelian so regular.

Remark 10. Let $e > 2$ and let G be a p-group of order $p^{p(e+2)-2}$ and exponent p^e. We claim that then $D = \mho^{e-1}(G)$ is regular. Indeed, we have $|G| = p^{(2p-2)+pe}$. If $|D| > p^{(2p-2)+p}$, then D is absolutely regular, by Lemma 4. It remains to show that if $|D| \leq p^{(2p-2)+p} = p^{3p-2}$, then D is also regular. Since $e > 2$, we get $D \leq \Phi(\Phi(G))$. If $p = 2$, then D is of order 2^4 so it is abelian, by Remark 9. Now let $p > 2$. Let $D > \mathrm{K}_2(D) = D' > \mathrm{K}_3(D) > \dots$ be the lower central series of D. If $|D/\mathrm{K}_2(D)| < p^4$, then $|D| < p^4$ so D is abelian, by Remark 9. Therefore, one may assume that $|\mathrm{K}_2(D)| \leq \frac{|D|}{p^4} \leq p^{3p-6}$. It follows from Remark 9 that the length of the lower central series of D is at most $1 + \frac{3p-6}{3} = p - 1$, so D is regular, by Lemma 2(c). In particular, if G is a group of order p^m, $m = p(e+2) - 2$, and exponent $p^e > p$ and $\mho^{e-1}(G)$ is irregular, then $e = 2$. This supplements part (b) of the theorem.

B. Wilkens [W] has showed that if $p > 2$, then, for given numbers $e > 1$ and $n \in \mathbb{N}$, there exists a finite group G of exponent p^e such that $\mho^n(G) > \{1\}$, thereby solving Question 12 in [B4].

Question 1. Let G be a p-group. Suppose that $N \leq \Phi(G)$ is an irregular G-invariant subgroup of order p^{2p} if $p > 2$ and 2^5 if $p = 2$. Describe the structure of N. (If $p > 2$ and N exists, then indices of its lower central series are $p^3, p^2, p^2, \dots, p^2, p$.)

Question 2. Suppose that a p-group G is such that $|\mho_{i-1}(G) : \mho_1(\mho_{i-1}(G))| = p^p$ for some $i \in \mathbb{N}$. Does there exist a constant $c = c(p)$ such that $|\mho_{i-1}(G) : \mho_i(G)| \leq p^c$?

As Mann noticed, if, in Question 2, $|\mho_{i-1}(G) : \mho_1(\mho_{i-1}(G))| < p^p$, then $c < p$. Indeed, one may assume, without loss of generality, that $\mho_i(G) = \{1\}$ so $\exp(G) = p^i$. In the case under consideration, $\mho_{i-1}(G)$, which is generated by elements of order p, is absolutely regular, by hypothesis. It follows that $|\mho_{i-1}(G)| < p^p$, proving our claim.

QUESTION 3. Let a p-group $G = A \operatorname{wr} B$ be a standard wreath product. Find $\mho^n(G)$ in terms of A, B and n.

QUESTION 4. Let $G = \mathrm{UT}(n,p) \in \mathrm{Syl}_p(\mathrm{GL}(n,p))$. (i) Find the maximal $a = a(n)$ such that $\mho^a(G) > \{1\}$. (ii) Find the minimal $b = b(n)$ such that $\mho^b(G)$ is regular.

QUESTION 5. Let $e > 2$. Does there exist $c = c(e)$ such that $\exp(\mho^{c(e)}(G)) < p^e$ for all groups G of exponent p^e? (According to Wilkens [W], $c(2)$ does not exist if $p > 2$.)

QUESTION 6. Given $e > 1$ and $n \in \mathbb{N}$, does there exist a p-group G of exponent p^e such that $\Omega_e^*(G) = \langle x \in G \mid o(x) = p^e \rangle \leq \mho_n(G)$?

QUESTION 7. Let G be a p-group, $p > 2$, and let $N \leq \Phi(G)$ be a G-invariant subgroup of exponent p and order p^n. Suppose that the class of N equals $1 + [\frac{n-3}{2}] + \epsilon$, where $\epsilon = 0$ if n is odd and $\epsilon = 1$ if n is even. Describe the structure of N.

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Y. Berkovich
Department of Mathematics
University of Haifa
Haifa 31905
Israel
E-mail: berkov@mathcs2.haifa.ac.il

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