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SECOND-METACYCLIC FINITE 2-GROUPS

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Abstract. Second-metacyclic finite 2-groups are finite 2-groups with some non-metacyclic maximal subgroup and with all second-maximal subgroups being metacyclic. According to a known result there are only four non-metacyclic finite 2-groups with all maximal subgroups being metacyclic. The groups pointed in the title should contain some of these groups as a subgroup of index 2. There are seventeen second-maximal finite 2-groups, four among them being of order 16, ten of order 32 and three of order 64.

1. Introduction

A group G is called *metacyclic* if there exists a cyclic normal subgroup N of G with cyclic factorgroup G/N. A group with some non-metacyclic maximal subgroup and with all second-maximal subgroups being metacyclic we call a *second-metacyclic group*. The aim of this article is to determine all second-metacyclic finite $2-$groups.

The starting point is the following result of N. Blackburn.

Theorem 1.1. *(see Janko* [1, Th. 7.1]*) Let G be a minimal non-metacyclic 2-group. Then G is one of the following groups:*

(a) *The elementary abelian group E_8 of order 8,*
(b) *The direct product $Q_8 \times Z_2$,*
(c) *The central product $Q_8 * Z_4$ of order 2^4,*
(d) $G = \langle a, b, c \mid a^4 = b^4 = [a, b] = 1,\ c^2 = a^2b^2,\ a^c = a^{-1},\ b^c = a^2b^3\rangle$,

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where G is special of order 2^5 with $\exp(G) = 4$, $\Omega_1(G) = G' = Z(G) = \Phi(G) = \langle a^2, b^2\rangle \cong E_4$ *and* $M = \langle a\rangle \times \langle b\rangle \cong Z_4 \times Z_4$ *is the unique abelian maximal subgroup of* G.

For brevity, we denote the second-metacyclic groups as $MC(2)$-groups. It is clear that their non-metacyclic maximal subgroups are minimal non-metacyclic groups and thus each $MC(2)$-group contains some group from Th. 1.1 as maximal subgroup. Especially, an $MC(2)$-group G is of order 16, 32 or 64. Our main result is stated in the following theorem.

THEOREM 1.2. *Let G be a second-metacyclic group. Then G is one of the following 17 groups:*

(a) *four groups of order 16:*
$E_{16},\ Z_4 \times E_4,\ D_8 \times Z_2$, *or the semidirect product* $E_4 \cdot Z_4$;

(b) *ten groups of order 32:*

(b1) *G contains a subgroup H isomorphic with $Q_8 \times Z_2$ so that*

$$\begin{aligned} H &= \langle a, b, c | a^4 = 1, b^2 = a^2, c^2 = 1, a^b = a^{-1}, a^c = a, b^c = b\rangle \\ &= \langle a, b\rangle \times \langle c\rangle : \\ G_1 &= \langle H, d \mid d^2 = 1, [a, d] = [b, d] = 1, c^d = a^2 c\rangle \\ &= \langle a, b\rangle * \langle c, d\rangle \cong Q_8 * D_8, \\ G_2 &= \langle H, d \mid d^2 = 1, a^d = a, b^d = abc, c^d = a^2 c\rangle, \\ G_3 &= \langle H, d \mid d^2 = a^2, a^d = a^{-1}, b^d = ab, c^d = c\rangle, \\ G_4 &= \langle H, d \mid d^2 = a^2, a^d = a^{-1},\ b^d = bc, c^d = c\rangle, \\ G_5 &= \langle H, d \mid d^2 = c, a^d = a^{-1}, b^d = ab, c^d = c\rangle, \\ G_6 &= \langle H, d \mid d^2 = c, [a, d] = [b, d] = [c, d] = 1\rangle \\ &= \langle a, b\rangle \times \langle d\rangle \cong Q_8 \times Z_4, \\ G_7 &= \langle H, d \mid d^2 = a, a^d = a, b^d = bc, c^d = a^2 c\rangle; \end{aligned}$$

(b2) *G contains a subgroup H isomorphic with $Q_8 * Z_4$ so that*

$$\begin{aligned} H &= \langle a, b, c \mid a^4 = 1, b^2 = c^2 = a^2, a^b = a^{-1}, a^c = a, b^c = b\rangle \\ &= \langle a, b\rangle * \langle c\rangle \end{aligned}$$

and if $L < G$, then $L \not\cong Q_8 \times Z_2$:

$$\begin{aligned} G_8 &= \langle H, d \mid d^2 = c, [a, d] = [b, d] = [c, d] = 1\rangle, \\ G_9 &= \langle H, d \mid d^2 = 1, a^d = a^{-1}, b^d = ab, c^d = c\rangle, \\ G_{10} &= \langle H, d \mid d^2 = ac, a^d = a, b^d = ab, c^d = c\rangle; \end{aligned}$$

(c) *three groups of order 64:*

$G > H = \langle a, b, c \mid a^4 = b^4 = 1, c^2 = a^2b^2, a^b = a, a^c = a^{-1}, b^c = a^2b^3\rangle :$

$$G_1 = \langle H, d \mid d^2 = a^2, a^d = a^3b^2, b^d = b^{-1}, c^d = c\rangle,$$

$$G_2 = \langle H, d \mid d^2 = b^2,\ a^d = a^{-1},\ b^d = a^2b^3,\ c^d = ac\rangle,$$

$$G_3 = \langle H, d \mid d^2 = b, a^d = ab^2, b^d = b, c^d = ac\rangle.$$

We proceed by recalling some useful known results.

LEMMA 1.3. *Let E be an elementary abelian subgroup of a 2-group G, and $g \in G$, $g^2 \in E$. Then $|C_E(g)|^2 \geq |E|$.*

PROOF. Because of $g^2 \in E$ and E abelian, $x^{g^2} = x$ for any $x \in G$. Thus $(xx^g)^g = x^g x^{g^2} = x^g x = xx^g$, for all $x \in G$, and so $xx^g \in C_E(g)$. Now, for $y \in G$, we have $xx^g = yy^g$ if and only if $xy \in C_E(g)$, which is equivalent with $C_E(g)x = C_E(g)y$. Therefore $xx^g \neq yy^g$ if and only if $C_E(g)x \neq C_E(g)y$, and so $|C_E(g)| \geq |E : C_E(g)|$. It follows $|C_E(g)|^2 \geq |E|$. ∎

THEOREM 1.4. *(see Janko* [1, Proposition 1.9]*) Let G be a p-group with a non-abelian subgroup P of order p^3. If $C_G(P) \leq P$, then G is of maximal class. Especially, if $p = 2$, G is metacyclic.*

THEOREM 1.5. *(see Janko* [1, Proposition 1.14]*) A 2-group G is metacyclic if and only if G and all its subgroups are generated by two elements.*

THEOREM 1.6. *Let G be a group of order* 16*. Then G is isomorphic to some of the following groups:*

(α) *abelian groups: $Z_{16}, Z_8 \times Z_2, Z_4 \times Z_4, Z_4 \times E_4, E_{16}$;*

(β) *non-abelian groups containing a cyclic maximal subgroup:*

$G = \langle a, b \mid a^8 = b^2 = 1, a^b = a^{-1}\rangle \cong D_{16}$ *– dihedral group,*

$G = \langle a, b \mid a^8 = b^4 = 1, a^b = a^{-1}\rangle \cong Q_{16}$ *– quaternion group,*

$G = \langle a, b \mid a^8 = b^2 = 1, a^b = a^3\rangle \cong SD_{16}$ *– semidihedral group,*

$G = \langle a, b \mid a^8 = b^2 = 1, a^b = a^5\rangle \cong M_{16}$
– M-group, which is minimal non-abelian;

(γ) *non-abelian groups with $exp(G) = 4$:*

(γ_1) *G containing a subgroup isomorphic with E_8 :*

$G \cong D_8 \times Z_2,$

$G \cong E_4 \cdot Z_4,$ *the semidirect product of E_4 by Z_4;*

(γ_2) *G being E_8-free. Then G contains $H \cong Z_4 \times Z_2$. If $\Omega_1(G) \nleqslant H$, then $G \cong Q_8 * Z_4$. If $\Omega_1(G) \leqslant H$, then either $G \cong Q_8 \times Z_2$ or $G \cong Z_4 \cdot Z_4$, the semidirect product of Z_4 by Z_4, which is minimal non-abelian.*

2. Proof of the Theorem 1.2

(a) **Groups of order 16**.
As group E_8 is the only (minimal) nonmetacyclic group of order 8, the second metacyclic groups of order 16 are those among them which contain E_8. According to Th. 1.6, we have four such groups: $Z_4 \times E_4$, E_{16}, $D_8 \times Z_2$ and $E_4 \cdot Z_4$, the semidirect product of E_4 by Z_4.

(b) **Groups of order 32**.
According to Th. 1.1, such a group contains a subgroup isomorphic to $Q_8 \times Z_2$, or to $Q_8 * Z_4$, the central product of Q_8 by Z_4.

($b1$) G contains a subgroup H isomorphic with $Q_8 \times Z_2$:
Let $H = \langle a, b, c \mid a^4 = 1, b^2 = a^2, c^2 = 1, a^b = a^{-1}, [a, c] = [b, c] = 1\rangle = \langle a, b\rangle \times \langle c\rangle \cong Q_8 \times Z_2$. Now, $|G : H| = 2$, $G = \langle H, d\rangle$, $d^2 \in H$. Since $\Phi(H) = \mho_1(H) = \langle x^2 \mid x \in H\rangle = \langle a^2\rangle$ and $\Omega_1(H) = \langle x \in H \mid x^2 = 1\rangle = \langle a^2, c\rangle$, thus $\langle a^2\rangle, \langle a^2, c\rangle \lhd H$ and so $\langle a^2\rangle, \langle a^2, c\rangle \lhd G$. The maximal subgroups of H are the following ones: $\langle a, b\rangle \cong \langle a, bc\rangle \cong \langle ac, b\rangle \cong \langle ac, bc\rangle \cong Q_8$, and $\langle a, c\rangle \cong \langle b, c\rangle \cong \langle ab, c\rangle \cong Z_4 \times Z_2$.

We use the bar convention for subgroups and elements of factor groups. For $\overline{G} = G/\langle a^2\rangle$, we have $\overline{H} = H/\langle a^2\rangle = \langle \overline{a}, \overline{b}, \overline{c}\rangle \cong E_8$, and $\overline{G} = \langle \overline{H}, \overline{d}\rangle, \overline{d}^2 \in \overline{H}$. By Lemma 1.3, it is $|C_{\overline{H}}(\overline{d})|^2 \geq |H| = 8$, and so $|C_{\overline{H}}(\overline{d})| \geq 4$. On the other hand, $\langle a^2, c\rangle \lhd G$ implies $\langle \overline{c}\rangle \lhd \overline{G}$ and $\overline{c} \in C_{\overline{H}}(\overline{d})$. As $|C_{\overline{E}}(\overline{d}) \cap \langle \overline{a}, \overline{b}\rangle| \geq 2$, some of the elements $\overline{a}$, $\overline{b}$, or $\overline{a}\overline{b}$ is contained in $C_{\overline{E}}(\overline{d})$, and we can assume without loss that $\overline{a} \in C_{\overline{E}}(\overline{d})$. Now, $\overline{b}^{\overline{d}} \in \overline{H} \backslash \langle \overline{a}, \overline{c}\rangle = \langle \overline{a}, \overline{c}\rangle \cdot \overline{b}$, and so: $G = \langle H, d \mid d^2 \in H,\ a^d = az_1,\ b^d = a^\varepsilon c^\eta b z_2,\ c^d = cz_3\rangle$, $\varepsilon, \eta \in \{0, 1\}$, $z_1, z_2, z_3 \in \{1, a^2\}$.

There are 3 cases with respect to the element d.

1) $\exists d \in G \backslash H$, s.th. $d^2 = 1$.
Since $\langle a^2, c, d\rangle \not\cong E_8$ it must be $c^d \neq c$, and so $c^d = a^2 c$. If $a^d = a^3$, then $(ac)^d = a^d c^d = a^3 a^2 c = ac$, and replacing a with ac, we have without loss $a^d = a$, $c^d = a^2 c$. Now, $b^{d^2} = b^1 = b = (b^d)^d = (a^\varepsilon c^\eta b z_2)^d = a^\varepsilon a^{2\eta} c^\eta a^\varepsilon c^\eta b z_2 z_2 = a^{2(\varepsilon+\eta)} b$. Therefore $\varepsilon = \eta = 0$ or $\varepsilon = \eta = 1$. If $\varepsilon = \eta = 0$, then $b^d = bz_2$. For $z_2 = a^2$ it is $(bc)^d = ba^2a^2c = bc$, and replacing bc with b, we have without loss $b^d = b$. Thus

$$G_1 = \langle H, d \mid d^2 = 1, [a, d] = [b, d] = 1, c^d = a^2 c\rangle = \langle a, b\rangle * \langle c, d\rangle \cong Q_8 * D_8.$$

If $\varepsilon = \eta = 1$, then $b^d = acbz_2 = az_2bc$. Now, replacing a with az_2, we get without loss $b^d = abc$, and the group:

$$G_2 = \langle H, d \mid d^2 = 1, a^d = a, b^d = abc, c^d = a^2 c\rangle.$$

2) $x \in G \backslash H \Rightarrow x^2 \neq 1$, $\exists d \in G \backslash H$ s.th. $d^4 = 1$.
Now $d^2 \in H$ and d^2 is an involution, $d^2 = \{a^2, c, a^2 c\}$. As $a^2 c$ and c are interchangeable, we may assume that $d^2 \in \{a^2, c\}$. If $d^2 = c$, then $c^d = (d^2)^d = d^2 = c$. If $d^2 = a^2$, then $(cd)^2 = cd^2c^d = ca^2cz_2 = a^2 z_3 \neq 1$, by our assumption. Thus $z_3 = 1$, and so $c^d = c$ in both cases.

2a) Case $d^2 = a^2$.
Now, $(ad)^2 = ad^2a^d = a^3az_1 = z_1 \neq 1$, by our assumption. Thus $z_1 = a^2$ and so $a^d = a^3,\ c^d = c,\ d^2 = a^2$. For $\varepsilon = 1$, $b^d = ac^\eta bz_2$. Replacing a with az_2, we may assume $b^d = ac^\eta b$. If $\eta = 1,\ b^d = acb$, and replacing a with ac, we get $b^d = ab$, as in the case $\eta = 0$, and so

$$G_3 = \langle H, d \mid d^2 = a^2, a^d = a^{-1}, b^d = ab, c^d = c\rangle.$$

For $\varepsilon = 0$, it is $b^d = c^\eta bz_2$. If $z_2 = a^2$, then $b^{ad} = (a^2b)^d = a^2c^\eta ba^2 = c^\eta b$. Replacing d with ad, we get $b^d = c^\eta b$. For $\eta = 0,\ b^d = b$ and $(bd)^2 = bd^2b^d = ba^2b = 1$, a contradiction. Thus, $\eta = 1$, $b^d = bc$, and we have:

$$G_4 = \langle H, d \mid d^2 = a^2, a^d = a^{-1}, b^d = bc, c^d = c\rangle.$$

2b) Case $d^2 = c$.
For $\varepsilon = 1,\ b^d = ac^\eta bz_2$, and replacing a with az_2, $b^d = ac^\eta b$. Again if $\eta = 1$, replacing a with ac, we get $b^d = ab$. Now, $b^{d^2} = b = (b^d)^d = (ab)^d = a^db^d = az_1ab \Rightarrow z_1 = a^2$, and so:

$$G_5 = \langle H, d \mid d^2 = c, a^d = a^{-1}, b^d = ab, c^d = c\rangle.$$

For $\varepsilon = 0,\ b^d = c^\eta bz_2$. If $\eta = 1$, then $b^d = cbz_2$, and $(bd)^2 = bd^2b^d = bccbz_2 = b^2z_2 = a^2z_2 \neq 1$, by our assumption. Thus $z_2 = 1$ and $(bd)^2 = a^2$, which leads to the case 2a). Thus we may assume that $\eta = 0$, and so $b^d = bz_2,\ a^d = az_1,\ c^d = c$. Thus $(a^d, b^d) \in \{(a, b), (a, b^3), (a^3, b), (a^3, b^3)\}$. As a, b and ab are interchangeable here, and for $a^d = a^3,\ b^d = b^3 \Rightarrow (ab)^d = a^3b^3 = ab$, there remain, without loss, only two cases: $a^d = a,\ b^d = b$ and $a^d = a,\ b^d = b^3$.
In the latter case $(ad)^2 = ad^2a^d = aca = a^2c,\ a^{ad} = a^d = a,\ b^{ad} = (b^3)^d = b^9 = b,\ (a^2c)^{ad} = (a^2c)^d = a^2c$, and replacing c with a^2c, and d with ad, we get without loss, that $a^d = a,\ b^d = b,\ c^d = c$, and thus

$$G_6 = \langle H, d \mid d^2 = c, [a, d] = [b, d] = [c, d] = 1\rangle.$$

3) $d \in G\backslash H \Rightarrow |d| = 8$.
Now, $d^2 \in H$ and $|d^2| = 4$. As all elements of order 4 in H are interchangeable, we may assume that $d^2 = a$, and so $a^d = a$. Now, $d^2 = a,\ a^d = a,\ b^d = a^\varepsilon c^\eta bz_2,\ c^d = cz_3$. If $\varepsilon = 1$, then $(bd)^2 = bd^2b^d = baac^\eta bz_2 = c^\eta z_2$, an involution, against our assumption. Therefore $\varepsilon = 0$, and $b^d = c^\eta bz_2$. Now, $b^{d^2} = b^a = b^3 = (b^d)^d = (c^\eta bz_2)^d = c^\eta z_3^\eta c^\eta bz_2z_2 = z_3^\eta b \Rightarrow z_3^\eta = b^2 = a^2 \Rightarrow z_3 = a^2,\ \eta = 1$. Thus $b^d = bcz_2,\ c^d = a^2c,\ a^d = a$. If $z_2 = a^2$, replacing c with a^2c, we get $b^d = bc$ and finally:

$$G_7 = \langle H, d \mid d^2 = a, a^d = ab^d = bc, c^d = a^2c\rangle.$$

(b2) G contains a subgroup H isomorphic with $Q_8 * Z_4$ and if $L < G$, then $L \ncong Q_8 \times Z_2$.
Let $H = \langle a, b, c \mid a^4 = 1, b^2 = c^2 = a^2, a^b = a^{-1}, [a, c] = [b, c] = 1\rangle$. Again, $G = \langle H, d\rangle,\ d^2 \in H$. Now $\mho_1(H) = \Phi(H) = \langle a^2\rangle,\ Z(H) = \langle c\rangle$. There are 8 elements of order 4: $a,\ a^3,\ b,\ a^2b,\ ab,\ a^3b,\ c,\ a^2c$, and 7 involutions:

$a^2,\ ac,\ a^3c,\ bc,\ a^2bc,\ abc,\ a^3bc$. The maximal subgroups of H are: $\langle a,b\rangle \cong Q_8,\ \langle a,c\rangle \cong \langle b,c\rangle \cong \langle ab,c\rangle \cong Z_4 \times Z_2$ and $\langle a,bc\rangle \cong \langle b,ac\rangle \cong \langle ab,ac\rangle \cong D_8$. Obviously, $\langle c\rangle, \langle a,b\rangle \lhd H$ and so $\langle c\rangle, \langle a,b\rangle \lhd G$. Again, for $\overline{G} = G/\langle a^2\rangle$, we have $\overline{H} \cong H/\langle a^2\rangle = \langle \overline{a}, \overline{b}, \overline{c}\rangle \cong E_8$, and $|C_{\overline{H}}(\overline{d})| \geq 4$ by Lemma 1.3. We may assume, without loss, that $C_{\overline{H}}(\overline{d}) \geq \langle \overline{a}, \overline{c}\rangle$, which implies $\overline{b}^{\overline{d}} \in \{\overline{b}, \overline{a}\overline{b}\}$. Returning to the originals, we get two cases:
1) $G = \langle H, d \mid d^2 \in H,\ a^d = az_1,\ b^d = bz_2,\ c^d = cz_3\rangle$,
2) $G = \langle H, d \mid d^2 \in H,\ a^d = az_1,\ b^d = abz_2,\ c^d = cz_3\rangle$,
where $z_1, z_2, z_3 \in \langle a^2\rangle$.

Case 1) 1a) $\exists d \in G\backslash H,\ |d| = 2$.
$\langle a^2, ac, d\rangle,\ \langle a^2, bc, d\rangle,\ \langle a^2, abc, d\rangle \ncong E_8$, thus $(ac)^d = acz_1z_3 \neq ac$, $(bc)^d = bcz_2z_3 \neq bc,\ (abc)^d = abcz_1z_2z_3 \neq abc \Rightarrow z_1z_3, z_2z_3, z_1z_2z_3 \neq 1 \Rightarrow z_1 = z_2 = 1,\ z_3 = a^2$, and so

$$G = \langle H, d \mid d^2 = 1, [a,d] = [b,d] = 1, c^d = a^2c\rangle = \langle a,b\rangle * \langle c,d\rangle \cong Q_8 * D_8.$$

But now $G > \langle a,b,d\rangle = \langle a,b\rangle \times \langle d\rangle \cong Q_8 \times Z_2$, against the assumption. G is isomorphic to G_1.

1b) $x \in G\backslash H \Rightarrow x^2 \neq 1,\ \exists d \in G\backslash H,\ |d| = 4$.
Now, d^2 is an involution on H. We may assume, without loss, that $d^2 = a^2$ or $d^2 = ac$. If $d^2 = a^2$, then $(ad)^2 = ad^2a^d = aa^2az_1 = z_1 \neq 1,\ (bd)^2 = z_2 \neq 1$, and $(abd)^2 = z_1z_2 \neq 1$, a contradiction. If $d^2 = ac$, then $b^{d^2} = b^{ac} = b^3 = (b^d)^d = (bz_2)^d = bz_2z_2 = b$, a contradiction again.

1c) $x \in G\backslash H \Rightarrow |x| = 8,\ d \in G\backslash H,\ d^2 \in H$.
We may assume, without loss, that $d^2 = a$, or $d^2 = c$. If $d^2 = a, b^{d^2} = (bz_2)^d = b = b^a = b^3$, a contradiction. Thus $d^2 = c,\ a^d = az_1$,
$b^d = bz_2,\ c^d = c$. Now, $(a^d, b^d) \in \{(a,b), (a,b^3), (a^3,b), (a^3,b^3)\}$. In the latter case $(ab)^d = a^3b^3 = ab$, and since a, b and ab may be replaced with each other, we may assume that: $a^d = a,\ b^d = b$ or $a^d = a,\ b^d = b^3$. In the latter case $(ad)^2 = a^2c,\ b^{ad} = b$, and thus replacing c with a^2c and d with ad, the second case is reduced to the first, and we get

$$G_8 = \langle H, d \mid d^2 = c, [a,d] = [b,d] = [c,d] = 1\rangle = \langle a,b\rangle * \langle d\rangle \cong Q_8 * Z_8.$$

Case 2)
Replacing a with az_2, we may assume that $b^d = ab$.

2a) $\exists d \in G\backslash H,\ |d| = 2$.
$\langle a^2, ac, d\rangle \ncong E_8 \Rightarrow (ac)^d = acz_1z_3 \neq ac \Rightarrow z_3 \neq z_1$. If $a^d = a$, then $b^{d^2} = b = (b^d)^d = (ab)^d = aab = b^3$, a contradiction. Therefore $a^d = a^3,\ c^d = c$, and

$$G_9 = \langle H, d \mid d^2 = 1, a^d = a^{-1}, b^d = ab, c^d = c\rangle.$$

2b) $x \in G\backslash H \Rightarrow x^2 \neq 1,\ \exists d \in G\backslash H,\ |d| = 4$.
Now, without loss $d^2 = a^2$ or $d^2 = ac$ or $d^2 = bc$. If $d^2 = a^2$, then $(ad)^2 = z_1 \neq 1$ and $(cd)^2 = z_3 \neq 1$, thus $z_1 = z_3 = a^2$. So $G = \langle H, d \mid d^2 = a^2,\ a^d = a^3,\ b^d = ab,\ c^d = c^3\rangle$. Here $(ac)^d = a^3c^3 = ac$, and $G > \langle a, d, ac\rangle =$

$\langle a, d\rangle \times \langle ac\rangle \cong Q_8 \times Z_2$, against the assumption. Actually, $G \cong G_2$. If $d^2 = ac$, then $b^{d^2} = (ab)^d = az_1ab = b^{ac} = b^3$, and so $z_1 = 1$. Now, $(ac)^d = ac = a^dc^d = acz_3$, thus also $z_3 = 1$. Therefore:

$$G_{10} = \langle H, d \mid d^2 = ac, a^d = a, b^d = ab, c^d = c\rangle.$$

If $d^2 = bc$, then $(bc)^d = bc = abcz_3 = az_3bc$, implying $az_3 = 1$, a contradiction.

2c) $x \in G\backslash H \Rightarrow |x| = 8,\ d \in G\backslash H,\ d^2 \in H$.

We may assume, without loss, that $d^2 = a$ or $d^2 = b$ or $d^2 = c$. For $d^2 = a$, we get $(bd)^2 = bd^2b^d = baab = 1$, a contradiction. As $b^d = ab$, it cannot be $d^2 = b$. If $d^2 = c$, then $(bd)^2 = bcab = b^2ca^b = a^2ca^{-1} = ac$, and so $|bd| = 4$, a contradiction again.

(c) **Groups of order 64**.

According a previous remark and by Th. 1.1(d), such a group G contains a subgroup

$$H = \langle a, b, c \mid a^4 = b^4 = 1, c^2 = a^2b^2, a^b = a, a^c = a^{-1}, b^c = a^2b^3\rangle,$$

where $\Omega_1(H) = \langle x \in H \mid x^2 = 1\rangle = Z(H) = \Phi(H) = \langle a^2, b^2\rangle \equiv K \cong E_4$. One can easily check that there are only 4 square roots for a^2 (that is, such $x \in H$, that $x^2 = a^2$), and 12 square roots for b^2 and a^2b^2 each. Thus $A = \langle a^2\rangle \lhd H$. The square roots of a^2 generate the subgroup $N = \langle a, b^2\rangle \cong Z_4 \times Z_2$. The group $L = \langle a, b\rangle$ is the unique subgroup of H isomorphic to $Z_4 \times Z_4$. Thus, A, K, N, L are all characteristic in H and consequently normal in G, as $H \lhd G$. It can easily be seen that

$$(*) \begin{cases} AutH = \Phi \sqcup \Psi, \text{where} \\ \Phi = \{\varphi | \varphi : a \mapsto a\zeta_1,\ b \mapsto b\zeta_2,\ c \mapsto a^\alpha b^{2\beta}c\}, \\ \Psi = \{\psi | \psi : a \mapsto a\zeta_1,\ b \mapsto ab\zeta_2,\ c \mapsto a^\alpha b^{1+2\beta}c\}, \\ \text{and } \zeta_1, \zeta_2 \in K,\ \alpha \in \{0,1,2,3\},\ \beta \in \{0,1\}. \end{cases}$$

As $A \lhd K \lhd N \lhd L \lhd H \lhd G$ is a normal chain G, we have:

$$(**)\ G = \langle H, d \mid (H),\ d^2 \in H,\ a^d = az_1,\ b^d = a^\varepsilon bz_2,\ c^d = a^\gamma b^\delta z_3c\rangle,$$

where $z_1, z_2, z_3 \in K$ and $\gamma, \delta, \varepsilon \in \{0, 1\}$ and (H) denotes the relations of H. We split our proof into several steps.

(i) If $T < G,\ |T| = 8$, than T is abelian and $T \not\cong E_8$.

Let T be a nonabelian subgroup of order 8 in G, thus $T \cong D_8$ or $T \cong Q_8$ and $|Z(T)| = 2$. If $C_G(T) \leq T$, then G is metacyclic by Th. 1.4, a contradiction. Hence $C_G(T) \nleq T$ and take in $C_G(T)$ a subgroup U of order 4 containing $Z(T)$. Now $\langle T, U\rangle = T * U$, the central product of T and U, is isomorfic to some of the groups $D_8 \times Z_2,\ Q_8 \times Z_2$, or $Q_8 * Z_4 \cong D_8 * Z_4$. Thus $\langle T, U\rangle$ would be a non-metacyclic subgroup of order 16 in G, a contradiction. Therefore every subgroup T of order 8 in G is abelian and, being metacyclic, it must be $T \not\cong E_8$.

(ii) If $g \in G \backslash H$, then $g^2 \neq 1$.
If $g^2 = 1$, then $T = \langle K, g\rangle = \langle a^2, b^2, g\rangle$ is abelian, by (i), and isomorphic to E_8, a contradiction.

(iii) $G/L \cong E_4$.
Else $G/L \cong Z_4$ and $G = \langle L, d \mid d^2 \in H \setminus L = Lc\rangle$. By (i) we see that we can assume without loss that $d^2 = c$, and so $d^4 = c^2 = a^2b^2$. Now $a^{d^2} = a^c = a^3 = (a^d)^d = (az_1)^d = az_1z_1^d$ implying $z_1^d = a^2z_1$ and thus $z_1 \in \{b^2, a^2b^2\}$ and $(b^2)^d = a^2b^2 = c^2 = (c^2)^d$, and so $b^2 = c^2$, a contradiction.

(iv) If there exists some $d \in G\backslash H$, s.th. $|d| = 4$, then either $G \cong G_1 = \langle a, b, c, d \mid (H),\ d^2 = a^2,\ a^d = a^3b^2,\ b^d = b^3,\ c^d = c\rangle$, or $G \cong G_2 = \langle a, b, c, d \mid (H),\ d^2 = b^2,\ a^d = a^3,\ b^d = a^2b^3,\ c^d = ac\rangle$. As $d^2 \in L$ and d^2 is an involution, we have $d^2 \in \Omega_1(L)^\sharp = a^2, b^2, a^2b^2 = K^\sharp$. By (i), $\langle a^2, b^2, d\rangle$ is abelian, and so $b^d = bz_2$. Now, $(c^2)^d = c^2 = (c^d)^2 = (a^\gamma b^\delta z_3c)^2 = a^\gamma b^\delta z_3c^2a^{3\gamma}a^{2\delta}b^{3\delta}z_3 = a^{2\delta}c^2$, implying $d = 0$ and $c^d = a^\gamma z_3c$. From $c^{ad} = (a^2c)^d = a^2c^d$, $c^{bd} = (a^2b^2c)^d = a^2b^2c^d$, and $c^{abd}(a^2c)^{bd} = (a^2a^2b^2c)^d = b^2c^d$, and, by (ii), $(ad)^2 = ad^2a^d = a^2d^2z_1 \neq 1$, $(bd)^2 = b^2d^2z_2 \neq 1$ and $(abd)^2 = a^2b^2d^2z_1z_2 \neq 1$, we conclude that

$$z_1 \neq a^2d^2,\ z_2 \neq b^2d^2,\ z_1z_2 \neq a^2b^2d^2, \tag{2.1}$$

and replacing, if needed, d by ad or bd or abd, we can assume, without loss, that $c^d = a^\gamma c$, and so:

$$c^d = c \quad or \quad c^d = ac. \tag{2.2}$$

Case 1) $c^d = c$.
Now $(cd)^2 = c^2d^2 = a^2b^2d^2 \neq 1$, by (ii), and so $d^2 \neq a^2b^2$. Therefore $d^2 \in \{a^2, b^2\}$. If $d^2 = b^2$, then $(cd)^2 = c^2d^2 = a^2$. Thus, replacing d by cd, we may assume without loss that $d^2 = a^2$. Besides of (1), we also have now the following conditions on z_1, z_2: $(acd)^2 = acd^2az_1c = ac^2d^2a^cz_1 = c^2d^2z_1 \neq 1$, and similarly $(bcd)^2 = b^2d^2z_2 \neq 1$, $(abcd)^2 = abcd^2az_1bz_2c = b^2d^2z_1z_2 \neq 1$, that is:

$$z_1 \neq c^2d^2,\ z_2 \neq b^2d^2,\ z_1z_2 \neq b^2d^2. \tag{2.3}$$

From (2.1) and (2.3) we get: $z_1 \neq 1, b^2$, $z_2 \neq a^2b^2$, $z_1z_2 \neq b^2, a^2b^2$. If $z_1 = a^2$, then $a^d = a^3$, and $\langle a, d \mid a^4 = 1,\ d^2 = a^2,\ a^d = a^3\rangle \cong Q_8$, a contradiction, by (i). Thus $z_1 = a^2b^2$, $z_2 \neq a^2b^2$, $a^2b^2z_2 \neq b^2, a^2b^2$, and so $z_2 = b^2$, giving: $G_1 = \langle a, b, c, d \mid (H), d^2 = a^2,\ a^d = a^3b^2,\ b^d = b^3,\ c^d = c\rangle$.

Case 2) $c^d = ac$.
We already know that $a^d = az_1$, $b^d = bz_2$ and $d^2 \in \{a^2, b^2, a^2b^2\}$. Now $c = c^{d^2} = (c^d)^d = (ac)^d = az_1ac = a^2z_1c$, so $z_1 = a^2$ and $a^d = a^3$. If $d^2 = a^2$, then $\langle a, d \mid a^4 = 1,\ d^2 = a^2,\ a^d = a^3\rangle \cong Q_8$, against (i). Therefore $d^2 \in \{b^2, a^2b^2\}$. Since $(bd)^2 = bd^2b^d = b^2z_2d^2$, we have $z_2 \neq b_2d_2$.

Case 2.1) $d^2 = b^2$.
From $z_2 \neq b^2d^2$ it follows $z_2 \in \{a^2, b^2, a^2b^2\}$. If $z_2 = a^2$, then $b^d = a^2b$ and $\langle a, bd \mid a^4 = 1,\ (bd)^2 = a^2,\ a^{bd} = a^3\rangle \cong Q_8$, against (i). Similarly, for $z_2 = b^2$ we have $\langle b, d \mid b^4 = 1,\ d^2 = b^2,\ b^d = b^3\rangle \cong Q_8$ again. It remains as the only possibility $z_2 = a^2b^2$, and we obtain the group

$$G_2 = \langle a, b, c, d \mid (H), d^2 = b^2,\ a^d = a^3,\ b^d = a^2b^3,\ c^d = ac\rangle.$$

Case 2.2) $d^2 = a^2b^2$.
Because of $z_2 \neq b^2d^2 = a^2$, we have now $z_2 \in \{1, b^2, a^2b^2\}$, that is $b^d \in \{b, b^3, a^2b^3\}$. If $b^d = b$, then $(bd)^2 = bd^2b^d = ba^2b^2b = a^2$, and $\langle a, bd\rangle \cong Q_8$, against (i). If $b^d = b^3$, then $(ab)^d = (ab)^3$ and $\langle ab, d\rangle \cong Q_8$, again the same contradiction. Therefore $b^d = a^2b^3$. Replacing a by $a' = a^3b^2$, b by $b' = ab$, c by $c' = bc$, we get:

$$d^2 = b'^2,\ a'^d = (a^3b^2)^d = ab^2 = a'^3,$$
$$b'^d = (ab)^d = a^3a^2b^3 = ab^3 = (a^3b^2)^2 \cdot (ab)^3 = a'^2b'^3,$$
$$c'^d = (bc)^d = a^2b^3ac = a^3b^2bc = a' \cdot c',$$

that is the relations of G_2. Thus, this group is isomorphic to G_2.

(v) If all elements in $G\backslash H$ are of order 8, then

$$G = G_3 = \langle a, b, c, d \mid (H), d^2 = b,\ a^d = ab^2,\ b^d = b,\ c^d = ac\rangle.$$

As $G/L \cong E_4$, $G = \langle H, d\rangle$, $|d| = 8$, it follows that d^2 is an element of order 4 in L. According to $(*)$, all such elements are replaceable by a or b, and so we may assume without loss that

$$G = \langle H, d \mid (H), d^2 \in \{a, b\},\ a^d = az_1,\ b^d = a^\varepsilon bz_2,\ c^d = a^\gamma b^\delta z_3c\rangle,$$

where $z_1, z_2, z_3 \in K$, $\gamma, \delta, \varepsilon \in \{0, 1\}$. Now $(c^2)^d = (c^d)^2 = (a^\gamma b^\delta z_3c)^2 = a^\gamma b^\delta z_3c^2a^{3\gamma}a^{2\delta}b^{3\delta}z_3 = a^{2\delta}c^2$.

Case 1) $d^2 = a$.
Now $a^d = a$. If $\varepsilon = 0$ then $z^d = z$ for $z \in K$, and from $(c^2)^d = c^2 = a^{2\delta}c^2$ it follows $\delta = 0$, $c^d = a^\gamma z_3c$. Similarly, from $c^{d^2} = c^a = a^2c = (a^\gamma z_3c)^d = a^\gamma z_3a^\gamma z_3c = a^{2\gamma}c$, we get $\gamma = 1$ and $c^d = az_3c$. But now $(cd)^2 = cd^2c^d = a^2c^2z_3 \in K$, and $|cd| = 4$, against our assumption. If $\varepsilon = 1$, then $b^d = abz_2$, $(b^2)^d = c^2$ and $(c^2)^d = b^2 = a^{2\delta}c^2$. Thus $\delta = 1$, and $c^d = a^\gamma bz_3c$. Therefore, $c^{d^2} = a^2c = (a^\gamma bz_3c)^d = a^\gamma \cdot abz_2z_3^da^\gamma bz_3c = a \cdot a^{2\gamma}bb^2z_2z_3z_3^dc = azc$, for some $z \in K$. This implies $z = a$, a contradiction because a is not in K.

Case 2) $d^2 = b$.
Now $b^d = b$, and $z^d = z$ for $z \in K$. From $(c^2)^d = c^2 = a^{2\delta}c^2$ it follows $\delta = 0$, $c^d = a^\gamma z_3c$. Since $[b, c] = b^{-1}b^c = a^2b^2 = [c, b]$, we have $c^{d^2} = c^b = a^2b^2c = (a^\gamma z_3c)^d = a^\gamma z_1^\gamma z_3a^\gamma z_3c$, implying $a^{2\gamma}z_1^\gamma = a^2b^2$. Thus $\gamma = 1, z_1 = b^2$ and so $a^d = ab^2$, $c^d = az_3c$. Replacing a by az_3, we get the group G_3 as stated above.

Remark 2.1. It is of some interest to check the maximal subgroups of second-metacyclic groups. We present their distribution in each of these groups in the following table:

(i) —G—=32:

$$\begin{aligned}
G_1 : &\quad 5 \cdot (Q_8 \times Z_2),\ 10 \cdot (Q_8 * Z_4);\\
G_2 : &\quad Q_8 \times Z_2,\ Q_8 * Z_4,\ 2 \cdot SD_{16},\ 2 \cdot Q_{16},\ M_{16};\\
G_3 : &\quad 2 \cdot (Q_8 \times Z_2),\ Z_8 \times Z_2,\ 4 \cdot Q_{16};\\
G_4 : &\quad 2 \cdot (Q_8 \times Z_2),\ Z_4 \times Z_4,\ 4 \cdot (Z_4 \cdot Z_4);\\
G_5 : &\quad Q_8 \times Z_2,\ Z_4 \cdot Z_4,\ Z_8 \times Z_2\\
G_6 : &\quad Q_8 \times Z_2,\ 3 \cdot (Z_4 \times Z_4),\ 3 \cdot (Z_4 \cdot Z_4);\\
G_7 : &\quad Q_8 \times Z_2,\ 2 \cdot M_{16};\\
G_8 : &\quad Q_8 * Z_4,\ 3 \cdot (Z_8 \times Z_2),\ 3 \cdot M_{16};\\
G_9 : &\quad 2 \cdot (Q_8 * Z_4),\ Z_8 \times Z_2,\ 2 \cdot SD_{16},\ Q_{16},\ D_{16};\\
G_{10} : &\quad Q_8 * Z_4,\ Z_4 \times Z_4,\ M_{16}.
\end{aligned}$$

(ii) $|G| = 64$:
Denoting $H_r = \langle a, b \mid a^8 = b^4 = 1,\ a^b = a^r \rangle$, we have:

$$\begin{aligned}
G_1 : &\quad 15 \cdot H;\\
G_2 : &\quad 2 \cdot H,\ 2 \cdot H_3,\ 2 \cdot H_7,\ Z_8 \times Z_4;\\
G_3 : &\quad H,\ 2 \cdot H_5.
\end{aligned}$$

The factor groups $\overline{G}_i = G_i/\langle a^2 \rangle,\ i = 1, 2, 3$ are isomorphic to the groups $G_1,\ G_4$ and G_7 of order 32 from (i), respectively.

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