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Universal Hyperbolic Geometry II: A pictorial overview

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ABSTRACT

This article provides a simple pictorial introduction to universal hyperbolic geometry. We explain how to understand the subject using only elementary projective geometry, augmented by a distinguished circle. This provides a completely algebraic framework for hyperbolic geometry, valid over the rational numbers (and indeed any field not of characteristic two), and gives us many new and beautiful theorems. These results are accurately illustrated with colour diagrams, and the reader is invited to check them with ruler constructions and measurements.

Key words: hyperbolic geometry, projective geometry, rational trigonometry, quadrance, spread, quadrea

MSC 2010: 51M10, 14N99, 51E99

Univerzalna hiperbolička geometrija II: slikovni pregled

SAŽETAK

Članak pruža jednostavan slikovni uvod u univerzalnu hiperboličku geometriju. Objašnjava se kako razumjeti sadržaj koristeći samo osnovnu projektivnu geometriju, proširenu jednom istaknutom kružnicom. Na taj se način dobiva potpuno algebarski okvir za hiperboličku geometriju, koji vrijedi nad poljem racionalnih brojeva (i u biti nad bilo kojim poljem karakteristike različite od 2) i daje mnoge nove lijepe teoreme. Ovi su rezultati prikazani crtežima u boji, a čitatelj je pozvan provjeriti ih konstruktivno i računski.

Ključne riječi: hiperbolička geometrija, projektivna geometrija, racionalna geometrija, kvadranca, širina, kvadrea

1 Introduction

This paper introduces *hyperbolic geometry* using only elementary mathematics, without any analysis, and in particular without transcendental functions. Classical hyperbolic geometry, (see for example [6], [7], [8], [9], [10], [11], [15], [17], [23]), is usually an advanced topic studied in the senior years of a university mathematics program, often built up from a foundation of differential geometry. In recent years, a new, simpler and completely algebraic understanding of this subject has emerged, building on the ideas of *rational trigonometry* ([18] and [19]). This approach is called *universal hyperbolic geometry*, because it extends the theory to more general settings, namely to arbitrary fields (usually characteristic not two), and because it generalizes to other quadratic forms (see [21]).

The basic reference is *Universal Hyperbolic Geometry I: Trigonometry* ([22]), which contains accurate definitions, many formulas and complete proofs, but no diagrams. This paper complements that one, providing a pictorial introduction to the subject with a minimum of formulas and no proofs, essentially relying only on planar projective geometry. The reader is encouraged to verify theorems by making explicit constructions and measurements; aside from a single base (null) circle, with only a ruler one can check most of the assertions of this paper in special cases. Alternatively a modern geometry program such as The Geometer's Sketchpad, C.A.R., Cabri, GeoAlgebra or Cinderella illustrates the subject with a little effort.

Our approach extends the classical *Cayley Beltrami Klein*, or *projective*, model of hyperbolic geometry, whose underlying space is the interior of a disk, with lines being straight line segments. In our formulation we consider also the boundary of the disk, which we call the *null circle*, also points outside the disk, and also points at infinity. The lines are now complete lines in the sense of projective geometry, not segments, and include also *null lines* which are tangent to the null circle, and lines which do not meet the null circle, including the line at infinity. This orientation is familiar to classical geometers (see for example [2], [3], [4], [5], [14]), but it is not well-known to students because of the current dominance of the differential geometric point of view. A novel aspect of this paper is that we introduce our metrical concepts—*quadrance*, *spread* and *quadrea*—

purely in projective terms. It means that only high school algebra suffices to set up the subject and make computations. The proofs however rely generally on computer calculations involving polynomial or rational function identities; some may be found in ([22]), others will appear elsewhere.

Many of the results are illustrated with two diagrams, one illustrating the situation in the classical setting using interior points of the null circle, and another with more general points. The fundamental metrical notions of *quadrance* between points and *spread* between lines are undefined when null points or null lines are involved, but most theorems involving them apply equally to points and lines interior or exterior of the null circle. The reader should be aware that in more advanced work the distinction between these two types of points and lines also becomes significant. Instead of area of hyperbolic triangles, we work with a rational analog called *quadrea*.

With this algebraic approach we can develop geometry over the *rational numbers*—in my view, always the most important field. The natural connections between geometry and number theory are then not suppressed, but enrich both subjects. Later in the series we will also illustrate hyperbolic geometry over *finite fields*, in the direction of [1] and [16], where *counting* becomes important.

2 The projective plane

Hyperbolic geometry may be visualized as the geometry of the projective plane, augmented by a distinguished circle c (in fact a more general conic may also be used). Since projective geometry is not these days as familiar as it was in former times, we begin by reviewing some of the basic notions. The starting point is the affine plane—familiar from Euclidean geometry and Cartesian coordinate geometry—containing the usual points $[x,y]$ and (straight) lines with equations $ax+by=c$. The affine plane is augmented by introducing a *new point* for every family of parallel lines. In this introductory section we use *parallel* in the usual sense of Euclidean geometry, so that the lines with equations $a_1x+b_1y=c_1$ and $a_2x+b_2y=c_2$ are parallel precisely when $a_1b_2-a_2b_1=0$. Later on we will see that there is a different, hyperbolic meaning of ‘parallel’ (which is different from the usage in classical hyperbolic geometry!) The new point, one for each family of parallel lines, is a ‘*point at infinity*’. We also introduce one *new line*, the ‘*line at infinity*’, which passes through every point at infinity.

Algebraically the projective plane may be defined without reference to the Euclidean plane, with points specified by homogeneous coordinates, or proportions, of the form $[x:y:z]$. Points $[x:y:1]$ correspond to the affine plane, and points at infinity are of the form $[x:y:0]$. The lines also are specified by homogeneous coordinates, now of the form $(a:b:c)$, with the pairing between the point $[x:y:z]$ and the line $(a:b:c)$ given by

$$ax+by-cz=0. \qquad (1)$$

This particular relation is the characterizing equation for hyperbolic geometry; for spherical/elliptic geometry a different convention between points and lines is used, where the line $\langle a:b:c\rangle$ passes through the point $[x:y:z]$ precisely when $ax+by+cz=0$. Note that we use round brackets for lines in hyperbolic geometry.

We will visualize the projective plane as an extension of the affine plane, with the usual property that any two distinct points a and b determine exactly one line which passes through them both, called the **join** of a and b, and denoted by ab, and with the *new* property that any two lines L and M determine exactly one point which lies on them both, called the **meet** of L and M, and denoted by LM.

In projective geometry, the notion of parallel lines disappears, since now *any* two lines meet. Familiar measurements, such as distance and angle, are also absent. It is really the *geometry of the straightedge*.

Despite its historical importance, intrinsic beauty and simplicity, projective geometry is these days sadly neglected in the school and university curriculum. Perhaps the wider realization that it actually underpins hyperbolic geometry will lead to a renaissance of the subject! Most readers will know the two basic theorems in the subject, which are illustrated in Figure 1.

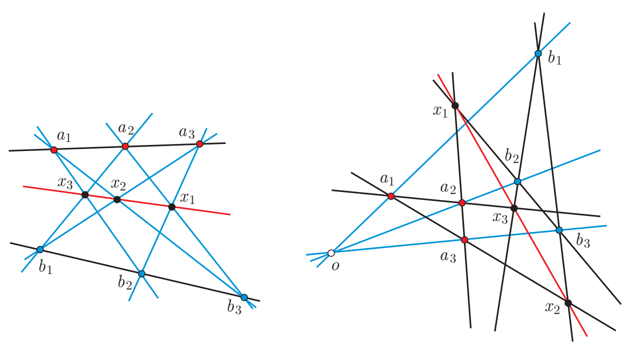

Figure 1: *Theorems of Pappus and Desargues*

Pappus' theorem asserts that if a_1, a_2 and a_3 are collinear, and b_1, b_2 and b_3 are collinear, then $x_1 \equiv (a_2b_3)(a_3b_2)$, $x_2 \equiv (a_3b_1)(a_1b_3)$ and $x_3 \equiv (a_1b_2)(a_2b_1)$ are collinear. *Desargues' theorem* asserts that if a_1b_1, a_2b_2 and a_3b_3 are concurrent, then $x_1 \equiv (a_2a_3)(b_2b_3)$, $x_2 \equiv (a_3a_1)(b_3b_1)$ and $x_3 \equiv (a_1a_2)(b_1b_2)$ are collinear. This is often stated in the form that if two triangles are perspective from a point, then they are also perspective from a line.

A further important notion concerns four collinear points a, b, c and d on a line L, in any order. Suppose we choose affine coordinates on L so that the coordinates of a,b,c and d are respectively x,y,z and w. Then the **cross-ratio** is defined to be the extended number (possibly ∞) given by the ratio of ratios:

$$(a,b:c,d) \equiv \left(\frac{a-c}{b-c}\right) / \left(\frac{a-d}{b-d}\right).$$

This is independent of the choice of affine coordinates on L.

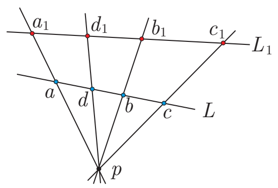

Figure 2: *Projective invariance of cross-ratio:* $(a,b:c,d) = (a_1,b_1:c_1,d_1)$

Moreover it is also projectively invariant, meaning that if a_1,b_1,c_1 and d_1 are also collinear points on a line L_1 which are perspective to a,b,c and d from some point p, as in Figure 2, then $(a,b:c,d) = (a_1,b_1:c_1,d_1)$.

The cross ratio is the most important invariant in projective geometry, and will be the basis of *quadrance* between points and *spread* between lines, but we give also analytic expressions for these quantities in homogeneous coordinates; both are *rational functions* of the inputs. The actual values assumed by the cross-ratio depend ultimately on the field over which our geometry is based, which is in principle quite arbitrary. To start it helps to restrict our attention to the *rational numbers*, invariably the most natural, familiar and important field. So we will adopt a scientific approach, identifying our sheet of paper with (part of) the *rational number plane*.

Here are a few more basic definitions. A **side** $\overline{a_1a_2} = \{a_1,a_2\}$ is a set of two points. A **vertex** $\overline{L_1L_2} = \{L_1,L_2\}$ is a set of two lines. A **couple** $\overline{aL} = \{a,L\}$ is a set consisting of a point a and a line L. A **triangle** $\overline{a_1a_2a_3} = \{a_1,a_2,a_3\}$ is a set of three points which are not collinear. A **trilateral** $\overline{L_1L_2L_3} = \{L_1,L_2,L_3\}$ is a set of three lines which are not concurrent. Every triangle $\overline{a_1a_2a_3}$ has three sides, namely $\overline{a_1a_2}$, $\overline{a_2a_3}$ and $\overline{a_1a_3}$, and similarly any trilateral $\overline{L_1L_2L_3}$ has three vertices, namely $\overline{L_1L_2}$, $\overline{L_2L_3}$ and $\overline{L_1L_3}$.

Since the points of a triangle and the lines of a trilateral are distinct, any triangle $\overline{a_1a_2a_3}$ determines an **associated trilateral** $\overline{L_1L_2L_3}$ where $L_1 \equiv a_2a_3$, $L_2 \equiv a_1a_3$ and $L_3 \equiv a_1a_2$. Conversely any trilateral $\overline{L_1L_2L_3}$ determines an **associated triangle** $\overline{a_1a_2a_3}$ where $a_1 \equiv L_2L_3$, $a_2 \equiv L_1L_3$ and $a_3 \equiv L_1L_2$.

3 Duality via polarity

We are now ready to introduce the **hyperbolic plane**, which is just the projective plane which we have just been describing, augmented by a distinguished Euclidean circle c in this plane, called the **null circle,** which appears in our diagrams always in blue. The points lying on c have a distinguished role, and are called **null points**. The lines tangent to c have a distinguished role, and are called **null lines**.

All other points and lines, including the points at infinity and the line at infinity, are for the purposes of elementary universal hyperbolic geometry treated in a non-preferential manner. In particular we do *not* restrict our attention to only **interior points** lying inside the circle c; this is a big difference with classical hyperbolic geometry; **exterior points** lying outside the circle are equally important. Similarly we do *not* restrict our attention to only **interior lines** which meet c in two points; **exterior lines** which do not meet c are equally important. Note also that these notions can be defined purely projectively once the null circle c has been specified: interior points do not lie on null lines, while exterior points do, and interior lines pass through null points, while exterior lines do not.

For those who prefer to work with coordinates, we may choose our circle to have homogeneous equation $x^2+y^2-z^2=0$, or in the plane $z=1$ with coordinates $X \equiv x/z$ and $Y \equiv y/z$, simply the unit circle $X^2+Y^2=1$.

The presence of the distinguished null circle c has as its main consequence a *complete duality* between points and lines of the projective plane, in the sense that every point a has associated to it a particular line $a^{\perp}$ and conversely. This duality is one of the ways in which universal hyperbolic geometry is very different from classical hyperbolic geometry, and it arises from a standard construction in projective geometry involving the distinguished null circle c—the notion of *polarity*. How polarity defines duality is central to the subject.

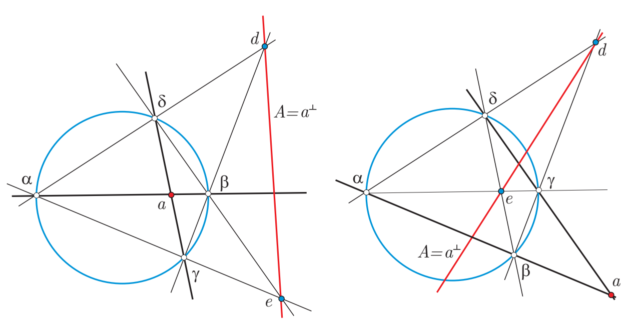

Figure 3: *Duality and pole-polar pairs*

To define the polar of a point a with respect to the null circle c, draw any two lines through a which each both pass through c in two distinct points, say α,β and γ,δ respectively; this is always possible. Now here is a beautiful fact from projective geometry: *if d and e are the other two diagonal points of the quadrilateral $\alpha\beta\gamma\delta$, then the line de does not depend on the two chosen lines through a, but only on a itself.* So we say that $de = A \equiv a^{\perp}$ is the **dual line** of the point a, and conversely $a = A^{\perp}$ is the **dual point** of the line A.

The picture is as in Figure 3, showing two different possible configurations for which the above prescriptions both hold. In case a is external to the circle, it is also possible to construct $A \equiv a^{\perp}$ from the tangents to the null circle c passing through a as in Figure 4, but this does not work for an interior point such as b.

The construction shows that there is in fact a symmetry between the initial point a and the diagonal points d and e, so that if we started with the point d, its dual would be the line ae, and if we started with the point e, its dual would be the point ad. So this shows another fundamental fact: *if d lies on the dual $a^{\perp}$ of the point a, then a lies on the dual $d^{\perp}$ of the point d.*

So to invert the construction, given a line A, choose two points d and e on it, find the dual lines $d^{\perp}$ and $e^{\perp}$, and define $a \equiv A^{\perp} = d^{\perp}e^{\perp}$. It is at this point that we need the projective plane, with its points and line at infinity, for if $d^{\perp}$ and $e^{\perp}$ were Euclidean parallel, then $d^{\perp}e^{\perp}$ would be a point at infinity. This situation occurs if we take our line A to be a diameter of the null circle c, in the sense of Euclidean geometry.

What happens when the point a is null? In that case the quadrilateral in the above construction degenerates, and the polar line $a^{\perp}$ is the null line tangent to the null circle at a. So every null line is dual to the unique null point which lies on it, as shown also in Figure 4.

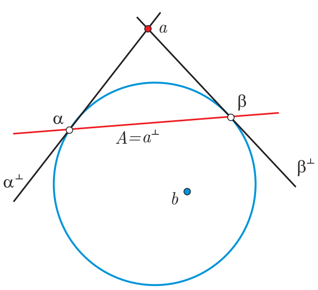

Figure 4: *Null points α,β and their dual null lines*

The duality between points and lines is surprisingly simple to describe in terms of homogeneous coordinates. The pole of the point $a \equiv [x:y:z]$ is just the line $a^{\perp} \equiv (x:y:z)$, so duality amounts to simply changing from one kind of brackets to the other! This explains why we chose the hyperbolic form of the pairing (1).

In the previous section we saw that every triangle $\overline{a_1a_2a_3}$ has an associated trilateral $\overline{L_1L_2L_3}$ and conversely. Now we see that there is another natural trilateral associated to $\overline{a_1a_2a_3}$, namely the **dual trilateral** $\overline{a_1^{\perp}a_2^{\perp}a_3^{\perp}}$. Conversely to any trilateral $\overline{L_1L_2L_3}$ there is associated the **dual triangle** $\overline{L_1^{\perp}L_2^{\perp}L_3^{\perp}}$.

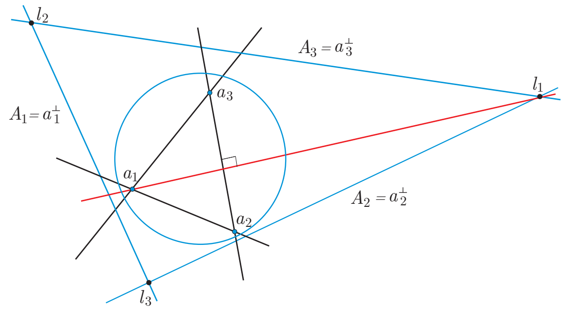

Figure 5: *A triangle $\overline{a_1a_2a_3}$ and its dual trilateral $\overline{A_1A_2A_3}$*

We also say that a couple $\overline{aL}$ is **dual** precisely when a and L are dual, and is otherwise **non-dual**.

So to summarize: *in (planar) universal hyperbolic geometry, duality implies that points and lines are treated completely symmetrically*. This is a significant departure both from Euclidean geometry and from classical hyperbolic geometry—in both of those theories, points and lines play quite different roles. In universal hyperbolic geometry, the above duality principle implies that every theorem can be dualized to create a (possibly) new theorem. We will state many theorems together with their duals, but to keep the length of this paper reasonable we will not extend this to all the theorems; the reader is encouraged to find statements and draw pictures of the dual results in these other cases.

4 Perpendicularity

The notions of perpendicular and parallel differ dramatically between Euclidean and hyperbolic geometries. In the affine geometry on which Euclidean geometry is based, the notion of *parallel lines* is fundamental, while *perpendicular lines* are determined by a quadratic form. In hyperbolic geometry, the situation is reversed—perpendicularity is more fundamental, and in fact parallelism is defined in terms of it!

Another novel feature is that perpendicularity applies not only to lines, but *also to points*. This is a consequence of the fundamental duality we have already established between points and lines.

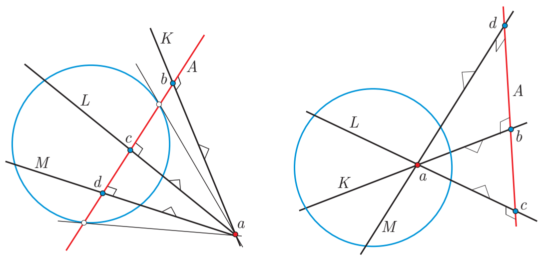

Figure 6: *Perpendicular points and lines*

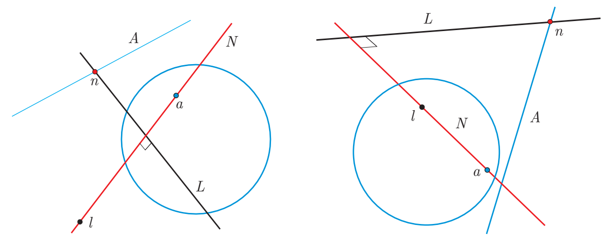

Figure 7: *Altitude line N and altitude point n of a couple $\overline{aL}$*

Perpendicularity in the hyperbolic setting is easy to introduce once we have duality. We say that a point b is **perpendicular** to a point a precisely when b lies on the dual line $a^{\perp}$ of a. This is equivalent to a lying on the dual line $b^{\perp}$ of b, so the relation is symmetric, and we write

$$a \perp b.$$

Dually a line L is perpendicular to a line M precisely when L passes through the dual point $M^{\perp}$ of M. This is equivalent to M passing through the dual point $L^{\perp}$ of L, and we write

$$L \perp M.$$

Figure 6 shows our pictorial conventions for perpendicularity: the line A is dual to the point a, so points b,c and d lying on A are perpendicular to a, and this is recorded by a small (right) corner placed on the join of the perpendicular points, and between them. Also the lines K,L and M pass through a, so are perpendicular to A, and this is recorded as usual by a small parallelogram at the meet of the perpendicular lines.

Our first theorem records two basic facts that are obvious from the definitions so far.

Theorem 1 (Altitude line and point) *For any non-dual couple $\overline{aL}$, there is a unique line N which passes through a and is perpendicular to L, namely $N \equiv aL^{\perp}$, and there is a unique point n which lies on L and is perpendicular to a, namely $n \equiv a^{\perp}L$. Furthermore N and n are dual.*

We call N the **altitude line** to L through a, and n the **altitude point** to a on L. In case a and L are dual, any line through a is perpendicular to L, and any point lying on L is perpendicular to a. While the former idea is familiar, the latter is not. Figure 7 shows that if we restrict ourselves to the inside of the null circle, altitude points are invisible, so it is no surprise that the concept is missing from classical hyperbolic geometry. Remember that we are obliged to respect the balance which duality provides us!

If a triangle $\overline{a_1a_2a_3}$ has say a_1 dual to a_2a_3, then any line through a_1 will be perpendicular to the opposite line a_2a_3, and we say the triangle is **dual**. A triangle is **non-dual** precisely when each of its points is not dual to the opposite line. Similar definitions apply to trilaterals.

Somewhat surprisingly, the next result is *not true* in classical hyperbolic geometry—a very conspicuous absence that too often goes unmentioned in books on the subject! The reason is that the orthocenter of a triangle of interior points might well be an exterior point, as the left diagram in Figure 8 shows. The absence of a distinguished orthocenter partially explains why the study of triangles in classical hyperbolic geometry is relatively undeveloped. With universal hyperbolic geometry, triangle geometry enters a rich new phase.

Theorem 2 (Orthocenter and ortholine) *The altitude lines of a non-dual triangle meet at a unique point o, called the **orthocenter** of the triangle. The altitude points of a non-dual trilateral join along a unique line O, called the **ortholine** of the trilateral. The ortholine O of the dual trilateral of a triangle is dual to the orthocenter o of the triangle.*

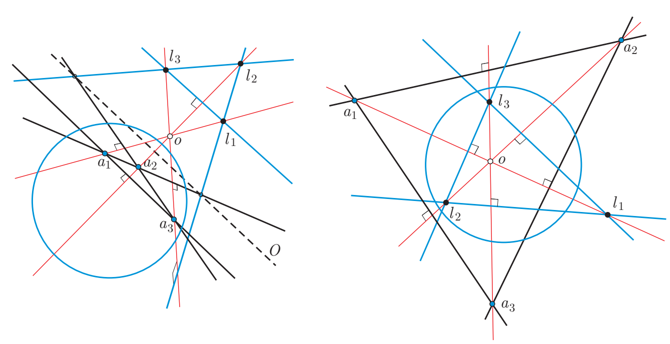

Figure 8: *Orthocenter o and ortholine O of a triangle $\overline{a_1a_2a_3}$ and its dual trilateral*

Figure 8 shows a triangle $\overline{a_1a_2a_3}$, its dual trilateral with associated triangle $\overline{l_1l_2l_3}$, and the corresponding orthocenter o and ortholine O.

5 Null points and lines

In modern treatments of hyperbolic geometry, points at infinity have a somewhat ambiguous role, and what we call null lines are rarely discussed, because they are essentially invisible in the Beltrami Poincaré models built from differential geometry. However earlier generations of classical geometers were well aware of them (see for example [14]).

In universal hyperbolic geometry null points and null lines play a particularly interesting and important role. Here is a first example, whose name comes from the definition that a triangle is **triply nil** precisely when each of its points is null.

Theorem 3 (Triply nil altitudes) *Suppose that α_1,α_2 and α_3 are distinct null points, with b any point lying on $\alpha_1\alpha_2$. Then the altitude lines to $\alpha_1\alpha_3$ and $\alpha_2\alpha_3$ through b are perpendicular.*

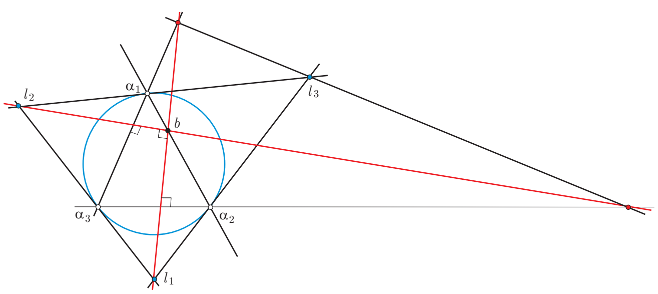

Figure 9: *The Triply nil altitudes theorem: $bl_1 \perp bl_2$*

Figure 9 shows that the Triply nil altitudes theorem may be recast in projective terms: if l_1, l_2 and l_3 are the poles of the lines of the triangle $\overline{\alpha_1\alpha_2\alpha_3}$ with respect to the conic c, then the points $(\alpha_1\alpha_3)(bl_1) = (bl_2)^{\perp}$, $(\alpha_2\alpha_3)(bl_2) = (bl_1)^{\perp}$ and l_3 are collinear.

In keeping with triangles and trilaterals, a (cyclic) set of four points is called a **quadrangle**, and a (cyclic) set of four lines a **quadrilateral**. The next result restates some facts that we already know about polarity of cyclic quadrilaterals in terms of perpendicularity.

Theorem 4 (Nil quadrangle diagonal) *Suppose that $\alpha_1,\alpha_2,\alpha_3$ and α_4 are distinct null points, with diagonal points $e \equiv (\alpha_1\alpha_2)(\alpha_3\alpha_4)$, $f \equiv (\alpha_1\alpha_3)(\alpha_2\alpha_4)$ and $g \equiv (\alpha_1\alpha_4)(\alpha_2\alpha_3)$. Then the lines ef, eg and fg are mutually perpendicular, and the points e, f and g are also mutually perpendicular.*

In classical hyperbolic geometry the null points $\alpha_1,\alpha_2,\alpha_3$ and α_4 would be considered to be 'at infinity', while the external diagonal points f and g in Figure 10 would be invisible. Let us repeat: *for us internal and external points are equally interesting.*

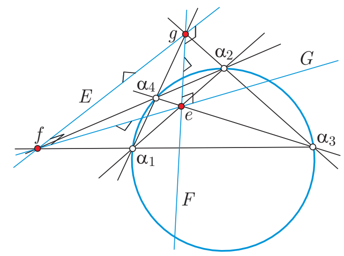

Figure 10: *Triply right diagonal triangle $\overline{efg}$*

6 Couples, parallels and bases

Points and lines are the basic objects in planar hyperbolic geometry. Given one point a, we may construct its dual line $A \equiv a^{\perp}$, and conversely given a line A we may construct its dual point $a \equiv A^{\perp}$. After that, there is in general nothing more to construct. For two objects, namely couples, sides and vertices, the situation is considerably more interesting, and gives us a chance to introduce some important additional concepts.

Given a non-dual couple $\overline{aL}$ we know we can construct the dual line $A \equiv a^{\perp}$ and the dual point $l \equiv L^{\perp}$, and the altitude line N and the altitude point n. Now we introduce another major point of departure from classical hyperbolic geometry, which provides an ironic twist to the oft-repeated history of hyperbolic geometry as a development arising from Euclid's Parallel Postulate.

Theorem 5 (Parallel line and point) *For any non-dual couple $\overline{aL}$ there is a unique line P which passes through a and is perpendicular to the altitude line N of $\overline{aL}$, namely $P \equiv a\left(a^{\perp}L\right)$, and there is a unique point p which lies on $a^{\perp}$ and is perpendicular to the altitude point n of $\overline{aL}$, namely $p \equiv a^{\perp}\left(aL^{\perp}\right)$. Furthermore P and p are dual.*

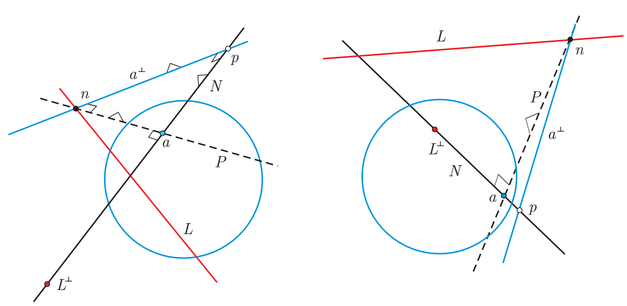

Figure 11: *Parallel line P and parallel point p of the couple $\overline{aL}$*

The line P is the **parallel line** of the couple $\overline{aL}$, and the point p is the **parallel point** of the couple $\overline{aL}$. These are shown in Figure 11. We may also refer to P as the **parallel line to the line** L **through** a. This is how we will henceforth use the term *parallel* in universal hyperbolic geometry—we do not say that *two lines are parallel*.

Theorem 6 (Base point and line) *For any non-dual couple $\overline{aL}$ there is a unique point b which lies on both L and the altitude line N of $\overline{aL}$, namely $b \equiv \left(aL^{\perp}\right)L$, and there is a unique line B which passes through both $L^{\perp}$ and the altitude point n of $\overline{aL}$, namely $B \equiv \left(a^{\perp}L\right)L^{\perp}$. Furthermore B and b are dual.*

The point b is the **base point** of the couple $\overline{aL}$, and the line B is the **base line** of the couple $\overline{aL}$. These are shown in Figure 12.

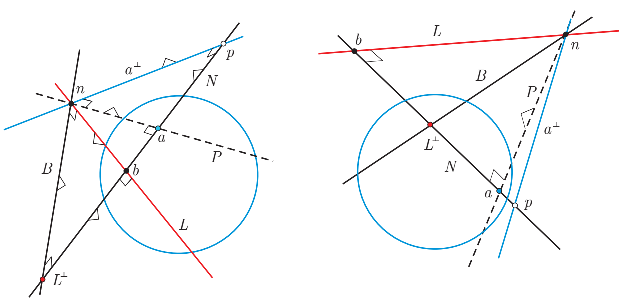

Figure 12: *Base point b and base line B of the couple $\overline{aL}$*

7 Sides, vertices and conjugates

There are also constructions for sides and vertices, and in fact the complete pictures associated with the general couple, side and vertex are essentially the same, but with different objects playing different roles.

Figure 13 shows the constructions possible if we start with a side $\overline{a_1a_2}$. Note that in this example one of these points is internal and one is external. A bit more terminology: we say that the side $\overline{a_1a_2}$ is **null** precisely when a_1a_2 is a null line, and **nil** precisely when at least one of a_1, a_2 is a null point. The vertex $\overline{L_1L_2}$ is **null** precisely when L_1L_2 is a null point, and **nil** precisely when at least one of L_1, L_2 is a null line.

Theorem 7 (Side conjugate points and lines) *For any side $\overline{a_1a_2}$ which is not both nil and null, there is a unique point $b_1 \equiv (a_1a_2)\,a_1^{\perp}$ which lies on a_1a_2 and is perpendicular to a_1, and there is a unique point $b_2 \equiv (a_1a_2)\,a_1^{\perp}$ which lies on a_1a_2 and is perpendicular to a_2.*

The points b_1 and b_2 are the **conjugate points** of the side $\overline{a_1a_2}$. The duals of these points are the lines $B_1 \equiv a_1\,(a_1a_2)^{\perp}$ and $B_2 \equiv a_2\,(a_1a_2)^{\perp}$, which are the **conjugate lines** of the side $\overline{a_1a_2}$. These relations are involutory: if the side $\overline{b_1b_2}$ is conjugate to the side $\overline{a_1a_2}$, then $\overline{a_1a_2}$ is also conjugate to $\overline{b_1b_2}$.

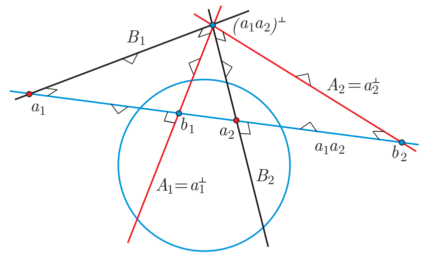

Figure 13: *Conjugate points and conjugate lines of the side $\overline{a_1a_2}$*

The same picture can also be reinterpreted by starting with a vertex.

Theorem 8 (Vertex conjugate points and lines) *For any vertex $\overline{A_1A_2}$ which is not both nil and null, there is a unique line $B_1 \equiv (A_1A_2)\,A_1^{\perp}$ which passes through A_1A_2 and is perpendicular to A_1, and there is a unique line $B_2 \equiv (A_1A_2)\,A_1^{\perp}$ which passes through A_1A_2 and is perpendicular to A_2.*

The lines B_1 and B_2 are the **conjugate lines** of the vertex $\overline{A_1A_2}$. The duals of these lines are the points $b_1 \equiv A_1\,(A_1A_2)^{\perp}$ and $b_2 \equiv A_2\,(A_1A_2)^{\perp}$, which are the **conjugate points** of the vertex $\overline{A_1A_2}$. This relation is also involutory: if the vertex $\overline{B_1B_2}$ is conjugate to the vertex $\overline{A_1A_2}$, then $\overline{A_1A_2}$ is also conjugate to $\overline{B_1B_2}$. This is shown also in Figure 14, which is essentially the same as Figures 12 and 13.

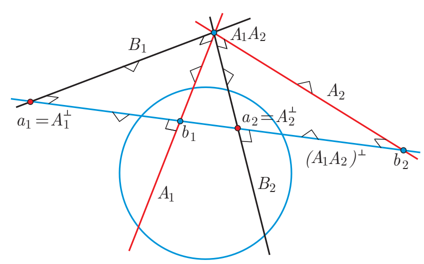

Figure 14: *Conjugate points and conjugate lines of the vertex $\overline{A_1A_2}$*

Furthermore the exact same configuration results if we had started with one of the sides $\overline{a_1b_2}$, or $\overline{a_2b_1}$, or $\overline{b_1b_2}$, or with one of the vertices $\overline{A_1B_2}$, or $\overline{A_2B_1}$, or $\overline{B_1B_2}$. However if we had started with a **right side**, meaning that the two points are perpendicular, such as $\overline{a_1 b_1}$, then we would obtain $L = a_1 b_1$ and its dual point l. But then the conjugate side of $\overline{a_1 b_1}$ would coincide with $\overline{a_1 b_1}$, and similarly the conjugate vertex of a right vertex such as $\overline{A_1 B_1}$ would coincide with $\overline{A_1 B_1}$.

8 Reflections

The basic symmetries of hyperbolic geometry are *reflections,* but they have a somewhat different character from Euclidean reflections. Hyperbolic reflections send points to points and lines to lines, preserving incidence, in other words they are *projective transformations*. There are two seemingly different notions, the reflection σ_a in a (non-null) point a, and the reflection σ_L in a (non-null) line L. It is an important fact that these two notions end up agreeing, in the sense that

$$\sigma_a = \sigma_A$$

when $A = a^\perp$.

The transformation σ_a is defined first by its action on null points, and then by its action on more general points and lines.

For a non-null point a, the reflection σ_a sends a null point α to the other null point α' on the line αa. We write

$$\alpha' = \alpha\sigma_a.$$

In case $a\alpha$ is a null line, in other words a tangent to the null circle c, then $\alpha' \equiv \alpha$. Note that if a was itself a null point, then this definition would yield a transformation that would send every null point to a, which will not be a symmetry in the sense we wish. However such transformations can still be useful.

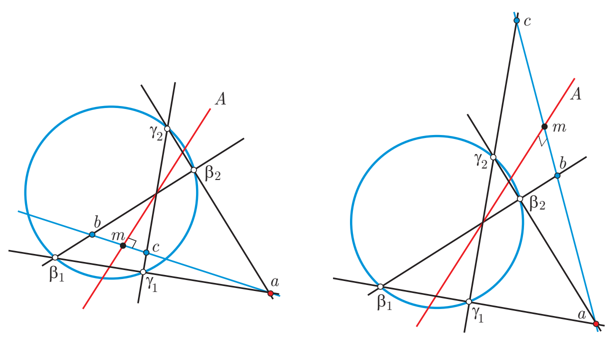

Figure 15: *Reflection in the point a or the dual line $A \equiv a^\perp$*

Once the action of a projective transformation on null points is known, it is determined on all points and lines, first of all on lines through two null points, and then on an arbitrary point by means of two such lines passing through it, and then on arbitrary lines. Figure 15 shows the reflection σ_a and its action on a point b to get $c \equiv b\sigma_a$. First find a line through b meeting the null circle at points β_1 and β_2, then construct $\gamma_1 \equiv \beta_1\sigma_a$ and $\gamma_1 \equiv \beta_1\sigma_a$, and then set

$$c \equiv b\sigma_a \equiv (ab)(\gamma_1\gamma_2).$$

For the reflection σ_L in a line L, the idea is dual to the above. It is defined first by its action on null lines, and then by its action on more general points and lines. For a non-null line L, the reflection σ_L sends a null line Π to the other null line Π' passing through the point $L\Pi$. We write

$$\Pi' = \Pi\sigma_L.$$

In case $L\Pi$ is a null point, $\Pi' \equiv \Pi$.

Once the action of a projective transformation on null lines is known, it is determined on all points and lines, since it is first of all determined on points lying on two null lines, and then on an arbitrary line by means of two such points lying on it, and then on arbitrary points.

Of course there is also a linear algebra/matrix approach to defining reflections. If $a = [u : v : w]$ then the action of $\sigma_a = \sigma_{a^\perp}$ on a point $b \equiv [x : y : z]$ is given by the projective matrix product

$$b\sigma_a = [x : y : z]\begin{bmatrix} u^2 - v^2 + w^2 & 2uv & 2uw \\ 2uv & -u^2 + v^2 + w^2 & 2vw \\ -2uw & -2vw & u^2 + v^2 - w^2 \end{bmatrix}$$

where the entries are only determined up to a scalar. The move to three dimensions simplifies the discussion.

9 Midpoints, midlines, bilines and bipoints

The notion of the *midpoint* of a side can be defined once we have the notion of a reflection. It also has a metrical formulation in terms of quadrance, which we have not yet introduced. There are three other closely related concepts, that of *midline*, *biline* and *bipoint*. Midpoints and midlines refer to sides, while bilines and bipoints refer to vertices. The existence of these objects reduces to questions in number theory—-whether or not certain quadratic equations have solutions.

If $c = b\sigma_d$ then we say d is a **midpoint** of the side $\overline{bc}$. In this case the point $e \equiv d^\perp(bc)$ is also a midpoint of $\overline{bc}$, and the two midpoints d and e of $\overline{bc}$ are perpendicular. The dual line $D \equiv d^\perp$ is a **midline** of $\overline{bc}$, meaning that it meets bc perpendicularly in a midpoint, namely e. The reflection σ_D in D of b is also c. Similarly the dual line $E \equiv e^\perp$ is also a midline of $\overline{bc}$. In Euclidean geometry midlines are called *perpendicular bisectors*. In hyperbolic geometry there are generally either zero or two midpoints between any two points, and so also zero or two midlines.

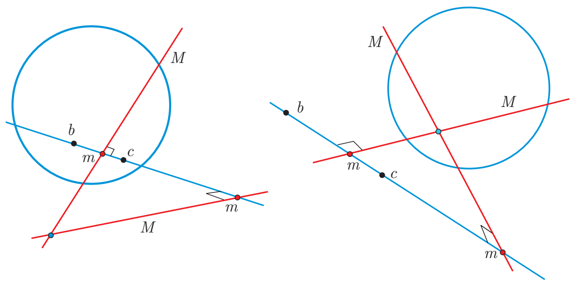

Figure 16: *Midpoints and midlines of $\overline{bc}$*

Figure 16 shows the midpoints m and midlines M of a side $\overline{bc}$. Figure 17 shows how to construct the midpoints and midlines of the side $\overline{bc}$ when such exist. For the midpoints of $\overline{bc}$ we first join b and c to the point $a \equiv (bc)^{\perp}$ to form lines M and N. If these are both interior lines, then their meets with the null circle give a completely nil quadrangle one of whose diagonal points is a, and the other two diagonal points d and e lie on bc and are the required midpoints. The duals D and E of d and e respectively are the midlines of $\overline{bc}$.

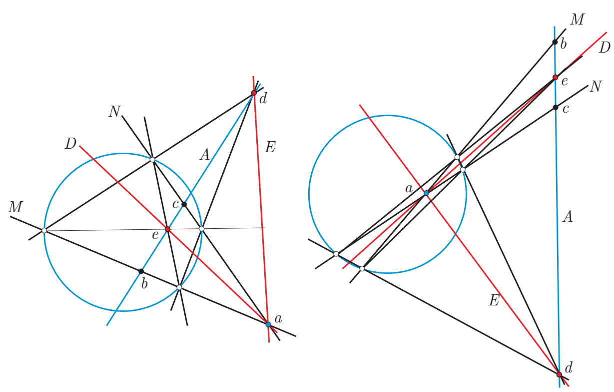

Figure 17: *Midpoints d and e of $\overline{bc}$, or bilines D and E of $\overline{MN}$*

The dual notions to midpoints and midlines of sides are the notions of bilines and bipoints of vertices. If M and N are lines and $M = N\sigma_D$ then the line D is a **biline** of the vertex $\overline{MN}$. In this case the line $E \equiv D^{\perp}(MN)$ is also a biline of $\overline{MN}$, and the two bilines D and E of $\overline{MN}$ are perpendicular.

The dual point $d \equiv D^{\perp}$ is a **bipoint** of $\overline{MN}$, and it joins MN perpendicularly in a biline, namely E. This implies that the reflection σ_d in d of M is also N. Similarly the dual point $e \equiv E^{\perp}$ is also a bipoint of $\overline{MN}$. In Euclidean geometry, bilines are called vertex bisectors or angle bisectors. Bipoints have no Euclidean analog.

Figure 17 can equally well be interpreted as illustrating the process of obtaining bilines D and E and bipoints d and e of the vertex $\overline{MN}$.

10 Hyperbolic triangle geometry

The richness of Euclidean triangle geometry is not reflected in the classical hyperbolic setting, but the situation is remedied with universal hyperbolic geometry. Here we give just a quick glimpse in this fascinating direction, which will be the focus of a subsequent paper in this series (see also [17]).

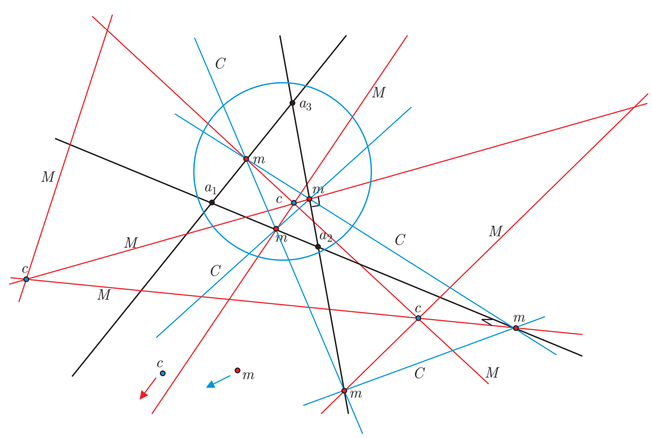

Figure 18: *Circumcenters and circumlines of a triangle $\overline{a_1a_2a_3}$*

Figure 18 shows a triangle $\overline{a_1a_2a_3}$ together with its six midpoints m (one is off the page) and corresponding six midlines M. The midpoints are collinear three at a time on four lines C called **circumlines**. The midlines are concurrent three at a time on four points c called **circumcenters**. The circumcenters are dual to the circumlines. Although we have not defined circles yet, a triangle generally has zero or four circumcircles, whose centers are at its circumcenters, if these exist.

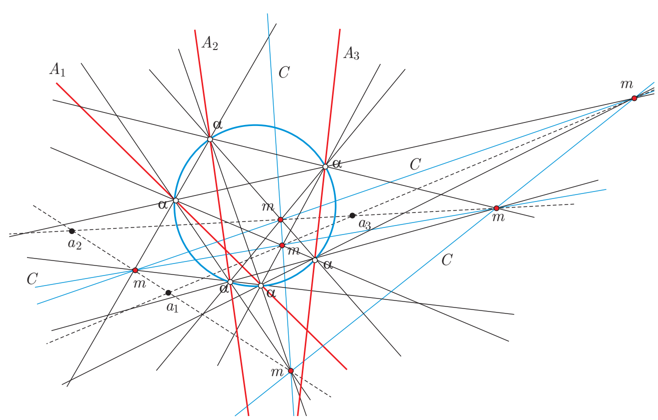

Figure 19: *Pascal's theorem via hyperbolic geometry*

An important application of circumlines is to *Pascal's theorem*, one of the great classical results of geometry. Figure 19 shows three lines A_1, A_2 and A_3 which meet the null circle at six null points α. The dual points of A_1, A_2 and A_3 are a_1, a_2 and a_3 respectively. The triangle $\overline{a_1a_2a_3}$ has six midpoints m (red) and the four circumlines C (blue) pass through three midpoints each.

Two of the original three lines, such as A_1 and A_2, determine four null points α, and the other two diagonal points formed by such a quadrangle of null points, not including A_1A_2, give two midpoints m, in this case of the side $\overline{a_1a_2}$. So Pascal's theorem is here seen as a consequence of the fact that the *six midpoints of a triangle are collinear three at a time* forming the circumlines.

The six null points α can be partitioned into three sets of two in 15 ways. By different choices of the lines A_1, A_2 and A_3, there are altogether 15 such diagrams associated to the same six null points, and 60 possible circumlines C playing the role of Pascal's line. Such a large configuration has many remarkable features, some of them projective, some of them metrical.

11 Parallels and the doubled triangle

Given a triangle $\overline{a_1a_2a_3}$, the **double triangle** $\overline{d_1d_2d_3}$ is the triangle whose lines are the parallels P_1, P_2 and P_3 to the lines L_1, L_2 and L_3 of $\overline{a_1a_2a_3}$ through the points a_1, a_2 and a_3 respectively. We retain the usual notational conventions, so that $d_1 = P_2P_3$ etc. The situation is shown in Figure 20.

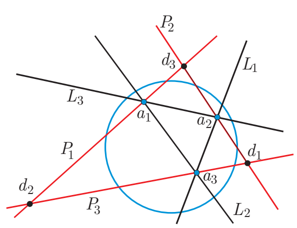

Figure 20: *A triangle $\overline{a_1a_2a_3}$ and its double triangle $\overline{d_1d_2d_3}$.*

The next theorem is surprising to me, and seems to require a somewhat involved computation.

Theorem 9 (Double median triangle) *If $\overline{d_1d_2d_3}$ is the double triangle of a triangle $\overline{a_1a_2a_3}$, then a_1, a_2 and a_3 are midpoints of the sides of $\overline{d_1d_2d_3}$.*

In Euclidean geometry the points in the next two theorems would both be the centroid of the triangle.

Theorem 10 (Double point) *If $\overline{d_1d_2d_3}$ is the double triangle of a triangle $\overline{a_1a_2a_3}$, then the lines a_1d_1, a_2d_2 and a_3d_3 are concurrent in a point x.*

Theorem 11 (Second double point) *If $\overline{d_1d_2d_3}$ is the double triangle of a triangle $\overline{a_1a_2a_3}$, and $\overline{g_1g_2g_3}$ is the double triangle of $\overline{d_1d_2d_3}$, then the lines a_1g_1, a_2g_2 and a_3g_3 are concurrent in a point y.*

These are shown in Figure 21; x is the **double point** of the triangle $\overline{a_1a_2a_3}$, and y is the **second double point** of the triangle $\overline{a_1a_2a_3}$.

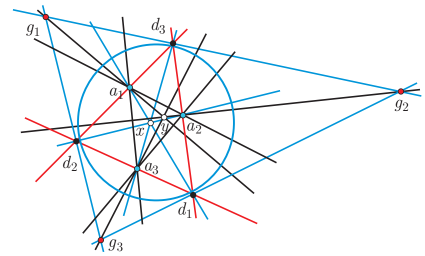

Figure 21: *First and second double points of a triangle $\overline{a_1a_2a_3}$*

It is not the case that the pattern continues in the obvious way: one cannot define a third double point in an analogous way. The study of the double triangle is clearly an interesting departure point from Euclidean triangle geometry.

12 Quadrance and spread

We now introduce the two basic measurements in universal hyperbolic geometry, the *quadrance* between points and the *spread* between lines. These are analogs of the corresponding notions in Euclidean *rational trigonometry*, but we assume no familiarity with this theory (although for a deeper understanding one should carefully compare the two). Our definition of quadrance and spread follows our projective orientation, and is given here in terms of the *cross-ratio* between four particular points or lines. The importance of this cross-ratio was shown in ([3]).

Suppose that a_1 and a_2 are points and that b_1 and b_2 are the conjugate points of the side $\overline{a_1a_2}$, as shown in Figure 22. Then define the **quadrance** between a_1 and a_2 to be the cross-ratio of points:

$$q(a_1, a_2) \equiv (a_1, b_2 : a_2, b_1).$$

The quadrance $q(a_1, a_2)$ is zero if $a_1 = a_2$. It is negative if a_1 and a_2 are both interior points, and approaches infinity as a_1 or a_2 approaches the null circle. It is undefined (or infinite) if one or both of a_1, a_2 is a null point. It is positive if one of a_1 and a_2 is an interior point and the other is an exterior point. It is negative if both a_1 and a_2 are exterior points and a_1a_2 is an interior line. It is zero if a_1a_2 is a null line. It is positive if both a_1 and a_2 are exterior points and a_1a_2 is an exterior line.

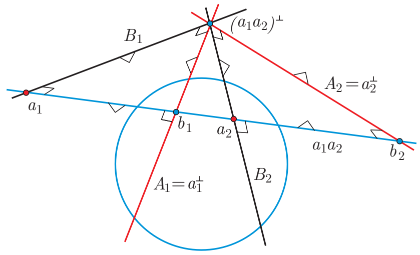

Figure 22: *Quadrance defined as cross-ratio:* $q(a_1,a_2) \equiv (a_1,b_2:a_2,b_1)$

The usual distance $d(a_1,a_2)$ between a_1 and a_2 in the Klein model is also defined in terms of a cross-ratio, involving two *other* points: the meets of a_1a_2 with the null circle. This is problematic in three ways. First of all for two general points there may be no such meet, and so the Klein distance does not extend to general points. But even if the meets exist, it is not easy to separate them algebraically to get four points in a prescribed and canonical order to apply the cross-ratio. Finally to get a quantity that acts somewhat linearly, one is forced to introduce a logarithm or inverse circular function. This is much more complicated analytically, and makes extending the theory to finite fields, for example, more problematic.

In any case it turns out that if a_1 and a_2 are interior points, there is a relation between quadrance and the Klein distance:

$$q(a_1,a_2) = -\sinh^2(d(a_1,a_2)). \tag{2}$$

To define the spread between lines, we proceed in a dual fashion. In Figure 22, B_1 and B_2 are the conjugate lines of the vertex $\overline{A_1A_2}$. Define the **spread** between the lines A_1 and A_2 to be the cross-ratio of lines:

$$S(A_1,A_2) \equiv (A_1,B_2:A_2,B_1).$$

This is positive if A_1 and A_2 are both interior lines that meet in an interior point. In fact if $\theta(A_1,A_2)$ is the usual angle between A_1 and A_2 in the Klein model, then it turns out that

$$S(A_1,A_2) = \sin^2(\theta(A_1,A_2)). \tag{3}$$

The relations (2) and (3) allow you to translate the subsequent theorems in this paper to formulas of classical hyperbolic trigonometry in the special case of interior points and lines.

The spread $S(A_1,A_2)$ between lines A_1 and A_2 is equal to the quadrance between the dual points, that is

$$S(A_1,A_2) = q\left(A_1^\perp,A_2^\perp\right).$$

So the basic duality between points and lines extends to the two fundamental measurements.

A **circle** is given by an equation of the form $q(x,a) = k$ for some fixed point a called the **center**, and a number k called the **quadrance**.

Here we show the circles centered at a point a of various quadrances. Figure 23 shows circles centered at a point a when a is an interior point. Figure 24 shows circles centered at an exterior point a. Both of these diagrams should be studied carefully. Note that the dual line $a^\perp$ of a is such a circle, of quadrance 1. Also note that the situation is dramatically different for a interior or a exterior. In the case of a an exterior point, there is a non-trivial circle of quadrance 0, namely the two null lines through a, and all circles meet the null circle at the two points where the dual line $a^\perp$ meets it. In classical hyperbolic geometry such curves are known as *constant width curves*—however from our point of view they are just circles.

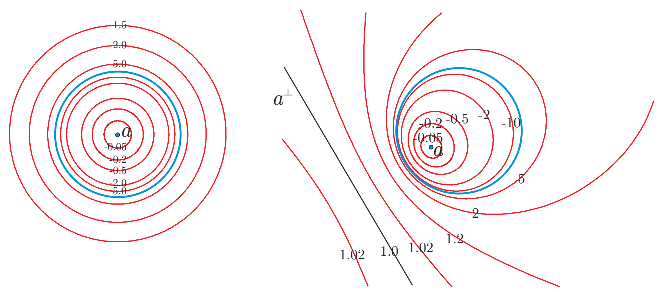

Figure 23: *Circles centered at a (interior)*

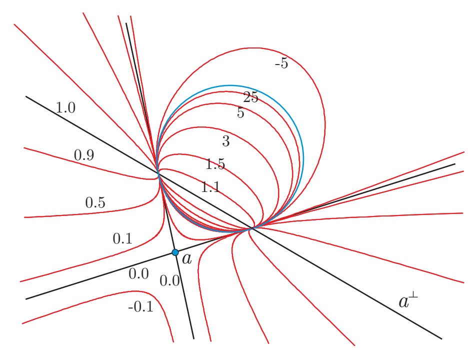

Figure 24: *Circles centered at a (exterior)*

There is a dual approach to circles where we use the relation $S(X,L) = k$ for a fixed line L and a variable line X. We leave it to the reader to show that we obtain the envelope of a circle as defined in terms of points. So the notion of a circle is essentially self-dual.

13 Basic trigonometric laws

For most calculations, we need explicit analytic formulae for the main measurements.

Theorem 12 *The* ***quadrance*** *between points* $a_1 \equiv [x_1 : y_1 : z_1]$ *and* $a_2 \equiv [x_2 : y_2 : z_2]$ *is*

$$q(a_1,a_2) = 1 - \frac{(x_1x_2 + y_1y_2 - z_1z_2)^2}{(x_1^2 + y_1^2 - z_1^2)(x_2^2 + y_2^2 - z_2^2)}.$$

Theorem 13 *The* ***spread*** *between lines* $L_1 \equiv (l_1 : m_1 : n_1)$ *and* $L_2 \equiv (l_2 : m_2 : n_2)$ *is*

$$S(L_1,L_2) = 1 - \frac{(l_1l_2 + m_1m_2 - n_1n_2)^2}{(l_1^2 + m_1^2 - n_1^2)(l_2^2 + m_2^2 - n_2^2)}.$$

These expressions are not defined if one or more of the points or lines involved is null, and reinforce the fact that the duality between points and lines extends to quadrances and spreads, and so every metrical result can be expected to have a dual formulation.

For a triangle $\overline{a_1a_2a_3}$ with associated trilateral $\overline{L_1L_2L_3}$ we will use the usual convention that $q_1 \equiv q(a_2,a_3)$, $q_2 \equiv q(a_1,a_3)$ and $q_3 \equiv q(a_1,a_2)$, and $S_1 \equiv S(L_2,L_3)$, $S_2 \equiv S(L_1,L_3)$ and $S_3 \equiv S(L_1,L_2)$. This notation will also be used in the degenerate case when a_1,a_2 and a_3 are collinear, or L_1,L_2 and L_3 are concurrent.

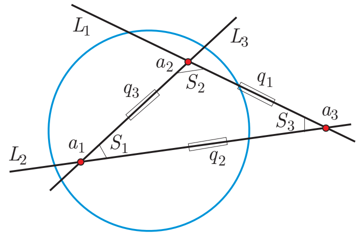

Figure 25: *Quadrance and spreads in a hyperbolic triangle*

With this notation, here are the main trigonometric laws in the subject. These are among the most important formulas in mathematics.

Theorem 14 (Triple quad formula) *If* a_1,a_2 *and* a_3 *are collinear points then*

$$(q_1 + q_2 + q_3)^2 = 2(q_1^2 + q_2^2 + q_3^2) + 4q_1q_2q_3.$$

Theorem 15 (Triple spread formula) *If* L_1,L_2 *and* L_3 *are concurrent lines then*

$$(S_1 + S_2 + S_3)^2 = 2(S_1^2 + S_2^2 + S_3^2) + 4S_1S_2S_3.$$

Theorem 16 (Pythagoras) *If* L_1 *and* L_2 *are perpendicular lines then*

$$q_3 = q_1 + q_2 - q_1q_2.$$

Theorem 17 (Pythagoras' dual) *If* a_1 *and* a_2 *are perpendicular points then*

$$S_3 = S_1 + S_2 - S_1S_2.$$

Theorem 18 (Spread law)

$$\frac{S_1}{q_1} = \frac{S_2}{q_2} = \frac{S_3}{q_3}.$$

Theorem 19 (Spread dual law)

$$\frac{q_1}{S_1} = \frac{q_2}{S_2} = \frac{q_3}{S_3}.$$

Theorem 20 (Cross law)

$$(q_1q_2S_3 - (q_1 + q_2 + q_3) + 2)^2 = 4(1-q_1)(1-q_2)(1-q_3).$$

Theorem 21 (Cross dual law)

$$(S_1S_2q_3 - (S_1 + S_2 + S_3) + 2)^2 = 4(1-S_1)(1-S_2)(1-S_3).$$

There are three symmetrical forms of Pythagoras' theorem, the Cross law and their duals, obtained by rotating indices. A proper appreciation for the beauty and power of these formulas requires some familiarity with rational trigonometry in the plane (see [18]), together with rolling up one's sleeves and solving many trigonometric problems in the hyperbolic setting. For students of geometry, this is an excellent undertaking.

14 Right triangles and trilaterals

Right triangles and trilaterals have some additional important properties besides the fundamental Pythagoras theorem we have already mentioned. We leave the dual results to the reader. Thales' theorem shows that there is an aspect of similar triangles in hyperbolic geometry. It also helps explain why spread is the primary measurement between lines in rational trigonometry.

Theorem 22 (Thales) *Suppose that* $\overline{a_1a_2a_3}$ *is a right triangle with* $S_3 = 1$. *Then*

$$S_1 = \frac{q_1}{q_3} \quad \text{and} \quad S_2 = \frac{q_2}{q_3}.$$

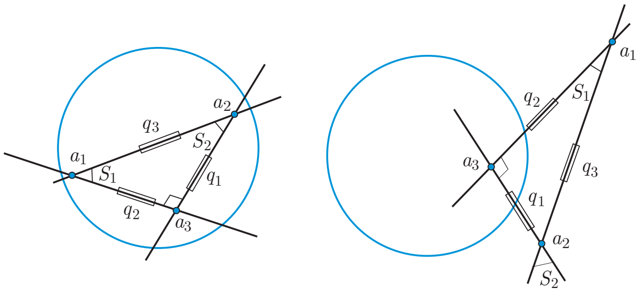

Figure 26: *Thales' theorem:* $S_1 = q_1/q_3$

The Right parallax theorem generalizes, and dramatically simplifies, a famous formula of Bolyai and Lobachevsky (see [6]) which usually requires exponential and circular functions, hence a prior understanding of real numbers.

Theorem 23 (Right parallax) *If a right triangle* $\overline{a_1a_2a_3}$ *has spreads* $S_1 = 0$, $S_2 = S$ *and* $S_3 = 1$, *then it will have only one defined quadrance* $q_1 = q$ *given by*

$$q = \frac{S-1}{S}.$$

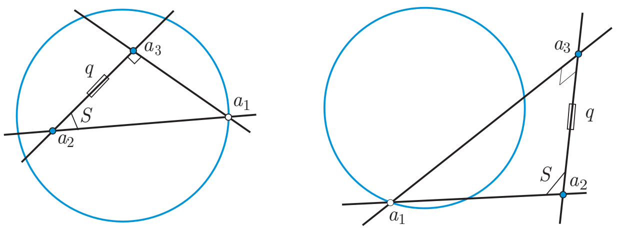

Figure 27: *Right parallax theorem:* $q = (S-1)/S$

We may restate this result in the form

$$S = \frac{1}{1-q}.$$

Napier's Rules are much simpler in the universal setting, where only high school algebra is required.

Theorem 24 (Napier's Rules) *Suppose a right triangle* $\overline{a_1a_2a_3}$ *has quadrances* q_1, q_2 *and* q_3, *and spreads* S_1, S_2 *and* $S_3 = 1$. *Then any two of the five quantities* S_1, S_2, q_1, q_2 *and* q_3 *determine the other three, solely by the three basic equations from Thales' theorem and Pythagoras' theorem:*

$$S_1 = \frac{q_1}{q_3} \qquad S_2 = \frac{q_2}{q_3} \qquad q_3 = q_1 + q_2 - q_1q_2.$$

15 Triangle proportions and barycentric coordinates

The following theorems implicitly involve barycentric coordinates. These are quite useful both in universal and classical hyperbolic geometry, see for example [17].

Theorem 25 (Triangle proportions) *Suppose that* $\overline{a_1a_2a_3}$ *is a triangle with quadrances* q_1, q_2 *and* q_3, *corresponding spreads* S_1, S_2 *and* S_3, *and that* d *is a point lying on the line* a_1a_2, *distinct from* a_1 *and* a_2. *Define the quadrances* $r_1 \equiv q(a_1, d)$ *and* $r_2 \equiv q(a_2, d)$, *and the spreads* $R_1 \equiv S(a_3a_1, a_3d)$ *and* $R_2 \equiv S(a_3a_2, a_3d)$. *Then*

$$\frac{R_1}{R_2} = \frac{S_1}{S_2}\frac{r_1}{r_2} = \frac{q_1}{q_2}\frac{r_1}{r_2}.$$

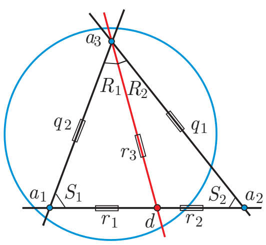

Figure 28: *Triangle proportions:*
$R_1/R_2 = (S_1/S_2) \times (r_1/r_2)$

Theorem 26 (Menelaus) *Suppose that* $\overline{a_1a_2a_3}$ *is a non-null triangle, and that* L *is a non-null line meeting* a_2a_3, a_1a_3 *and* a_1a_2 *at the points* b_1, b_2 *and* b_3 *respectively. Define the quadrances*

$$\begin{aligned} r_1 &\equiv q(a_2, b_1) \qquad & t_1 &\equiv q(b_1, a_3) \\ r_2 &\equiv q(a_3, b_2) \qquad & t_2 &\equiv q(b_2, a_1) \\ r_3 &\equiv q(a_1, b_3) \qquad & t_3 &\equiv q(b_3, a_2). \end{aligned}$$

Then $r_1r_2r_3 = t_1t_2t_3$.

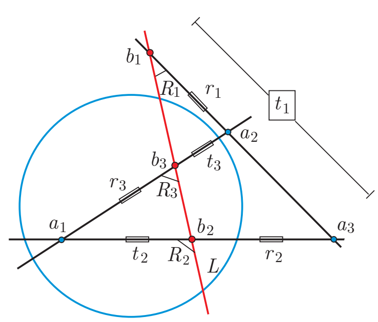

Figure 29: *Menelaus' theorem:* $r_1r_2r_3 = t_1t_2t_3$

Theorem 27 (Menelaus' dual) *Suppose that $\overline{A_1A_2A_3}$ is a non-null trilateral, and that l is a non-null point joining A_2A_3, A_1A_3 and A_1A_2 on the lines B_1, B_2 and B_3 respectively. Define the spreads*

$$\begin{aligned} R_1 &\equiv S(A_2,B_1) & T_1 &\equiv S(B_1,A_3)\\ R_2 &\equiv S(A_3,B_2) & T_2 &\equiv S(B_2,A_1)\\ R_3 &\equiv S(A_1,B_3) & T_3 &\equiv S(B_3,A_2). \end{aligned}$$

Then $R_1R_2R_3 = T_1T_2T_3$.

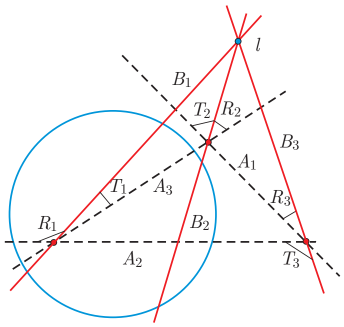

Figure 30: *Menelaus dual theorem: $R_1R_2R_3 = T_1T_2T_3$*

Theorem 28 (Ceva) *Suppose that the triangle $\overline{a_1a_2a_3}$ has non-null lines, that a_0 is a point distinct from a_1, a_2 and a_3, and that the lines a_0a_1, a_0a_2 and a_0a_3 meet the lines a_2a_3, a_1a_3 and a_1a_2 respectively at the points b_1, b_2 and b_3. Define the quadrances*

$$\begin{aligned} r_1 &\equiv q(a_2,b_1) & t_1 &\equiv q(b_1,a_3)\\ r_2 &\equiv q(a_3,b_2) & t_2 &\equiv q(b_2,a_1)\\ r_3 &\equiv q(a_1,b_3) & t_3 &\equiv q(b_3,a_2). \end{aligned}$$

Then $r_1r_2r_3 = t_1t_2t_3$.

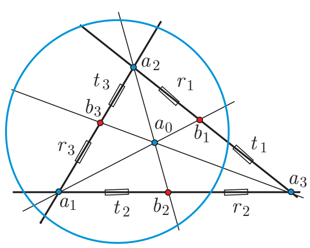

Figure 31: *Ceva's theorem: $r_1r_2r_3 = t_1t_2t_3$*

Theorem 29 (Ceva dual) *Suppose that the trilateral $\overline{A_1A_2A_3}$ is non-null, and that A_0 is a line distinct from A_1, A_2 and A_3, and that the points A_0A_1, A_0A_2 and A_0A_3 join the points A_2A_3, A_1A_3 and A_1A_2 respectively on the lines B_1, B_2 and B_3. Define the spreads*

$$\begin{aligned} R_1 &\equiv S(A_2,B_1) & T_1 &\equiv S(B_1,A_3)\\ R_2 &\equiv S(A_3,B_2) & T_2 &\equiv S(B_2,A_1)\\ R_3 &\equiv S(A_1,B_3) & T_3 &\equiv S(B_3,A_2). \end{aligned}$$

Then $R_1R_2R_3 = T_1T_2T_3$.

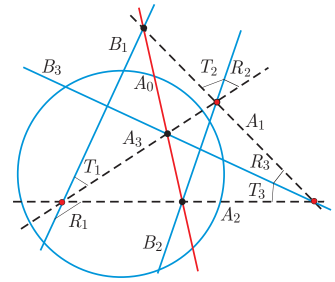

Figure 32: *Ceva's dual theorem: $R_1R_2R_3 = T_1T_2T_3$*

16 Isosceles triangles

Theorem 30 (Pons Asinorum) *Suppose that the non-null triangle $\overline{a_1a_2a_3}$ has quadrances q_1, q_2 and q_3, and corresponding spreads S_1, S_2 and S_3. Then $q_1 = q_2$ precisely when $S_1 = S_2$.*

Theorem 31 (Isosceles right) *If $\overline{a_1a_2a_3}$ is an isosceles triangle with two right spreads $S_1 = S_2 = 1$, then also $q_1 = q_2 = 1$ and $S_3 = q_3$.*

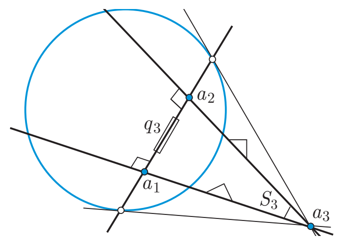

Figure 33: *Isosceles right triangle: $q_1 = q_2 = 1$ and $S_3 = q_3$*

Theorem 32 (Isosceles triangle) *Suppose a non-null isosceles triangle $\overline{a_1a_2a_3}$ has quadrances $q_1 = q_2 \equiv q$ and q_3, and corresponding spreads $S_1 = S_2 \equiv S$ and S_3. Then the following relations hold:*

$$q_3 = \frac{4(1-S)q(1-q)}{(1-Sq)^2} \qquad \text{and} \qquad S_3 = \frac{4S(1-S)(1-q)}{(1-Sq)^2}.$$

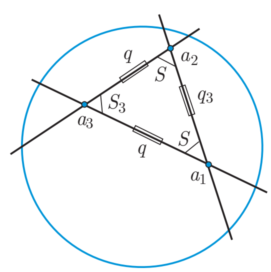

Figure 34: *An isosceles triangle: $q_1 = q_2 = q$, $S_1 = S_2 = S$*

Theorem 33 (Isosceles parallax) *If $\overline{a_1a_2a_3}$ is an isosceles triangle with a_1 a null point, $q_1 \equiv q$ and $S_2 = S_3 \equiv S$, then*

$$q = \frac{4(S-1)}{S^2}.$$

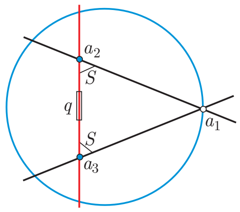

Figure 35: *Isosceles parallax: $q = 4(S-1)/S^2$*

17 Equilateral triangles

Theorem 34 (Equilateral quadrance spread) *Suppose that a triangle $\overline{a_1a_2a_3}$ is equilateral with common quadrance $q_1 = q_2 = q_3 \equiv q$, and with common spread $S_1 = S_2 = S_3 \equiv S$. Then*

$$(1-Sq)^2 = 4(1-S)(1-q).$$

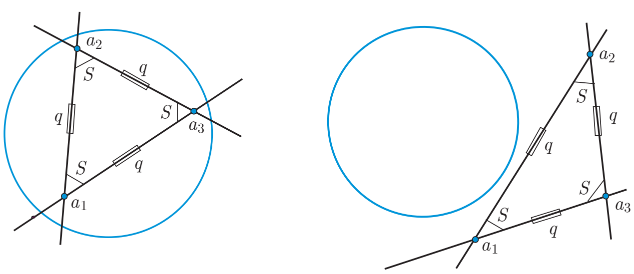

Figure 36: *Equilateral quadrance spread theorem:*
$(1-Sq)^2 = 4(1-S)(1-q)$

18 Lambert quadrilaterals

Theorem 35 (Lambert quadrilateral) *Suppose a quadrilateral $\overline{abcd}$ has all three spreads at a, b and c equal to 1. Suppose that $q \equiv q(a,b)$ and $p \equiv q(b,c)$. Then*

$$q(c,d) = y = \frac{q(1-p)}{1-qp} \qquad q(a,d) = x = \frac{p(1-q)}{1-qp}$$

$$q(a,c) = s = q+p-qp \qquad q(b,d) = r = \frac{q+p-2qp}{1-qp}$$

and

$$S(ba,bd) = \frac{x}{r} \qquad S(bc,bd) = \frac{y}{r}$$

$$S(ac,ab) = \frac{p}{s} \qquad S(cb,ca) = \frac{q}{s}$$

$$S(ac,ad) = \frac{q(1-p)}{s} \qquad S(ca,cd) = \frac{p(1-q)}{s}$$

and

$$S(da,dc) = S = 1-pq.$$

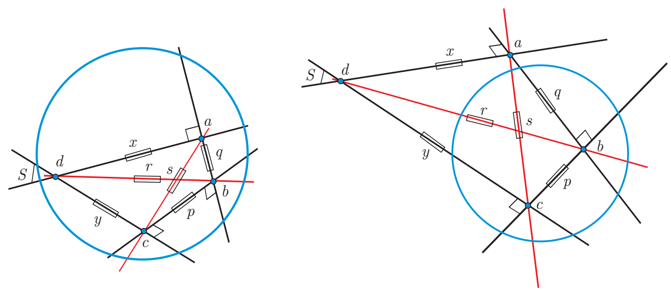

Figure 37: *Lambert quadrilateral $\overline{abcd}$*

19 Quadrea and triangle thinness

If $\overline{a_1a_2a_3}$ is a triangle with quadrances q_1, q_2 and q_3, and spreads S_1, S_2 and S_3, then from the Spread law the quantity

$$\mathcal{A} \equiv S_1q_2q_3 = S_2q_1q_3 = S_3q_1q_2$$

is well-defined, and called the **quadrea** of the triangle $\overline{a_1a_2a_3}$. It is the analog of the squared area in universal hyperbolic geometry. In Figure 38 several triangles with their associated quadreas are shown. Note that the quadrea is positive for a triangle of internal points, but may also be negative otherwise.

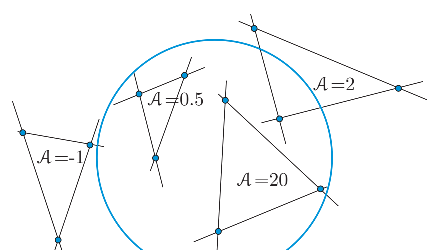

Figure 38: *Examples of triangles with quadreas*
$\mathcal{A} = -1, 0.5, 20$ *and* 2

An interesting aspect of hyperbolic geometry is that *triangles are thin*. Here are two ways of giving meaning to this, both involving the quadrea of a triangle.

Theorem 36 (Triply nil Cevian thinness) *Suppose that $\overline{\alpha_1\alpha_2\alpha_3}$ is a triply nil triangle, and that a is a point distinct from α_1,α_2 and α_3. Define the cevian points $c_1 \equiv (a\alpha_1)(\alpha_2\alpha_3)$, $c_2 \equiv (a\alpha_2)(\alpha_1\alpha_3)$ and $c_3 \equiv (a\alpha_3)(\alpha_1\alpha_2)$. Then $\mathcal{A}\left(\overline{c_1c_2c_3}\right) = 1$.*

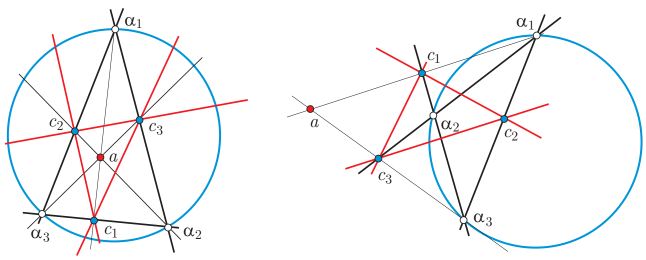

Figure 39: *Cevian triangle thinness: $\mathcal{A}\left(\overline{c_1c_2c_3}\right) = 1$*

Theorem 37 (Triply nil altitude thinness) *Suppose that $\overline{\alpha_1\alpha_2\alpha_3}$ is a triply nil triangle and that a is a point distinct from the duals of the lines. If the altitudes to the lines of this triangle from a meet the lines respectively at base points b_1,b_2 and b_3, then $\mathcal{A}\left(\overline{b_1b_2b_3}\right) = 1$.*

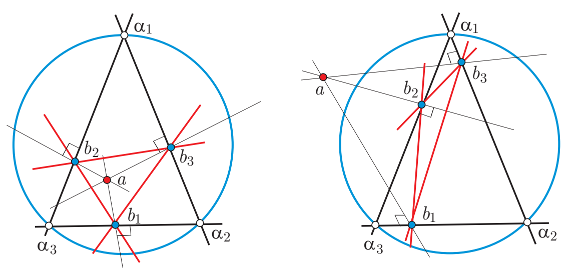

Figure 40: *Altitude triangle thinness: $\mathcal{A}\left(\overline{b_1b_2b_3}\right) = 1$*

20 Null perspective and null subtended theorems

There are many trigonometric results that prominently feature *null points* and *null lines*. We give a sample of these now. For some we include the dual formulations, for others these are left to the reader.

Theorem 38 (Null perspective) *Suppose that α_1,α_2 and α_3 are distinct null points, and b is any point on $\alpha_1\alpha_3$ distinct from α_1 and α_3. Suppose further that x and y are points lying on $\alpha_1\alpha_2$, and that $x_1 \equiv (\alpha_2\alpha_3)(xb)$ and $y_1 \equiv (\alpha_2\alpha_3)(yb)$. Then*

$$q(x,y) = q(x_1,y_1).$$

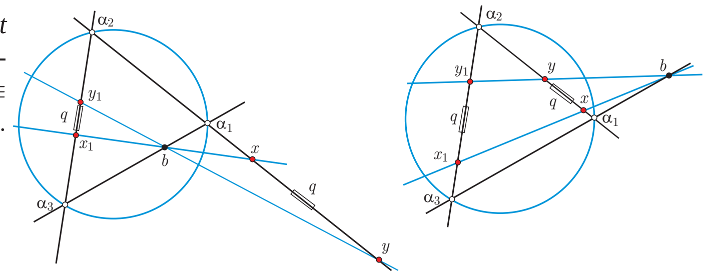

Figure 41: *Null perspective theorem: $q(x,y) = q(x_1,y_1)$*

Theorem 39 (Null subtended) *Suppose that the line L passes through the null points α_1 and α_2. Then for any other null point α_3 and any line M, let $a_1 \equiv (\alpha_1\alpha_3)M$ and $a_2 \equiv (\alpha_2\alpha_3)M$. Then $q \equiv q(a_1,a_2)$ and $S \equiv S(L,M)$ are related by*

$$qS = 1.$$

In particular q is independent of α_3.

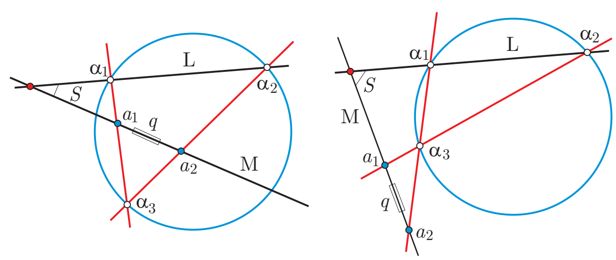

Figure 42: *Null subtended theorem: $qS = 1$*

Figure 42 shows two different examples; note that M need not pass through any null points. The Null subtended theorem allows you to create a hyperbolic ruler using just a straight-edge, in the sense that you can use it to repeatedly duplicate a given segment on a given line. Here is the dual result.

Theorem 40 (Null subtended dual) *Suppose that the point l lies on the null lines Λ_1 and Λ_2. Then for any other null line Λ_3 and any point m, let $A_1 \equiv (\Lambda_1\Lambda_3)m$ and $A_2 \equiv (\Lambda_2\Lambda_3)m$. Then $S \equiv S(L_1,L_2)$ and $q \equiv q(l,m)$ are related by*

$$Sq = 1.$$

Theorem 41 (Opposite subtended) *Suppose $\overline{\alpha\beta\gamma\delta}$ is a quadrangle of null points, and that υ,μ are also null points. Let $a \equiv (\alpha\mu)(\gamma\delta)$, $b \equiv (\beta\mu)(\gamma\delta)$, $c \equiv (\gamma\upsilon)(\alpha\beta)$ and $d \equiv (\delta\upsilon)(\alpha\beta)$. Then*

$$q(a,b) = q(c,d).$$

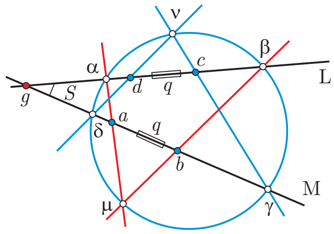

Figure 43: *Opposite subtended theorem:* $q(a,b) = q(c,d)$

Butterfly theorems have been investigated in the hyperbolic plane ([13]). The next theorems concern a related configuration of null points.

Theorem 42 (Butterfly quadrance) *Suppose that* $\overline{\alpha\beta\gamma\delta}$ *is a quadrangle of null points, with* $g \equiv (\alpha\gamma)(\beta\delta)$ *a diagonal point. Let* L *be any line passing through* g*, and suppose that* L *meets* $\alpha\delta$ *at* x *and* $\beta\gamma$ *at* y*. Then*

$$q(g,x) = q(g,y).$$

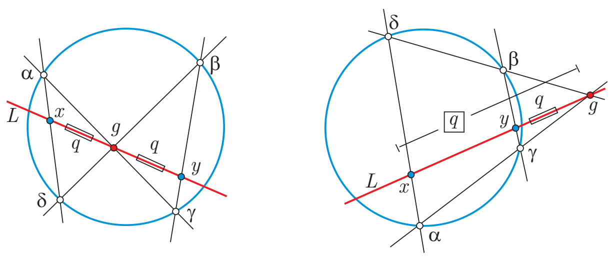

Figure 44: *Butterfly quadrance theorem:* $q(g,x) = q(g,y)$

Theorem 43 (Butterfly spread) *Suppose that* $\overline{\alpha\beta\gamma\delta}$ *is a quadrangle of null points, with* $g \equiv (\alpha\gamma)(\beta\delta)$ *a diagonal point. Let* L *be any line passing through* g*. Then*

$$S(L,\alpha\delta) = S(L,\beta\gamma).$$

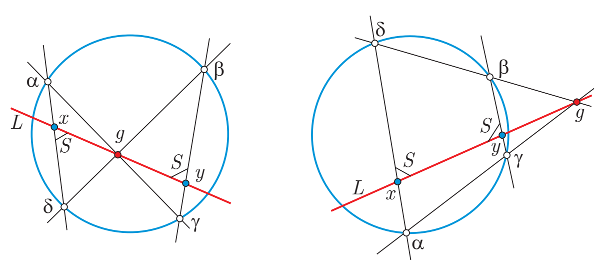

Figure 45: *Butterfly spread theorem:* $S(L,\alpha\delta) = S(L,\beta\gamma)$

21 The 48/64 theorems

In universal hyperbolic geometry we discover many constants of nature that express themselves in a geometrical way. Prominent among these are the numbers 48 and 64, but there are many others too!

Theorem 44 ($48/64$**)** *If the three spreads between opposite lines of a quadrangle* $\overline{\alpha_1\alpha_2\alpha_3\alpha_4}$ *of null points are* P,R *and* T*, then*

$$PR + RT + PT = 48$$

and

$$PRT = 64.$$

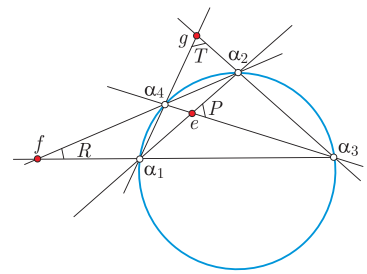

Figure 46: *The* $48/64$ *theorem:* $PR + RT + PT = 48$ *and* $PRT = 64$

It follows that

$$\frac{1}{R} + \frac{1}{S} + \frac{1}{T} = \frac{3}{4}.$$

In particular if we know two of these spreads, we get a linear equation for the third one.

Theorem 45 ($48/64$ **dual)** *If the three quadrances between opposite points of a quadrilateral* $\overline{\Lambda_1\Lambda_2\Lambda_3\Lambda_4}$ *of null lines are* p,r *and* t*, then*

$$pr + rt + pt = 48$$

and

$$prt = 64.$$

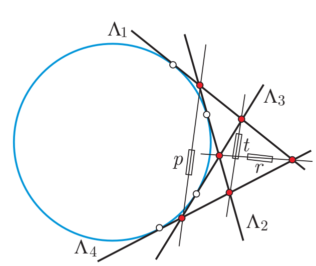

Figure 47: *The* $48/64$ *dual theorem:* $pr + rt + pt = 48$ *and* $prt = 64$

22 Pentagon theorems and extensions

The next theorem does not rely on null points, but is closely connected to a family of results that do.

Theorem 46 (Pentagon ratio) *Suppose $\overline{a_1a_2a_3a_4a_5}$ is a pentagon, meaning a cyclical list of five points, no three consecutive points collinear. Define* ***diagonal points***

$$\begin{aligned} b_1 &\equiv (\alpha_2\alpha_4)(\alpha_3\alpha_5), & b_2 &\equiv (\alpha_3\alpha_5)(\alpha_4\alpha_1), \\ b_3 &\equiv (\alpha_4\alpha_1)(\alpha_5\alpha_2), & b_4 &\equiv (\alpha_5\alpha_2)(\alpha_1\alpha_3), \\ &\textit{and} & b_5 &\equiv (\alpha_1\alpha_3)(\alpha_2\alpha_4), \end{aligned}$$

and subsequently ***opposite points***

$$\begin{aligned} c_1 &\equiv (a_1b_1)(a_2a_5), & c_2 &\equiv (a_2b_2)(a_3a_1), \\ c_3 &\equiv (a_3b_3)(a_4a_2), & c_4 &\equiv (a_4b_4)(a_5a_3), \\ &\textit{and} & c_5 &\equiv (a_5b_5)(a_1a_4). \end{aligned}$$

Then

$$\begin{aligned} &q(b_1,c_4)\,q(b_2,c_5)\,q(b_3,c_1)\,q(b_4,c_2)\,q(b_5,c_3) \\ &= q(b_2,c_4)\,q(b_3,c_5)\,q(b_4,c_1)\,q(b_5,c_2)\,q(b_1,c_3). \end{aligned}$$

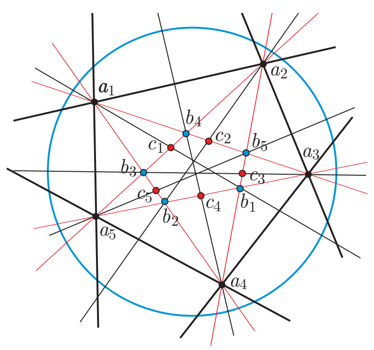

Figure 48: *Pentagon ratio theorem*

Since the pentagon is arbitrary, it follows by a scaling argument that *exactly the same theorem* holds in planar Euclidean geometry, where we replace the hyperbolic quadrance q with the Euclidean quadrance Q, since for five very close interior points, the hyperbolic quadrances and Euclidean quadrances are approximately proportional.

There are also some interesting additional features that occur in the special case of the pentagon when all the points a_i are null.

Theorem 47 (Pentagon null product) *Suppose $\overline{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}$ is a pentagon of null points. Define diagonal points*

$$\begin{aligned} b_1 &\equiv (\alpha_2\alpha_4)(\alpha_3\alpha_5), & b_2 &\equiv (\alpha_3\alpha_5)(\alpha_4\alpha_1), \\ b_3 &\equiv (\alpha_4\alpha_1)(\alpha_5\alpha_2), & b_4 &\equiv (\alpha_5\alpha_2)(\alpha_1\alpha_3), \\ &\textit{and} & b_5 &\equiv (\alpha_1\alpha_3)(\alpha_2\alpha_4). \end{aligned}$$

Then

$$q(b_1,b_2)\,q(b_2,b_3)\,q(b_3,b_4)\,q(b_4,b_5)\,q(b_5,b_1) = -\frac{1}{4^5}.$$

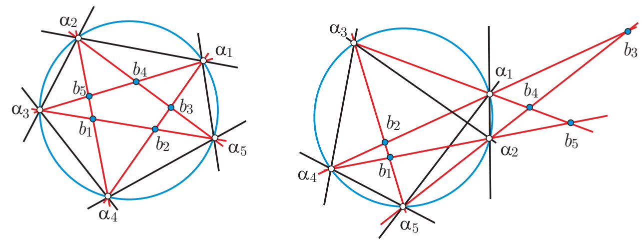

Figure 49: *Pentagon null product theorem:*
$q(b_1,b_2)\,q(b_2,b_3)\,q(b_3,b_4)\,q(b_4,b_5)\,q(b_5,b_1) = -\frac{1}{4^5}$

Theorem 48 (Pentagon null symmetry) *With notation as in the Pentagon ratio theorem, suppose that $\overline{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}$ is a pentagon of null points, then*

$$\begin{aligned} q(b_1,c_4) &= q(b_5,c_2), & q(b_2,c_5) &= q(b_1,c_3), \\ q(b_3,c_1) &= q(b_2,c_4), & q(b_4,c_2) &= q(b_3,c_5) \\ &\textit{and} & q(b_5,c_3) &= q(b_4,c_1). \end{aligned}$$

Furthermore if we fix $\alpha_2,\alpha_3,\alpha_4$ and α_5, then the quadrance $q(b_4,c_2) = q(b_3,c_5)$ is constant, independent of α_1.

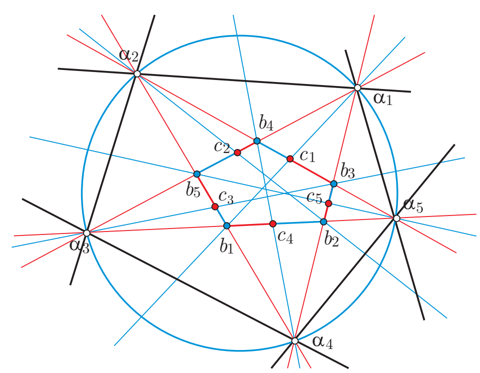

Figure 50: *Pentagon null symmetry theorem:*
$q(b_1,c_4) = q(b_5,c_2)$ *etc*

Since five points determine a conic, here is an analog to the Pentagon ratio theorem for general septagons.

Theorem 49 (Septagon conic ratio) *Suppose $\overline{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6\alpha_7}$ is a septagon of points lying on a conic. Define diagonal points*

$$\begin{aligned} b_1 &\equiv (\alpha_3\alpha_5)(\alpha_4\alpha_6), & b_2 &\equiv (\alpha_4\alpha_6)(\alpha_5\alpha_7), \\ b_3 &\equiv (\alpha_5\alpha_7)(\alpha_6\alpha_1), & b_4 &\equiv (\alpha_6\alpha_1)(\alpha_7\alpha_2), \\ b_5 &\equiv (\alpha_7\alpha_2)(\alpha_1\alpha_3), & b_6 &\equiv (\alpha_1\alpha_3)(\alpha_2\alpha_4), \\ &\textit{and} & b_7 &\equiv (\alpha_2\alpha_4)(\alpha_3\alpha_5), \end{aligned}$$

and opposite points

$$\begin{aligned} c_1 &\equiv (\alpha_1b_1)(\alpha_7\alpha_2), & c_2 &\equiv (\alpha_2b_2)(\alpha_1\alpha_3), \\ c_3 &\equiv (\alpha_3b_3)(\alpha_2\alpha_4), & c_4 &\equiv (\alpha_4b_4)(\alpha_3\alpha_5), \\ c_5 &\equiv (\alpha_5b_5)(\alpha_4\alpha_6), & c_6 &\equiv (\alpha_6b_6)(\alpha_5\alpha_7), \\ &\textit{and} & c_7 &\equiv (\alpha_7b_7)(\alpha_6\alpha_1). \end{aligned}$$

Then

$$q(c_1,b_5)\,q(c_2,b_6)\,q(c_3,b_7)\,q(c_4,b_1)\,q(c_5,b_2)\,q(c_6,b_3)\,q(c_7,b_4)$$
$$=q(c_1,b_4)\,q(c_2,b_5)\,q(c_3,b_6)\,q(c_4,b_7)\,q(c_5,b_1)\,q(c_6,b_2)\,q(c_7,b_3).$$

Since the notion of a conic is projective, a scaling argument shows that the same theorem holds also in the Euclidean case. Figure 51 shows the special case of a septagon of null points (top), and a more general case where the septagon lies on a conic (bottom), in this case a Euclidean circle.

I conjecture that the Septagon conic ratio theorem extends to all odd polygons.

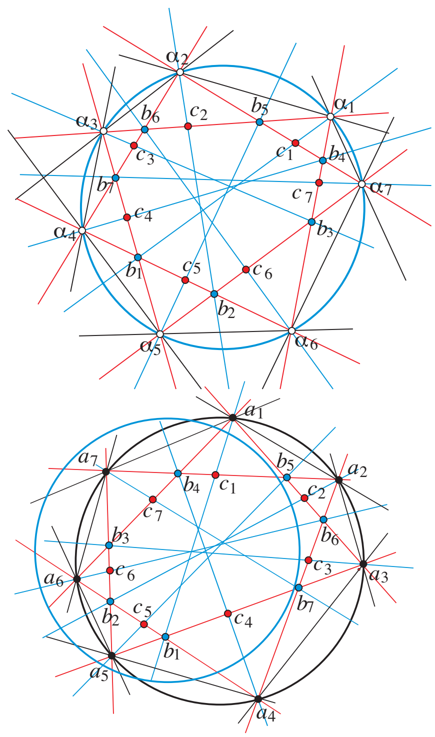

Figure 51: *Septagon conic ratio theorem*

23 Conics in hyperbolic geometry

The previous result used the fact that conics are well-defined in hyperbolic geometry, since they can be defined projectively, and we are working in a projective setting. A natural question is: can we also study conics metrically as we do in the Euclidean plane? In fact we can, and the resulting theory is both more intricate and richer than the Euclidean theory, nevertheless incorporating the Euclidean case as a limiting special case.

We have already mentioned (hyperbolic) circles and illustrated them in Figures 23 and 24. Let us now just briefly outline some results for a **(hyperbolic) parabola**, which may be defined as the locus of a point a satisfying $q(a,f)=q(a,D)$ where f is a fixed point called a **focus**, and D is a fixed line called a **directrix**, and where $q(a,D)$ is the quadrance from the point a to the base point b of the altitude to D through a. The following theorems summarize some basic facts about such a hyperbolic parabola, some similar to the Euclidean situation, others quite different. The situation is illustrated in Figure 52. Careful examination reveals many more interesting features of this situation, which will be discussed in a further paper in this series.

Theorem 50 (Parabola focus directrix pair) *If a (hyperbolic) parabola p has focus f_1 and directrix D_1, then it also has another focus $f_2 \equiv D_1^{\perp}$ and another directrix $D_2 \equiv f_1^{\perp}$.*

Theorem 51 (Parabola tangents) *Suppose that b_1 is a point on D_1 such that the two midlines of the side $\overline{b_1 f_1}$ exist. Then these midlines meet the altitude line to D_1 through b_1 at two points (both labelled a_1 in the Figure) lying on the parabola, and are the tangents to the parabola at those points.*

We note that in addition if the two midlines of the side $\overline{b_1 f_1}$ exist, then both midlines meet at the point $b_2 \equiv (b_1 f_1)^{\perp}$ lying on D_2 and the corresponding midlines of the side $\overline{b_2 f_2}$ meet the altitude line to D_2 through b_2 at two points (both labelled a_2 in the Figure) lying on the parabola, and themselves meet at b_1. This gives a pairing between some of the points b_1 lying on D_1 and some of the points b_2 lying on D_2. Of the four points labelled a_1 and a_2 lying on the parabola, one of the a_1 points and one of the a_2 points are (somewhat mysteriously) perpendicular. The entire situation is very rich, and emphasizes once again (see [20]) that the theory of conics is not a closed book, but rather a rich mine which has only been partly explored so far.

Recall that in Euclidean geometry the locus of a point a satisfying $q(a,f_1)+q(a,f_2)=k$ for two fixed points f_1 and f_2 and some fixed number k is a circle.

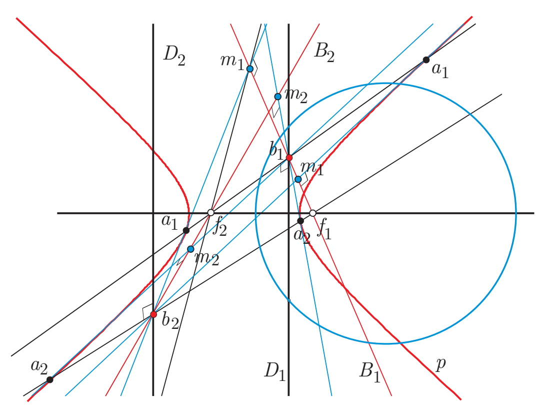

Figure 52: *Construction of a (hyperbolic) parabola*

Theorem 52 (Sum of two quadrances) *The (hyperbolic) parabola p described in the previous theorem may also be defined as the locus of those points a satisfying*

$$q(a, f_1) + q(a, f_2) = 1.$$

Many classical theorems for the Euclidean parabola hold also for the hyperbolic parabola p. Here are two, illustrated in Figure 53.

Theorem 53 (Parabola chord spread) *If a and b are two points on the hyperbolic parabola p with directrix D and focus f, and if c is the meet of D with the tangent of p at a, while d is the meet of D with the tangent to p at b, then $S(cf, fd) = S(af, fb)$.*

Theorem 54 (Parabola chord tangents perpendicular) *If a and b are two points on the hyperbolic parabola p with directrix D and focus f, and if e is the meet of D with ab, while g is the meet of the tangents to p at a and b, then ef is perpendicular to gf.*

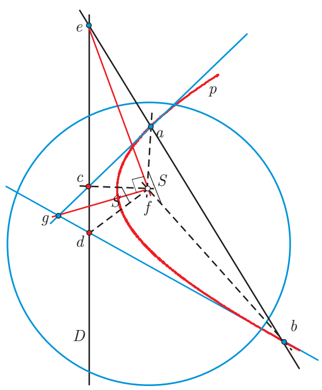

Figure 53: *Hyperbolic parabola with focus f and directrix D*

24 Bolyai's construction of limiting lines

Here is a universal version of a famous construction of J. Bolyai, to find the limiting lines U and V to an interior line L through a point a, where limiting means that U and V meet L on the null circle.

Start by constructing the altitude line K from a to L, meeting L at c, then the parallel line P through a to L, namely that line perpendicular to K. Now let m denote the midpoints of $\overline{ac}$, there are either two such points or none. If there are two, choose any point b on L, construct the altitude N to P through b, and reflect b in both midpoints m to get d and e on P. The side $\overline{ed}$ has a a midpoint, and the hyperbolic circle centered at a through d and e meets N at the points u and v. Then $U \equiv au$ and $V \equiv av$ are the required limiting lines as shown.

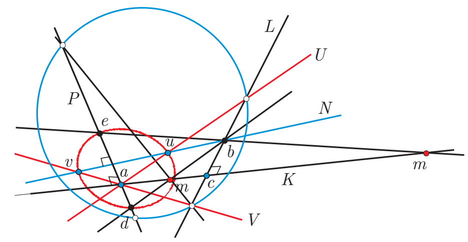

Figure 54: *A variant on J. Bolyai's construction of the limiting lines from a to L*

This construction seems to not be possible with only a straightedge, as we use a hyperbolic circle; this would correspond to the fact that there are two solutions. The question of what can and cannot be constructed with only a straightedge seems also an interesting one.

25 Canonical points

Both the Canonical points theorem in this section and the Jumping Jack theorem of the next section involve *cubic relations* between certain quadrances. I predict both will open up entirely new directions in hyperbolic geometry.

The Canonical points theorem has rather many aspects, one of which is a classical theorem of projective geometry.

Theorem 55 (Canonical points) *Suppose that α_1 and α_2 are distinct null points, and that x_3 and y_3 are points lying on $\alpha_1\alpha_2$. For any third null point α_3, and any point b_1 lying on $\alpha_2\alpha_3$, define $x_2 = (\alpha_1\alpha_3)(y_3b_1)$ and $y_2 = (\alpha_1\alpha_3)(x_3b_1)$. Similarly for any point b_2 lying on $\alpha_1\alpha_3$ define $x_1 = (\alpha_2\alpha_3)(y_3b_2)$ and $y_1 = (\alpha_2\alpha_3)(x_3b_2)$. Then $b_3 \equiv (x_1y_2)(x_2y_2)$ lies on $\alpha_1\alpha_2$. Now define points*

$$c_1 = (x_2x_3)(y_2y_3) \qquad c_2 = (x_1x_3)(y_1y_3) \qquad c_3 = (x_1x_2)(y_1y_2)$$

and corresponding points

$$\begin{aligned} z_3 &= (c_1b_1)(\alpha_1\alpha_2) \qquad & w_2 &= (c_1b_1)(\alpha_1\alpha_3) \\ z_1 &= (c_2b_2)(\alpha_2\alpha_3) \qquad & w_3 &= (c_2b_2)(\alpha_1\alpha_2) \\ z_2 &= (c_3b_3)(\alpha_1\alpha_3) \qquad & w_1 &= (c_3b_3)(\alpha_2\alpha_3). \end{aligned}$$

Then z_3 and w_3 depend only on x_3 and y_3, and not on α_3, b_1 and b_2. Furthermore b_1, z_2, w_3 are collinear, as are b_2, z_3, w_1, and b_3, z_1, w_2.

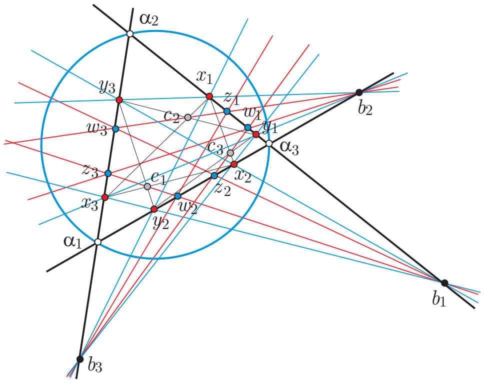

Figure 55: *Canonical points theorem:* $\overline{x_3 y_3}$ *determines* $\overline{z_3 w_3}$

In particular note that the theorem implies that any two points x and y whose join passes through two null points determine canonically two points z and w lying on xy in this fashion. We call z and w the **canonical points** of x and y. In Figure 55 z_3 and w_3 are the canonical points of x_3 and y_3, while z_1 and w_1 are the canonical points of x_1 and y_1, and z_2 and w_2 are the canonical points of x_2 and y_2.

Theorem 56 (Canonical points cubic) *With notation as above, the quadrances* $q \equiv q(x_3, y_3)$ *and* $r \equiv q(x_3, z_3)$ *satisfy the cubic relation*

$$(q - 4r)^2 = 8qr(2r - q). \tag{4}$$

We call the algebraic curve

$$(x - 4y)^2 = 8xy(2y - x)$$

the **Canonical points cubic**. The graph is shown in Figure 56. It is perhaps interesting that the point $[9/8, 9/8]$ is the apex of one of the branches of this algebraic curve.

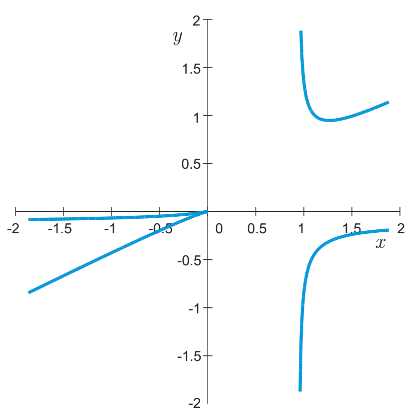

Figure 56: *The Canonical points cubic:* $(x - 4y)^2 = 8xy(2y - x)$

26 The Jumping Jack theorem

Here is my personal favourite theorem. Although one can give a computational proof of it, the result begs for a conceptual framework that explains it, and points to other similar facts (if they exist!)

Theorem 57 (Jumping Jack) *Suppose that* $\overline{\alpha_1\alpha_2\alpha_3\alpha_4}$ *is a quadrangle of null points, with* $g \equiv (\alpha_1\alpha_3)(\alpha_2\alpha_4)$ *a diagonal point, and let* L *be any line through* g*. Then for an arbitrary null point* α_5*, define the meets* $x \equiv (\alpha_1\alpha_3)(\alpha_4\alpha_5)$, $y \equiv L(\alpha_4\alpha_5)$, $z \equiv (\alpha_2\alpha_4)(\alpha_3\alpha_5)$ *and* $w \equiv L(\alpha_3\alpha_5)$*. If* $r \equiv q(x, y)$ *and* $s \equiv q(z, w)$ *then*

$$16rs(3 - 4(s + r)) = 1.$$

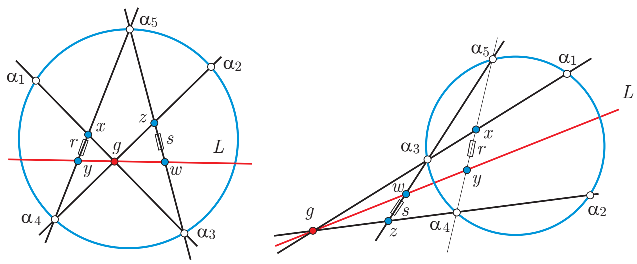

Figure 57: *Jumping Jack theorem:* $16rs(3 - 4(s + r)) = 1$

We call the algebraic curve

$$16xy(3 - 4(x + y)) = 1$$

the **Jumping Jack cubic**. The Jumping Jack theorem shows that it has an infinite number of rational solutions, which include a parametric description with 6 independent parameters.

The graph is shown in Figure 58. Note the isolated solution $[1/4, 1/4]$, which is the centroid of the trilateral formed by the three asymptotes.

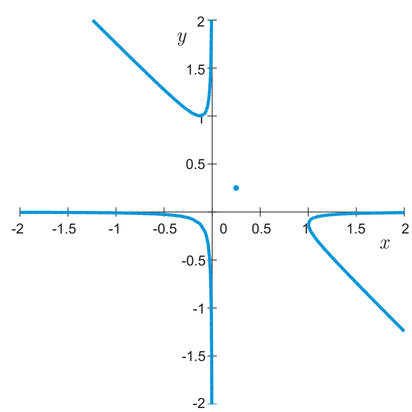

Figure 58: *Jumping Jack cubic:* $16xy(3 - 4(x + y)) = 1$

27 Conclusion

Universal hyperbolic geometry provides a new framework for a classical subject. It provides a more logical foundation for this geometry, as now analysis is not used, but only high school algebra with polynomials and rational functions. The main laws of trigonometry require only quadratic equations for their solutions. Theorems extend now beyond the familiar interior of the unit disk, and also to geometries over finite fields. Although we have not stressed this, it turns out that almost all the theorems we have described also hold in elliptic geometry! That is because the algebraic treatment turns out to be essentially independent of the projective quadratic form in the three dimensional space that is implicitly used to set up the theory in (1). We have shown how many classical results can be enlarged to fit into this new framework, and also described novel and interesting results.

So there are many opportunities for researchers to make essential discoveries at this early stage of the subject. When it comes to hyperbolic geometry, we are all beginners now.

References

[1] J. ANGEL, Finite upper half planes over finite fields, *Finite Fields and their Applications* **2** (1996), 62–86.

[2] W. BENZ, *Classical Geometries in Modern Contexts: Geometry of Real Inner Product Spaces*, Birkhäuser, Basel, 2007.

[3] H. BRAUNER, *Geometrie Projektiver Räume I, II*, Bibliographisches Institut, Mannheim, 1976.

[4] J. L. COOLIDGE, *The Elements of Non-Euclidean Geometry*, Merchant Books, 1909.

[5] H. S. M. COXETER, *Non-Euclidean Geometry*, 6th ed., Mathematical Association of America, Washington D. C., 1998.

[6] M. J. GREENBERG, *Euclidean and Non-Euclidean Geometries: Development and History*, 4th ed., W. H. Freeman and Co., San Francisco, 2007.

[7] R. HARTSHORNE, *Geometry: Euclid and Beyond*, Springer, New York, 2000.

[8] B. IVERSEN, *Hyperbolic Geometry*, London Mathematical Society Student Texts 25, Cambridge University Press, Cambridge, 1992.

[9] H. LENZ, *Nichteuklidische Geometrie*, Bibliographishes Institut, Mannheim, 1967.

[10] J. MCCLEARY, *Geometry from a Differentiable Viewpoint*, Cambridge University Press, New York, 1994.

[11] J. MILNOR, Hyperbolic Geometry—The First 150 Years, *Bulletin of the AMS* **6** (1982), 9-24.

[12] A. PRÉKOPA and E. MOLNAR, eds., *Non-Euclidean Geometries: János Bolyai Memorial Volume*, Springer, New York, 2005.

[13] A. SLIEPČEVIĆ and E. JURKIN, The Butterfly Theorems in the Hyperbolic Plane, *Annales Universitatis Scientiarum Budapestinensis* **48** (2005), 109–117.

[14] D. M. Y. SOMMERVILLE, *The Elements of Non-Euclidean Geometry*, G. Bell and Sons, Ltd., London, 1914, reprinted by Dover Publications, New York, 2005.

[15] J. STILLWELL, *Sources of Hyperbolic Geometry*, American Mathematical Society, Providence, R. I., 1996.

[16] A. TERRAS, *Fourier Analysis on Finite Groups and Applications*, Cambridge University Press, Cambridge,1999.

[17] A. UNGAR, *Hyperbolic Triangle Centers: The Special Relativistic Approach*, FTP, volume 166, Springer, 2010.

[18] N. J. WILDBERGER, *Divine Proportions: Rational Trigonometry to Universal Geometry*, Wild Egg Books, Sydney, 2005.

[19] N. J. WILDBERGER, One dimensional metrical geometry, *Geometriae Dedicata* **128** (1) (2007), 145–166.

[20] N. J. WILDBERGER, Chromogeometry and Relativistic Conics, *KoG* **13** (2009), 43–50.

[21] N. J. WILDBERGER, Affine and Projective Universal Geometry, to appear, *J. of Geometry*, `http://arxiv.org/abs/math/0612499v1`.

[22] N. J. WILDBERGER, Universal Hyperbolic Geometry I: Trigonometry, `http://arxiv.org/abs/0909.1377v1`.

[23] H. E. WOLFE, *Introduction to Non-Euclidean Geometry*, Holt, Rinehart and Winston, New York, 1945.

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