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GLOBAL IN TIME ESTIMATES FOR ONE-DIMENSIONAL COMPRESSIBLE VISCOUS MICROPOLAR FLUID MODEL

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Abstract. An initial-boundary value problem for one-dimensional flow of a compressible viscous heat-conducting micropolar fluid is considered. It is assumed that the fluid is thermodynamically perfect and polytropic. A problem has a unique strong solution on $]0,1[\times]0,T[$, for each $T>0$. Using this result we obtain a priori estimates for the solution independent of T.

1. Introduction

Micropolar fluids are fluids with microstructure. They belong to a class of fluids with nonsymmetric stress tensor that we call polar fluids. The model of micropolar fluids is introduced in [3] by C.A. Eringen as a significant generalization of the classical Navier-Stokes model. Some problems for three-dimensional flow of a incompressible viscous micropolar fluid are presented in the book [6], in particular their mathematical theories.

We consider nonstationary one-dimensional flow of a compressible and heat-conducting micropolar fluid. The equations of motion for this fluid are derived from integral form of conservation laws for polar fluids, under a number of supplementary assumtions such as selection of constitutive equations, Fourier's law, Boyle's law and polytropy (see [8]). A corresponding initial-boundary value problem has a unique strong solution on $]0,1[\times]0,T[$, for each $T>0$ ([9]). Considering stabilisation problem, one has to prove a priori estimates for the solution independent of T, what is the main difficulty. We obtain these estimates using the results from [9]. In our proof we follow some ideas of S.N. Antontsev, A.V. Kazhykhov and V.N. Monakhov, applied to 1-D initial-boundary value problem for a classical fluid ([1]).

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2. Statement of the problem and the main result

Let ρ, v, ω and θ denote, respectively, the mass density, velocity, microrotation velocity and temperature of the fluid in the Lagrangean description.

Governing equations of the flow under consideration are as follows ([9]):

$$\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial v}{\partial x} = 0, \tag{2.1}$$

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}(\rho \frac{\partial v}{\partial x}) - K \frac{\partial}{\partial x}(\rho\theta), \tag{2.2}$$

$$\rho \frac{\partial \omega}{\partial t} = A[\rho \frac{\partial}{\partial x}(\rho \frac{\partial \omega}{\partial x}) - \omega], \tag{2.3}$$

$$\rho \frac{\partial \theta}{\partial t} = -K\rho^2\theta \frac{\partial v}{\partial x} + \rho^2 (\frac{\partial v}{\partial x})^2 + \rho^2 (\frac{\partial \omega}{\partial x})^2 + \omega^2 + D\rho \frac{\partial}{\partial x}(\rho \frac{\partial \theta}{\partial x}) \tag{2.4}$$

in $]0,1[\times R^+$, where K, A and D are positive constants. We take the homogeneous boundary conditions

$$v(0,t) = v(1,t) = 0, \tag{2.5}$$

$$\omega(0,t) = \omega(1,t) = 0, \tag{2.6}$$

$$\frac{\partial \theta}{\partial x}(0,t) = \frac{\partial \theta}{\partial x}(1,t) = 0 \tag{2.7}$$

for $t > 0$ and non-homogeneous initial conditions

$$\rho(x,0) = \rho_0(x), \tag{2.8}$$

$$v(x,0) = v_0(x), \tag{2.9}$$

$$\omega(x,0) = \omega_0(x), \tag{2.10}$$

$$\theta(x,0) = \theta_0(x) \tag{2.11}$$

for $x \in \Omega =]0,1[$, where ρ_0, v_0, ω_0 and θ_0 are given functions. We assume that there exist $m, M \in R^+$, such that

$$0 < m \le \rho_0(x) \le M, \;\; m \le \theta_0(x) \le M, \;\; x \in \Omega. \tag{2.12}$$

Let

$$\rho_0, \theta_0 \in H^1(\Omega), \; v_0, \omega_0 \in H_0^1(\Omega). \tag{2.13}$$

Then for each $T > 0$ the problem (2.1)-(2.11) has a unique strong solution ([9])

$$(x,t) \to (\rho, v, \omega, \theta)(x,t), \; (x,t) \in Q_T = \Omega\times]0,T[\tag{2.14}$$

with the properties:

$$\rho \in L^\infty(0,T; H^1(\Omega)) \cap H^1(Q_T), \tag{2.15}$$

$$v, \omega, \theta \in L^\infty(0, T; H^1(\Omega)) \cap H^1(Q_T) \cap L^2(0, T; H^2(\Omega)), \tag{2.16}$$

$$\rho > 0,\ \theta > 0 \ \text{ on } \ \bar{Q}_T. \tag{2.17}$$

From embedding and interpolation theorems ([5]) one can conclude that from (2.15) and (2.16) it follows:

$$\rho \in L^\infty(0, T; C(\bar{\Omega})) \cap C([0, T]; L^2(\Omega)), \tag{2.18}$$

$$v, \omega, \theta \in L^2(0, T; C^1(\bar{\Omega})) \cap C([0, T], H^1(\Omega)), \tag{2.19}$$

$$\rho, v, \omega, \theta \in C(\bar{Q}_T). \tag{2.20}$$

Our purpose is to prove the following result.

THEOREM 2.1. *Let the initial functions satisfy (2.12) and (2.13). Then the problem (2.1)-(2.11) has a solution*

$$(x, t) \to (\rho, v, \omega, \theta)(x, t),\ \ (x, t) \in Q = \Omega \times]0, \infty[, \tag{2.21}$$

having the properties:

$$\rho \in L^\infty(0, \infty; H^1(\Omega)), \tag{2.22}$$

$$\frac{\partial \rho}{\partial t} \in L^\infty(0, \infty; L^2(\Omega)) \cap L^2(Q),\ \ \frac{\partial \rho}{\partial x} \in L^2(0, \infty; L^2(\Omega)) \tag{2.23}$$

$$v, \omega \in L^\infty(0, \infty; H^1(\Omega)) \cap L^2(0, \infty; H^2(\Omega)) \cap H^1(Q), \tag{2.24}$$

$$\theta \in L^\infty(0, \infty; H^1(\Omega)), \tag{2.25}$$

$$\frac{\partial \theta}{\partial x} \in L^2(0, \infty; H^1(\Omega)),\ \ \frac{\partial \theta}{\partial t} \in L^2(Q). \tag{2.26}$$

3. Some properties of the solution (2.14)

In that what follows we denote by $C > 0$ a generic constant, not depending of $T > 0$ and having possibly different values at different places. We use some of considerations from [1, 9, 10]. In these cases we omit proofs, making reference to correspondent pages of [1, 9, 10]. Let

$$\frac{1}{2}\|v_0\|^2 + \frac{1}{2A}\|\omega_0\|^2 + \|\theta_0\|_{L^1(\Omega)} = E_1, \tag{3.1}$$

$$\frac{1}{2A}\|\omega_0\|^2 = E_2, \tag{3.2}$$

where $\|\| = \|\|_{L^2(\Omega)}$.

Lemma 3.1. *([9],pp.201, 205) For $t > 0$ it holds*

$$\frac{1}{2}\|v(.,t)\|^2 + \frac{1}{2A}\|\omega(.,t)\|^2 + \|\theta(.,t)\|_{L^1(\Omega)} = E_1, \tag{3.3}$$

$$\frac{1}{2A}\|\omega(.,t)\|^2 + \int_0^t \int_0^1 \rho(\frac{\partial \omega}{\partial x})^2 dx d\tau + \int_0^t \int_0^1 \frac{\omega^2}{\rho} dx d\tau = E_2. \tag{3.4}$$

Let

$$I_0 = \int_0^1 [\frac{1}{2K}v_0^2 + \frac{1}{2AK}\omega_0^2 + \rho_0^{-1}(\rho_0 \ln \rho_0 - \rho_0 + 1) + \frac{1}{K}(\theta_0 - \ln \theta_0 - 1)]dx, \tag{3.5}$$

$$U(t) = \int_0^1 [\frac{1}{2K}v^2 + \frac{1}{2AK}\omega^2 + \rho^{-1}(\rho \ln \rho - \rho + 1) + \frac{1}{K}(\theta - \ln \theta - 1)]dx, \tag{3.6}$$

$$V(t) = \frac{1}{K}\int_0^1 [\frac{\rho}{\theta}(\frac{\partial v}{\partial x})^2 + \frac{\rho}{\theta}(\frac{\partial \omega}{\partial x})^2 + \frac{\omega^2}{\rho\theta} + D\frac{\rho}{\theta^2}(\frac{\partial \theta}{\partial x})^2]dx. \tag{3.7}$$

Lemma 3.2 ([10]). *For $t > 0$ it holds*

$$U(t) + \int_0^t V(\tau) d\tau = I_0. \tag{3.8}$$

Let

$$\int_0^1 \frac{1}{\rho_0(x)} dx = \alpha, \tag{3.9}$$

$$\sigma = \rho\frac{\partial v}{\partial x} - K\rho\theta. \tag{3.10}$$

Lemma 3.3 ([1], pp. 84-85). *For each $t > 0$ there exists at least one point $x_0 = x_0(t) \in \bar{\Omega}$ such that*

$$\begin{aligned} \int_0^t \sigma(x_0(t), \tau) d\tau = &-\frac{1}{\alpha}\int_0^t \int_0^1 (v^2 + K\theta) dx d\tau \\ &+ \frac{1}{\alpha}\int_0^1 v_0(x) \int_x^1 \frac{d\xi}{\rho_0(\xi)} dx - \int_0^{x_0(t)} v_0(\xi) d\xi. \end{aligned} \tag{3.11}$$

Let

$$m_\rho(t) = \min_{x \in \bar{\Omega}} \rho(x,t), \ M_\theta(t) = \max_{x \in \bar{\Omega}} \theta(x,t) \tag{3.12}$$

and

$$G(t) = \int_0^1 \frac{\rho}{\theta^2}(\frac{\partial \theta}{\partial x})^2 dx. \tag{3.13}$$

LEMMA 3.4. *There exists a constant $C \in R^+$ such that for each $t > 0$ the inequality*

$$M_\theta(t) \le C(1 + m_\rho^{-1}(t)G(t)) \tag{3.14}$$

holds true.

PROOF. Let

$$\psi(x,t) = \sqrt{\theta(x,t)} - \int_0^1 \sqrt{\theta(\xi,t)}d\xi. \tag{3.15}$$

Then for $t > 0$ it holds $\int_0^1 \psi(x,t)dx = 0$ and therefore there exists $x_1 = x_1(t) \in \bar{\Omega}$ such that $\psi(x_1(t),t) = 0$. With the help of (3.15) and the Hölder inequality we find that

$$\begin{aligned} \psi(x,t) &= \int_{x_1}^x \frac{\partial}{\partial \xi}\psi(\xi,t)d\xi \le \frac{1}{2}\int_0^1 \frac{1}{\sqrt{\theta(\xi,t)}}|\frac{\partial\theta}{\partial\xi}|d\xi \\ &\le \frac{1}{2}(\int_0^1 \frac{\rho}{\theta^2}(\frac{\partial\theta}{\partial\xi})^2 d\xi)^{1/2}(\int_0^1 \rho^{-1}\theta d\xi)^{1/2} \end{aligned} \tag{3.16}$$

and hence

$$\theta(x,t) \le C(\int_0^1 \theta(\xi,t)d\xi + \int_0^1 \frac{\rho}{\theta^2}(\frac{\partial\theta}{\partial\xi})^2 d\xi \cdot m_\rho^{-1}(t)\int_0^1 \theta(\xi,t)d\xi). \tag{3.17}$$

Taking into account (3.3) and (3.13) from (3.17) we obtain (3.14). $\square$

LEMMA 3.5. *The following equality holds*

$$\rho(x,t) = \frac{\rho_0(x)\exp\{\frac{1}{\alpha}\int_0^t\int_0^1(v^2+K\theta)dxd\tau + A(x,t)\}}{1 + K\rho_0(x)\int_0^t \theta(x,\tau)\exp\{\frac{1}{\alpha}\int_0^\tau\int_0^1(v^2+K\theta)dxds + A(x,\tau)\}d\tau} \tag{3.18}$$

where

$$A(x,t) = \int_0^x v_0(\xi)d\xi - \int_{x_0(t)}^x v(\xi,t)d\xi - \frac{1}{\alpha}\int_0^1 v_0(x)\int_x^1 \frac{d\xi}{\rho_0(\xi)}dx. \tag{3.19}$$

PROOF. Taking into account (3.10) from (2.2) we get

$$\frac{\partial v}{\partial t} = \frac{\partial\sigma}{\partial x}. \tag{3.20}$$

Integrating (3.20) over $]0,t[, t > 0$ and (for a given t) over $]x_0(t),x[, x \in [0,1]$, (where $x_0(t)$ is defined in Lemma 3.3) we obtain

$$\int_{x_0(t)}^x (v - v_0)d\xi = \int_0^t \sigma(x,\tau)d\tau - \int_0^t \sigma(x_0(t),\tau)d\tau. \tag{3.21}$$

Using (2.1) for the function σ we have

$$\int_0^t \sigma d\tau = -\int_0^t \frac{1}{\rho}\frac{\partial\rho}{\partial t}d\tau - K\int_0^t \rho\theta d\tau = -\ln\frac{\rho}{\rho_0} - K\int_0^t \rho\theta d\tau. \tag{3.22}$$

With the help of (3.11) and (3.22) from (3.21) we get

$$\begin{aligned}\rho(x,t)\exp\{K\int_0^t \rho\theta d\tau\} = \\ \rho_0(x)\exp\{\frac{1}{\alpha}\int_0^t\int_0^1 (v^2+K\theta)dxd\tau + A(x,t)\}.\end{aligned} \tag{3.23}$$

Multiplying (3.23) by $K\theta$ and integrating over $]0,t[, t>0$, we have

$$\begin{aligned}&\exp\{K\int_0^t \rho\theta d\tau\} \\ &= 1 + K\rho_0(x)\int_0^t \theta(x,\tau)\exp\{\frac{1}{\alpha}\int_0^\tau\int_0^1 (v^2+K\theta)dxds + A(x,\tau)\}d\tau.\end{aligned} \tag{3.24}$$

From (3.24) and (3.23) it follows (3.18). □

LEMMA 3.6. *There exists a constant $C\in R^+$ such that for each $t>0$ it holds*

$$m_\rho(t) \geq C. \tag{3.25}$$

PROOF. Let $K\geq 2$. We introduce the non-negative function $B_1(t), t>0$, by

$$B_1(t) = 2(\int_0^t\int_0^1 \rho(\frac{\partial\omega}{\partial x})^2 dxd\tau + \int_0^t\int_0^1 \frac{\omega^2}{\rho}dxd\tau) + (K-2)\int_0^1 \theta dx. \tag{3.26}$$

With the help of (3.4) from (3.3) we obtain

$$\|v(.,t)\|^2 + K\int_0^1 \theta(x,t)dx = C + B_1(t). \tag{3.27}$$

Taking into account (3.27), (2.12), (3.12) and the estimate $|A(x,t)| \leq 2\|v_0\| + \|v(.,t)\| \leq C$ from (3.18) we find the inequality

$$\begin{aligned}\rho^{-1}(x,t) &\leq \frac{C(1+\exp\{\frac{1}{\alpha}\int_0^t B_1(s)ds\}\int_0^t M_\theta(\tau)\exp\{\frac{1}{\alpha}C\tau\}d\tau)}{\exp\{\frac{1}{\alpha}Ct\}\exp\{\frac{1}{\alpha}\int_0^t B_1(\tau)d\tau\}} \\ &\leq C\exp\{-\frac{1}{\alpha}Ct\}(1+\int_0^t M_\theta(\tau)\exp\{\frac{1}{\alpha}C\tau\}d\tau).\end{aligned} \tag{3.28}$$

Using (3.14) from (3.28) it follows

$$m_\rho^{-1}(t) \leq C(1 + \exp\{-\frac{1}{\alpha}Ct\} \int_0^t m_\rho^{-1}(\tau) G(\tau) \exp\{\frac{1}{\alpha}C\tau\} d\tau) \tag{3.29}$$
$$\leq C(1 + \int_0^t m_\rho^{-1}(\tau) G(\tau) d\tau)$$

or, because of Gronwall's inequality,

$$m_\rho^{-1}(t) \leq C \exp\{\int_0^t G(\tau) d\tau\}. \tag{3.30}$$

With the help of (3.13) and (3.8) we conclude that $G \in L^1(]0, \infty[)$. Then the estimate (3.25) follows immediately from (3.30).

Let now be $K < 2$ and

$$B_2(t) = K(\int_0^t \int_0^1 \rho(\frac{\partial \omega}{\partial x})^2 dx d\tau + \int_0^t \int_0^1 \frac{\omega^2}{\rho} dx d\tau) + \frac{2-K}{2} \|v(.,t)\|^2. \tag{3.31}$$

Using (3.3) and (3.4) we obtain

$$\|v(.,t)\|^2 + K \int_0^1 \theta(x,t) dx = C + B_2(t). \tag{3.32}$$

We have again $B_2(t) \geq 0, t > 0$. Inserting (3.32) into (3.18) in the same way as before we get (3.25). ∎

Let

$$M_\rho(t) = \max_{x \in \bar{\Omega}} \rho(x,t), \quad m_\theta(t) = \min_{x \in \bar{\Omega}} \theta(x,t). \tag{3.33}$$

Lemma 3.7. *There exists a constant $C \in R^+$ such that for each $t > 0$ it holds*

$$M_\rho(t) \leq C. \tag{3.34}$$

Proof. With the help of the Hölder inequality we obtain

$$v^2(x,t) \leq (\int_0^1 |\frac{\partial v}{\partial x}| dx)^2 \leq \int_0^1 \frac{\rho}{\theta} (\frac{\partial v}{\partial x})^2 dx \cdot m_\rho^{-1}(t) \int_0^1 \theta dx, \tag{3.35}$$

$$\omega^2(x,t) \leq (\int_0^1 |\frac{\partial \omega}{\partial x}| dx)^2 \leq \int_0^1 \frac{\rho}{\theta} (\frac{\partial \omega}{\partial x})^2 dx \cdot m_\rho^{-1}(t) \int_0^1 \theta dx \tag{3.36}$$

and using (3.25), (3.3), (3.7) and (3.8) we conclude that there exists $C \in R^+$ such that for each $t > 0$ it holds

$$\int_0^t \max_{x \in \bar{\Omega}} v^2(x,\tau) d\tau \leq C, \int_0^t \max_{x \in \bar{\Omega}} \omega^2(x,\tau) d\tau \leq C. \tag{3.37}$$

Now, we write the equality (3.3) in the form

$$\int_0^1 \theta(x,t)dx = E_1 - \frac{1}{2}\|v(.,t)\|^2 - \frac{1}{2A}\|\omega(.,t)\|^2 \tag{3.38}$$

and with the help of (3.37) we conclude that

$$E_1 t - C \leq \int_0^t \int_0^1 \theta(x,\tau)dxd\tau \leq E_1 t + C. \tag{3.39}$$

Using the estimates (2.12), (3.37) and (3.39) from (3.18) we obtain

$$M_\rho(t) \leq \frac{C\exp\{\frac{KE_1}{\alpha}t\}}{1+\int_0^t m_\theta(\tau)\exp\{\frac{KE_1}{\alpha}\tau\}d\tau}. \tag{3.40}$$

Now we intend to get the estimate for the function $m_\theta(t), t > 0$. Therefore we introduce the function ψ_1 by

$$\psi_1(m_\theta(t), h(t)) = \int_{m_\theta(t)}^{h(t)} \frac{(s-\ln s)^{1/2}}{s}ds, \tag{3.41}$$

where $h(t) = E_1 - \frac{1}{2}\|v(.,t)\|^2 - \frac{1}{2A}\|\omega(.,t)\|^2 = \int_0^1 \theta(x,t)dx$. One can easy conclude that it holds

$$\psi_1(m_\theta(t), h(t)) \geq \frac{1}{E_1}(h(t) - m_\theta(t)). \tag{3.42}$$

Taking into account that there exist $x_1(t), x_2(t) \in \bar{\Omega}$ such that $m_\theta(t) = \theta(x_1(t),t)$ and $h(t) = \theta(x_2(t),t)$ and applying the Hölder inequality, from (3.41) and (3.42) we find that

$$\begin{aligned}\frac{1}{E_1}(h(t)-m_\theta(t)) \leq \psi_1(h(t), m_\theta(t)) &= \int_{m_\theta}^{h(t)} \frac{(s-\ln s)^{1/2}}{s}ds \\ &= \int_{x_1}^{x_2} \frac{(\theta(x,t)-\ln\theta(x,t))^{1/2}}{\theta(x,t)} \cdot \frac{\partial\theta}{\partial x}dx \\ &\leq (\int_0^1 (\theta-\ln\theta)dx)^{1/2} \cdot (\int_0^1 \frac{\rho}{\theta^2}(\frac{\partial\theta}{\partial x})^2 dx)^{1/2} \cdot m_\rho^{-1/2}(t).\end{aligned} \tag{3.43}$$

Using (3.6), (3.8), (3.13) and (3.25) from (3.43) we get immediately

$$m_\theta(t) \geq h(t) - CG^{1/2}(t), \quad t \geq 0. \tag{3.44}$$

Applying the Young inequality with a parameter $\varepsilon > 0$ on the second term on the right-hand side of (3.44) we have

$$m_\theta(t) \geq E_1 - \frac{1}{2}\|v(.,t)\|^2 - \frac{1}{2A}\|\omega(.,t)\|^2 - \varepsilon - C_\varepsilon G(t). \tag{3.45}$$

Because $m_\theta(t) > 0, t > 0$, we conclude that

$$m_\theta(t) \geq c - \min\{c, f_0(t)\}, \tag{3.46}$$

where

$$c = E_1 - \varepsilon, \quad f_0(t) = \frac{1}{2}\|v(.,t)\|^2 + \frac{1}{2A}\|\omega(.,t)\|^2 + C_\varepsilon G(t) \in L^1(]0,\infty[). \tag{3.47}$$

Obviously,

$$f_1(t) = \min\{c, f_0(t)\} \in L^1(]0,\infty[). \tag{3.48}$$

Let $\varepsilon > 0$ be such that $0 < \frac{c\alpha}{KE_1} < 1$. Using (3.48) and (3.46) from (3.40) we obtain

$$\begin{aligned} M_\rho(t) &\le \frac{C\exp\{\frac{KE_1}{\alpha}t\}}{1 + \frac{c\alpha}{KE_1}(\exp\{\frac{KE_1}{\alpha}t\} - 1) - \int_0^t f_1(\tau)\exp\{\frac{KE_1}{\alpha}\tau\}d\tau} \\ &\le \frac{C}{1 - \frac{KE_1}{c\alpha}\varphi(t)} \end{aligned} \tag{3.49}$$

where

$$\varphi(t) = \exp\{-\frac{KE_1}{\alpha}t\}\int_0^t f_1(\tau)\exp\{\frac{KE_1}{\alpha}\tau\}d\tau. \tag{3.50}$$

One can easily conclude that $\lim_{t\to\infty}\varphi(t) = 0$. Because of it a function φ that is non-negative and continuous on $]0,\infty[$, takes its maximum at a point $t_0 \in]0,\infty[$. It holds

$$\begin{aligned} \varphi(t_0) = \sup_{t>0}\varphi(t) &= \exp\{-\frac{KE_1}{\alpha}t_0\}\int_0^{t_0} f_1(\tau)\exp\{\frac{KE_1}{\alpha}\tau\}d\tau \\ &\le \exp\{-\frac{KE_1}{\alpha}t_0\}\frac{c\alpha}{KE_1}(\exp\{\frac{KE_1}{\alpha}t_0\} - 1) < \frac{c\alpha}{KE_1} \end{aligned} \tag{3.51}$$

and we get

$$1 - \frac{KE_1}{c\alpha}\varphi(t_0) > 0. \tag{3.52}$$

The estimate (3.34) follows from (3.52) and (3.49). ∎

4. Proof of Theorem 2.1.

The proof is based on a priori estimates (obtained in Section 3) for the function (2.14) not depending of T.

Lemma 4.1. *It holds*

$$\rho \in L^\infty(Q), \tag{4.1}$$

$$v, \omega \in L^\infty(0,\infty; L^2(\Omega)) \cap L^2(0,\infty; L^2(\Omega)), \tag{4.2}$$

$$\frac{\partial \omega}{\partial x} \in L^2(0,\infty; L^2(\Omega)), \tag{4.3}$$

$$\theta \in L^\infty(0,\infty; L^2(\Omega)), \tag{4.4}$$

$$\frac{\partial \theta}{\partial x} \in L^2(0,\infty;L^2(\Omega)). \tag{4.5}$$

Proof. The conclusions (4.1), (4.2) and (4.3) follow immediately from (3.3), (3.4), (3.37), (3.25) and (3.34).

As in [9] we introduce a total energy

$$\phi = \frac{1}{2}v^2 + \frac{1}{2A}\omega^2 + \theta. \tag{4.6}$$

We multiply the equations (2.2), (2.3) and (2.4) respectively by $v, A^{-1}\rho^{-1}\omega$ and ρ^{-1}. After addition of the obtained equations, we find that

$$\frac{\partial \phi}{\partial t} = \frac{\partial}{\partial x}(D\rho\frac{\partial \theta}{\partial x} + \rho v\frac{\partial v}{\partial x} + \rho\omega\frac{\partial \omega}{\partial x}) - K\frac{\partial}{\partial x}(\rho\theta v). \tag{4.7}$$

Multiplying (4.7) by ϕ and integrating over $]0,1[$ we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\phi(.,t)\|^2 &+ \int_0^1 \rho(D\frac{\partial \theta}{\partial x} + v\frac{\partial v}{\partial x} + \omega\frac{\partial \omega}{\partial x})(v\frac{\partial v}{\partial x} + \frac{1}{A}\omega\frac{\partial \omega}{\partial x} + \frac{\partial \theta}{\partial x})dx \\ &= K\int_0^1 \rho\theta v^2\frac{\partial v}{\partial x}dx + \frac{K}{A}\int_0^1 \rho\theta v\omega\frac{\partial \omega}{\partial x}dx + K\int_0^1 \rho v\theta\frac{\partial \theta}{\partial x}dx. \end{aligned} \tag{4.8}$$

From (4.8) it follows

$$\begin{aligned} \frac{d}{dt}\|\phi(.,t)\|^2 &+ 2D\int_0^1 \rho(\frac{\partial \theta}{\partial x})^2dx + 2\int_0^1 \rho v^2(\frac{\partial v}{\partial x})^2dx + \frac{2}{A}\int_0^1 \rho\omega^2(\frac{\partial \omega}{\partial x})^2dx \\ &= -2(D+1)\int_0^1 \rho\frac{\partial \theta}{\partial x}v\frac{\partial v}{\partial x}dx - 2(\frac{D}{A}+1)\int_0^1 \rho\frac{\partial \theta}{\partial x}\omega\frac{\partial \omega}{\partial x}dx \\ &\quad - 2(\frac{1}{A}+1)\int_0^1 \rho\omega\frac{\partial \omega}{\partial x}v\frac{\partial v}{\partial x}dx + 2K\int_0^1 \rho\theta v^2\frac{\partial v}{\partial x}dx \\ &\quad + \frac{2K}{A}\int_0^1 \rho\theta v\omega\frac{\partial \omega}{\partial x}dx + 2K\int_0^1 \rho v\theta\frac{\partial \theta}{\partial x}dx. \end{aligned} \tag{4.9}$$

Using the Young inequality with a parameter $\varepsilon_i > 0$ $(i = 1, 2, ..., 6)$, for the terms on the right-hand side of (4.9) we find the estimates as follows:

$$\left|\int_0^1 \rho\frac{\partial \theta}{\partial x}v\frac{\partial v}{\partial x}dx\right| \le \frac{\varepsilon_1^2}{2}\int_0^1 \rho v^2(\frac{\partial v}{\partial x})^2dx + \frac{\varepsilon_1^{-2}}{2}\int_0^1 \rho(\frac{\partial \theta}{\partial x})^2dx, \tag{4.10}$$

$$\left|\int_0^1 \rho\frac{\partial \theta}{\partial x}\omega\frac{\partial \omega}{\partial x}dx\right| \le \frac{\varepsilon_2^2}{2}\int_0^1 \rho\omega^2(\frac{\partial \omega}{\partial x})^2dx + \frac{\varepsilon_2^{-2}}{2}\int_0^1 \rho(\frac{\partial \theta}{\partial x})^2dx, \tag{4.11}$$

$$\left|\int_0^1 \rho\omega\frac{\partial \omega}{\partial x}v\frac{\partial v}{\partial x}dx\right| \le \frac{\varepsilon_3^2}{2}\int_0^1 \rho v^2(\frac{\partial v}{\partial x})^2dx + \frac{\varepsilon_3^{-2}}{2}\int_0^1 \rho\omega^2(\frac{\partial \omega}{\partial x})^2dx, \tag{4.12}$$

$$\left|\int_0^1 \rho\theta v^2 \frac{\partial v}{\partial x}dx\right| \le \frac{\varepsilon_4^2}{2}\int_0^1 \rho v^2 (\frac{\partial v}{\partial x})^2 dx + \frac{\varepsilon_4^{-2}}{2}\int_0^1 \rho\theta^2 v^2 dx, \tag{4.13}$$

$$\left|\int_0^1 \rho\theta v\omega \frac{\partial \omega}{\partial x}dx\right| \le \frac{\varepsilon_5^2}{2}\int_0^1 \rho \omega^2 (\frac{\partial \omega}{\partial x})^2 dx + \frac{\varepsilon_5^{-2}}{2}\int_0^1 \rho\theta^2 v^2 dx, \tag{4.14}$$

$$\left|\int_0^1 \rho v\theta \frac{\partial \theta}{\partial x}dx\right| \le \frac{\varepsilon_6^2}{2}\int_0^1 \rho (\frac{\partial \theta}{\partial x})^2 dx + \frac{\varepsilon_6^{-2}}{2}\int_0^1 \rho\theta^2 v^2 dx. \tag{4.15}$$

Taking $\varepsilon_6, \varepsilon_1$ and ε_2 such that

$$K\varepsilon_6^2 + (D+1)\varepsilon_1^{-2} + (DA^{-1}+1)\varepsilon_2^{-2} = D \tag{4.16}$$

and using (4.10)–(4.15), from (4.9) we find that

(4.17)

$$\frac{d}{dt}\|\phi(.,t)\|^2 + D\int_0^1 \rho(\frac{\partial \theta}{\partial x})^2 dx + 2\int_0^1 \rho v^2(\frac{\partial v}{\partial x})^2 dx + \frac{2}{A}\int_0^1 \rho\omega^2(\frac{\partial \omega}{\partial x})^2 dx$$
$$\le C_1\int_0^1 \rho v^2(\frac{\partial v}{\partial x})^2 dx + C_2\int_0^1 \rho\omega^2(\frac{\partial \omega}{\partial x})^2 dx + C_3\int_0^1 \rho\theta^2 v^2 dx.$$

Now, multiplying the equation (2.2) by $4v^3$, integrating over $]0,1[$ and applying the Young inequality with a parameter $\varepsilon > 0$ we get

$$\frac{d}{dt}\int_0^1 v^4 dx + 12\int_0^1 \rho v^2(\frac{\partial v}{\partial x})^2 dx = 12K\int_0^1 \rho\theta v^2 \frac{\partial v}{\partial x}dx \tag{4.18}$$
$$\le \frac{\varepsilon^2}{2}\int_0^1 \rho v^2(\frac{\partial v}{\partial x})^2 dx + \frac{144K^2\varepsilon^{-2}}{2}\int_0^1 \rho\theta^2 v^2 dx.$$

Setting $\varepsilon^2 = 2$ in (4.18) we have

$$\frac{d}{dt}\int_0^1 v^4 dx + 11\int_0^1 \rho v^2(\frac{\partial v}{\partial x})^2 dx \le C\int_0^1 \rho\theta^2 v^2 dx. \tag{4.19}$$

Multiplying the equation (2.3) by $4\omega^3\rho^{-1}A^{-1}$ and integrating over $]0,1[$ we obtain easily

$$\frac{1}{A}\frac{d}{dt}\int_0^1 \omega^4 dx + 12\int_0^1 \rho\omega^2(\frac{\partial \omega}{\partial x})^2 dx + 4\int_0^1 \frac{\omega^4}{\rho}dx = 0. \tag{4.20}$$

Multiplying (4.19) and (4.20) respectively by $\frac{C_1}{11}$ and $\frac{C_2}{12}$, after addition of the obtained inequalities with (4.17) we find that

(4.21)

$$\frac{d}{dt}\|\phi(.,t)\|^2 + \frac{C_1}{11}\frac{d}{dt}\int_0^1 v^4 dx + \frac{C_2}{12A}\frac{d}{dt}\int_0^1 \omega^4 dx + D\int_0^1 \rho(\frac{\partial \theta}{\partial x})^2 dx$$
$$+ 2\int_0^1 \rho v^2(\frac{\partial v}{\partial x})^2 dx + \frac{2}{A}\int_0^1 \rho\omega^2(\frac{\partial \omega}{\partial x})^2 dx + \frac{C_2}{3}\int_0^1 \frac{\omega^4}{\rho}dx \le C\int_0^1 \rho\theta^2 v^2 dx$$

and after integration over $]0,t[, t>0$, from (4.21) it follows

$$\|\phi(.,t)\|^2 + \|v(.,t)\|^4_{L^4(\Omega)} + \|\omega(.,t)\|^4_{L^4(\Omega)} \tag{4.22}$$

$$\leq C(\int_0^t M_\rho(\tau)\max_{x\in\bar{\Omega}} v^2\|\theta(.,\tau)\|^2 d\tau + \|\phi(.,0)\|^2 + \|v_0\|^4_{L^4(\Omega)} + \|\omega_0\|^4_{L^4(\Omega)}).$$

Note that

$$v_0, \omega_0 \in L^4(\Omega), \quad \phi(.,0) \in L^2(\Omega). \tag{4.23}$$

Taking into account (4.6) and (3.34) from (4.22) we get

$$\|\phi(.,t)\|^2 \leq C(\int_0^t \max_{x\in\bar{\Omega}} v^2(x,t)\|\phi(.,\tau)\|^2 d\tau + 1). \tag{4.24}$$

With the help of (3.37) and Gronwall's inequality from (4.24) for each $t>0$ we obtain

$$\|\phi(.,t)\|^2 \leq C \tag{4.25}$$

and hence (4.4).

Because of (3.25) and (3.34) from (4.21) for $t>0$ we obtain immediately

$$\int_0^t \|\frac{\partial\theta}{\partial x}(.,\tau)\|^2 d\tau \leq C(\int_0^t \max_{x\in\bar{\Omega}} v^2(x,\tau)\|\theta(.,\tau)\|^2 d\tau + 1). \tag{4.26}$$

Using (4.4) and (3.37) from (4.26) one can easily conclude that the inclusion (4.5) is true. □

LEMMA 4.2. *([1], pp.93-94) It holds*

$$\frac{\partial\rho}{\partial x} \in L^\infty(0,\infty;L^2(\Omega)), \tag{4.27}$$

$$\theta^{1/2}\frac{\partial\rho}{\partial x} \in L^2(0,\infty;L^2(\Omega)), \tag{4.28}$$

$$\frac{\partial v}{\partial x} \in L^2(0,\infty;L^2(\Omega)). \tag{4.29}$$

LEMMA 4.3. *([1], pp.94-95) It holds*

$$\frac{\partial v}{\partial x} \in L^\infty(0,\infty;L^2(\Omega)), \quad \frac{\partial^2 v}{\partial x^2} \in L^2(0,\infty;L^2(\Omega)). \tag{4.30}$$

LEMMA 4.4. *It holds*

$$\frac{\partial\omega}{\partial x} \in L^\infty(0,\infty;L^2(\Omega)), \quad \frac{\partial^2 \omega}{\partial x^2} \in L^2(0,\infty;L^2(\Omega)), \tag{4.31}$$

$$\frac{\partial\theta}{\partial x} \in L^\infty(0,\infty;L^2(\Omega)), \quad \frac{\partial^2 \theta}{\partial x^2} \in L^2(0,\infty;L^2(\Omega)). \tag{4.32}$$

PROOF. We consider the equations (2.3) and (2.4) rewriting them in the next form:

$$\frac{1}{A}\frac{\partial\omega}{\partial t}=\frac{\partial\rho}{\partial x}\frac{\partial\omega}{\partial x}+\rho\frac{\partial^2\omega}{\partial x^2}-\frac{\omega}{\rho}, \tag{4.33}$$

$$\frac{\partial\theta}{\partial t}=-K\rho\theta\frac{\partial v}{\partial x}+\rho(\frac{\partial v}{\partial x})^2+\rho(\frac{\partial\omega}{\partial x})^2+\frac{\omega^2}{\rho}+D\frac{\partial\rho}{\partial x}\frac{\partial\theta}{\partial x}+D\rho\frac{\partial^2\theta}{\partial x^2}. \tag{4.34}$$

Multiplying (4.33) and (4.34) respectively by $\frac{\partial^2\omega}{\partial x^2}$ and $\frac{\partial^2\theta}{\partial x^2}$ and integrating over $]0,1[$ we obtain

$$\frac{1}{2A}\frac{d}{dt}\int_0^1(\frac{\partial\omega}{\partial x})^2dx+\int_0^1\rho(\frac{\partial^2\omega}{\partial x^2})^2dx=\int_0^1\frac{\omega}{\rho}\frac{\partial^2\omega}{\partial x^2}dx-\int_0^2\frac{\partial\rho}{\partial x}\frac{\partial\omega}{\partial x}\frac{\partial^2\omega}{\partial x^2}dx, \tag{4.35}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1(\frac{\partial\theta}{\partial x})^2dx+D\int_0^1\rho(\frac{\partial^2\theta}{\partial x^2})^2dx&\\ =K\int_0^1\rho\theta\frac{\partial v}{\partial x}\frac{\partial^2\theta}{\partial x^2}dx-\int_0^1\rho(\frac{\partial v}{\partial x})^2\frac{\partial^2\theta}{\partial x^2}dx-\int_0^1\rho(\frac{\partial\omega}{\partial x})^2\frac{\partial^2\theta}{\partial x^2}dx&\\ -\int_0^1\frac{\omega^2}{\rho}\frac{\partial^2\theta}{\partial x^2}dx-D\int_0^1\frac{\partial\rho}{\partial x}\frac{\partial\theta}{\partial x}\frac{\partial^2\theta}{\partial x^2}dx.&\end{aligned} \tag{4.36}$$

Taking into account the inequalities

$$\omega^2\le 2\|\omega\|\|\frac{\partial\omega}{\partial x}\|,\ \ (\frac{\partial\omega}{\partial x})^2\le 2\|\frac{\partial\omega}{\partial x}\|\|\frac{\partial^2\omega}{\partial x^2}\|\le 2\|\frac{\partial^2\omega}{\partial x^2}\|^2 \tag{4.37}$$

and applying on the right-hand side of (4.35) the Young inequality with a parameter $\varepsilon_i>0$ $(i=1,2)$ we get

$$\begin{aligned}\frac{1}{2A}\frac{d}{dt}\int_0^1(\frac{\partial\omega}{\partial x})^2dx+\int_0^1\rho(\frac{\partial^2\omega}{\partial x^2})^2dx&\\ \le C\|\omega\|\|\frac{\partial^2\omega}{\partial x^2}\|+\max_{x\in\bar\Omega}|\frac{\partial\omega}{\partial x}|\|\frac{\partial\rho}{\partial x}\|\|\frac{\partial^2\omega}{\partial x^2}\|&\\ \le\varepsilon_1\|\frac{\partial^2\omega}{\partial x^2}\|^2+C_{\varepsilon_1}\|\omega\|^2+C\|\frac{\partial\omega}{\partial x}\|^{1/2}\|\frac{\partial^2\omega}{\partial x^2}\|^{3/2}\|\frac{\partial\rho}{\partial x}\|&\\ \le\varepsilon_1\|\frac{\partial^2\omega}{\partial x^2}\|^2+C_{\varepsilon_1}\|\omega\|^2+\varepsilon_2\|\frac{\partial^2\omega}{\partial x^2}\|^2+C_{\varepsilon_2}\|\frac{\partial\rho}{\partial x}\|^4\|\frac{\partial\omega}{\partial x}\|^2.&\end{aligned} \tag{4.38}$$

Let $\varepsilon_i>0$ $(i=1,2)$ be small enough. Integrating (4.38) over $]0,t[$ and using (2.13), (3.25), (3.37), (4.27) and (4.3) we conclude that

$$\sup_{t\in R^+}\|\frac{\partial\omega}{\partial x}(.,t)\|^2+\int_0^\infty\|\frac{\partial^2\omega}{\partial x^2}(.,\tau)\|^2d\tau\le C. \tag{4.39}$$

Therefore (4.31) holds true.

Taking into account (3.34), (4.37) and the inequalities

$$(\frac{\partial v}{\partial x})^2 \leq 2\|\frac{\partial v}{\partial x}\|\|\frac{\partial^2 v}{\partial x^2}\| \leq 2\|\frac{\partial^2 v}{\partial x^2}\|^2, \;\; (\frac{\partial \theta}{\partial x})^2 \leq 2\|\frac{\partial \theta}{\partial x}\|\|\frac{\partial^2 \theta}{\partial x^2}\| \tag{4.40}$$

and using the Young inequality with a parameter $\varepsilon_i > 0$ ($i = 1, 2, ..., 5$), for the terms on right-hand side of (4.36) we find the estimates as follows:

$$\begin{aligned} \left| K \int_0^1 \rho\theta \frac{\partial v}{\partial x}\frac{\partial^2 \theta}{\partial x^2} dx \right| &\leq C \max_{x\in\bar{\Omega}} |\frac{\partial v}{\partial x}| \|\theta\| \|\frac{\partial^2 \theta}{\partial x^2}\| \\ &\leq C\|\frac{\partial^2 v}{\partial x^2}\|\|\theta\|\|\frac{\partial^2 \theta}{\partial x^2}\| \\ &\leq \varepsilon_1 \|\frac{\partial^2 \theta}{\partial x^2}\|^2 + C_{\varepsilon_1}\|\frac{\partial^2 v}{\partial x^2}\|^2\|\theta\|^2, \end{aligned} \tag{4.41}$$

$$\begin{aligned} \left| \int_0^1 \rho(\frac{\partial v}{\partial x})^2 \frac{\partial^2 \theta}{\partial x^2} dx \right| &\leq \varepsilon_2 \|\frac{\partial^2 \theta}{\partial x^2}\|^2 + C_{\varepsilon_2}\|\frac{\partial v}{\partial x}\|^4_{L^4(\Omega)} \\ &\leq \varepsilon_2 \|\frac{\partial^2 \theta}{\partial x^2}\|^2 + C\|\frac{\partial v}{\partial x}\|^2\|\frac{\partial^2 v}{\partial x^2}\|^2, \end{aligned} \tag{4.42}$$

$$\begin{aligned} \left| \int_0^1 \rho(\frac{\partial \omega}{\partial x})^2 \frac{\partial^2 \theta}{\partial x^2} dx \right| &\leq \varepsilon_3 \|\frac{\partial^2 \theta}{\partial x^2}\|^2 + C_{\varepsilon_3}\|\frac{\partial \omega}{\partial x}\|^4_{L^4(\Omega)} \\ &\leq \varepsilon_3 \|\frac{\partial^2 \theta}{\partial x^2}\|^2 + C_{\varepsilon_3}\|\frac{\partial \omega}{\partial x}\|^2\|\frac{\partial^2 \omega}{\partial x^2}\|^2, \end{aligned} \tag{4.43}$$

$$\begin{aligned} \left| \int_0^1 \frac{\omega^2}{\rho} \frac{\partial^2 \theta}{\partial x^2} dx \right| &\leq \varepsilon_4 \|\frac{\partial^2 \theta}{\partial x^2}\|^2 + C_{\varepsilon_4}\|\omega\|^4_{L^4(\Omega)} \\ &\leq \varepsilon_4 \|\frac{\partial^2 \theta}{\partial x^2}\|^2 + C_{\varepsilon_4}\|\omega\|^2\|\frac{\partial \omega}{\partial x}\|^2, \end{aligned} \tag{4.44}$$

$$\begin{aligned} \left| \int_0^1 \frac{\partial \rho}{\partial x}\frac{\partial \theta}{\partial x}\frac{\partial^2 \theta}{\partial x^2} dx \right| &\leq \max_{x\in\bar{\Omega}} |\frac{\partial \theta}{\partial x}| \|\frac{\partial \rho}{\partial x}\| \|\frac{\partial^2 \theta}{\partial x^2}\| \\ &\leq C\|\frac{\partial \rho}{\partial x}\|\|\frac{\partial \theta}{\partial x}\|^{1/2}\|\frac{\partial^2 \theta}{\partial x^2}\|^{3/2} \\ &\leq \varepsilon_5 \|\frac{\partial^2 \theta}{\partial x^2}\|^2 + C_{\varepsilon_5}\|\frac{\partial \rho}{\partial x}\|^4\|\frac{\partial \theta}{\partial x}\|^2. \end{aligned} \tag{4.45}$$

Let $\varepsilon_i > 0$ $(i = 1, 2, ..., 5)$ be small enough. Taking into account (4.41)–(4.45) and integrating (4.36) over $]0, t[$ we get

$$\begin{aligned}
&\|\frac{\partial\theta}{\partial x}(.,t)\|^2 + \int_0^t \|\frac{\partial^2\theta}{\partial x^2}\|^2 d\tau \\
&\qquad \le C(\sup_{t\in R^+} \|\theta(.,t)\|^2 \int_0^t \|\frac{\partial^2 v}{\partial x^2}\|^2 d\tau + \sup_{t\in R^+} \|\frac{\partial v}{\partial x}(.,t)\|^2 \int_0^t \|\frac{\partial^2 v}{\partial x^2}\|^2 d\tau \\
&\qquad\quad + \sup_{t\in R^+} \|\frac{\partial \omega}{\partial x}(.,t)\|^2 \int_0^t \|\frac{\partial^2 \omega}{\partial x^2}\|^2 d\tau + \sup_{t\in R^+} \|\omega(.,t)\|^2 \int_0^t \|\frac{\partial \omega}{\partial x}\|^2 d\tau \\
&\qquad\quad + \sup_{t\in R^+} \|\frac{\partial \rho}{\partial x}(.,t)\|^4 \int_0^t \|\frac{\partial \theta}{\partial x}\|^2 d\tau) + \|\frac{d}{dx}\theta_0\|^2.
\end{aligned} \tag{4.46}$$

Using the inclusions (2.13), (4.3), (4.4), (4.2), (4.5), (4.27), (4.30) and (4.31) from (4.46) we obtain (4.32). □

Lemma 4.5.

$$\frac{\partial \rho}{\partial t} \in L^\infty(0,\infty; L^2(\Omega)) \cap L^2(0,\infty; L^2(\Omega)), \tag{4.47}$$

$$\frac{\partial v}{\partial t}, \frac{\partial \omega}{\partial t}, \frac{\partial \theta}{\partial t} \in L^2(0,\infty; L^2(\Omega)). \tag{4.48}$$

Proof. Taking into account (3.34), from the equation (2.1) we get immediately

$$\|\frac{\partial \rho}{\partial t}(.,t)\|^2 \le C\|\frac{\partial v}{\partial x}(.,t)\|^2 \tag{4.49}$$

and using (4.30) and (4.29) we obtain (4.47).

With the help of the property ([1], pp.95)

$$\theta^2(x,t) \le C[\theta(x,t) + \|\frac{\partial\theta}{\partial x}(.,t)\|^2], \tag{4.50}$$

the estimate (3.34) and inequality (4.40), from (2.2) we get

$$\begin{aligned}
\|\frac{\partial v}{\partial t}\|^2 \le& C(\|\frac{\partial^2 v}{\partial x^2}\|^2 \|\frac{\partial \rho}{\partial x}\|^2 + \|\frac{\partial^2 v}{\partial x^2}\|^2 \\
&+ \int_0^1 (\frac{\partial \rho}{\partial x})^2 \theta dx + \|\frac{\partial \theta}{\partial x}\|^2 \|\frac{\partial \rho}{\partial x}\|^2 + \|\frac{\partial \theta}{\partial x}\|^2).
\end{aligned} \tag{4.51}$$

Integrating (4.51) over $]0, t[$ and using (4.30), (4.27), (4.28) and (4.5) we conclude that

$$\int_0^t \|\frac{\partial v}{\partial t}(.,\tau)\|^2 d\tau \le C \tag{4.52}$$

for each $t > 0$.
With the help of (4.37), (3.34) and (3.25) from (2.3) we find that

$$\|\frac{\partial\omega}{\partial t}\|^2 \leq C[\|\frac{\partial^2\omega}{\partial x^2}\|^2\|\frac{\partial\rho}{\partial x}\|^2 + \|\frac{\partial^2\omega}{\partial x^2}\|^2 + \|\omega\|^2] \tag{4.53}$$

and taking into account (4.27), (4.31) and (4.2), for each $t > 0$ we have

$$\int_0^t \|\frac{\partial\omega}{\partial t}(.,\tau)\|d\tau \leq C. \tag{4.54}$$

Finally, using (3.34), (4.40) and (4.37), from the equation (2.4) we find that

$$\begin{aligned}\|\frac{\partial\theta}{\partial t}\|^2 \leq & C(\|\frac{\partial^2 v}{\partial x^2}\|^2\|\theta\|^2 + \|\frac{\partial^2 v}{\partial x^2}\|^2\|\frac{\partial v}{\partial x}\|^2 + \|\frac{\partial^2\omega}{\partial x^2}\|^2\|\frac{\partial\omega}{\partial x}\|^2 \\ & + \|\frac{\partial\omega}{\partial x}\|^2\|\omega\|^2 + \|\frac{\partial^2\theta}{\partial x^2}\|^2\|\frac{\partial\rho}{\partial x}\|^2 + \|\frac{\partial^2\theta}{\partial x^2}\|^2).\end{aligned} \tag{4.55}$$

Using the inclusions (4.30), (4.31), (4.32), (4.2), (4.3), (4.4) and (4.27), from (4.55) we get easily

$$\int_0^t \|\frac{\partial\theta}{\partial t}(.,\tau)\|^2 d\tau \leq C. \tag{4.56}$$

□

Lemma 4.6. *It holds*

$$\frac{\partial\rho}{\partial x} \in L^2(0,\infty; L^2(\Omega)). \tag{4.57}$$

Proof. From the equations (2.1) and (2.2) it follows

$$\frac{\partial}{\partial t}(\frac{\partial \ln\rho}{\partial x}) + K\theta\frac{\partial\rho}{\partial x} = -\frac{\partial v}{\partial t} - K\rho\frac{\partial\theta}{\partial x}. \tag{4.58}$$

Multiplying (4.58) by $\frac{\partial \ln\rho}{\partial x}$ and integrating over $]0,1[$ we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\frac{\partial \ln\rho}{\partial x}\|^2 + K\int_0^1 \rho\theta(\frac{\partial \ln\rho}{\partial x})^2 dx \\ = -\int_0^1 \frac{\partial v}{\partial t}\frac{\partial \ln\rho}{\partial x}dx - K\int_0^1 \rho\frac{\partial\theta}{\partial x}\frac{\partial \ln\rho}{\partial x}dx.\end{aligned} \tag{4.59}$$

The second term at the right-hand side we estimate with the help of the Cauchy inequality:

$$\begin{aligned}\rho\left|\frac{\partial\theta}{\partial x}\frac{\partial \ln\rho}{\partial x}\right| & \leq \rho[\frac{1}{2}\theta(\frac{\partial \ln\rho}{\partial x})^2 + \frac{1}{2\theta}(\frac{\partial\theta}{\partial x})^2] \\ & \leq \rho[\frac{\theta}{2}(\frac{\partial \ln\rho}{\partial x})^2 + \frac{1}{4}(1+\frac{1}{\theta^2})(\frac{\partial\theta}{\partial x})^2].\end{aligned} \tag{4.60}$$

Taking into account (4.60) and integrating (4.59) over $]0,t[$ we get

$$\|\frac{\partial \ln\rho}{\partial x}(.,t)\|^2 + K\int_0^t\int_0^1 \rho\theta(\frac{\partial \ln\rho}{\partial x})^2 dx d\tau \le 2\int_0^t \|\frac{\partial v}{\partial t}\|\|\frac{\partial \ln\rho}{\partial x}\| d\tau + \frac{K}{2}\int_0^t\int_0^1 \rho(\frac{\partial\theta}{\partial x})^2 dx d\tau + \frac{K}{2}\int_0^t\int_0^1 \frac{\rho}{\theta^2}(\frac{\partial\theta}{\partial x})^2 dx d\tau + \|\frac{d\ln\rho_0}{dx}\|^2. \tag{4.61}$$

Applying the Young inequality with a parameter $\varepsilon > 0$ on the first integral on the right-hand side and using the properties (4.5), (3.8), (3.34), (3.25) and (2.13) from (4.61) we find that

$$\|\frac{\partial \ln\rho}{\partial x}(.,t)\|^2 + C\int_0^t m_\theta(\tau)\|\frac{\partial \ln\rho}{\partial x}(.,\tau)\|^2 d\tau \le \varepsilon\int_0^t \|\frac{\partial \ln\rho}{\partial x}(.,\tau)\|^2 d\tau + C_\varepsilon\int_0^t \|\frac{\partial v}{\partial t}(.,\tau)\|^2 d\tau + C. \tag{4.62}$$

With the help of (3.46) we obtain

$$C\int_0^t \|\frac{\partial \ln\rho}{\partial x}(.,\tau)\|^2 d\tau \le \sup_{t\in R^+}\|\frac{\partial \ln\rho}{\partial x}(.,t)\|^2\int_0^t f_1(\tau)d\tau + \varepsilon\int_0^t \|\frac{\partial \ln\rho}{\partial x}(.,\tau)\|^2 d\tau + C_\varepsilon\int_0^t \|\frac{\partial v}{\partial t}(.,\tau)\|^2 d\tau + C, \tag{4.63}$$

where f_1 is defined by (3.48). Taking ε small enough and using (4.27) and (4.48) from (4.63), for each $t > 0$ we find that

$$\int_0^t \|\frac{\partial \ln\rho}{\partial x}(.,\tau)\|^2 d\tau \le C. \tag{4.64}$$

Using (3.25) from (4.64) we get immediately (4.57). □

Theorem 2.1 is an immediate consequence of the above lemmas.

Remark. In the second paper we intend to prove (by using Theorem 2.1) that a solution of the problem (2.1)–(2.11) converges to a stationary solution $(\alpha^{-1}, 0, 0, E_1)$ (where E_1 and α are defined by (3.1) and (3.9) respectively) in the space $(H^1(\Omega))^4$, when $t \to \infty$.

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