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A REMARK ON CONCENTRATION OF THE ERROR BETWEEN A FUNCTION AND ITS BEST POLYNOMIAL APPROXIMANTS. II: A PROBLEM OF HASSON

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Abstract. In the present paper we construct a function to give a positive answer to a problem raised by Hasson [4], that says, the conclusion of a result ([4, Theorem 1.3]) cannot be strengthened.

1. Introduction

Let $C_{[-1,1]}$ be the space of all real valued continuous functions defined on the interval $[-1,1]$, and $C^k_{[-1,1]}$ be the space of all functions on $[-1,1]$ which have k continuous derivatives. Write $E_n(f)$ to indicate the best approximation of $f \in C_{[-1,1]}$ by polynomials of degree n. As usual, by $\|\cdot\|$ we denote the usual supremum norm on $[-1,1]$, while $\|\cdot\|_{[a,b]}$ stands for the supremum norm on $[a,b]$ as specified. Also, $D_-f(x)$, $D^-f(x)$, $D_+f(x)$ and $D^+f(x)$ indicate the four Dini derivatives of $f \in C_{[-1,1]}$ at the point $x \in (-1,1)$.

Considering the generalization of the classical alternation theorem ([1]), Hasson [4] studied the concentration of the error between a function and its polynomial approximants recently. Hasson proved the following two main results.

Theorem 1.1 (Hasson [4, Theorem 1.1]). *Let k be a positive integer, $f(x) \in C^{k-1}_{[-1,1]}$. If there exists an $a \in (-1,1)$ with one of the following properties:*

$$f^{(k)}_+(a) > f^{(k)}_-(a),$$

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or

$$f_-^{(k)}(a) > f_+^{(k)}(a),$$

where $f_-^{(k)}(x)$ *and* $f_+^{(k)}(x)$ *are the* k*th left derivative and* k*th right derivative of* $f(x)$ *at the point* x*, and if* $P_n(x)$ *is a sequence of polynomials of degree* n *such that*

$$\|f - P_n\| = O(E_n(f)) = O(n^{-k}),$$

then there exist positive constants r *and* C *such that*

$$\|f - P_n\|_{[a-r/n, a+r/n]} \geq Cn^{-k}, \quad n = 1, 2, 3, \ldots.$$

Theorem 1.2 (Hasson [4, Theorem 1.3]). *Let* k *be a positive integer,* $f \in C^{k-1}_{[-1,1]}$*. If there exists an* $a \in (-1, 1)$ *with one of the following properties:*

$$D_+ f^{(k-1)}(a) \neq D^- f^{(k-1)}(a)$$

or

$$D_- f^{(k-1)}(a) \neq D^+ f^{(k-1)}(a),$$

and if $P_n(x)$ *is a sequence of polynomials of degree at most* n *such that*

$$\|f - P_n\| = O(E_n(f)) = O(n^{-k}),$$

then there exist positive constants r *and* C *such that*

$$\|f - P_n\|_{[a-r/n, a+r/n]} \geq Cn^{-k}$$

holds for infinitely many n*.*

Regarding the sharpness of Theorem 1.2, Hasson [4] wrote: "It would be also of interest to build a function g satisfying the hypotheses of Theorem 1.3 (Theorem 1.2) and for which, necessarily,

$$\|g - P_n\|_{[a-r/n, a+r/n]} \geq Cn^{-k}$$

infinitely often, but for which

$$\|g - P_{n_l}\|_{[a-r/n_l, a+r/n_l]} = o(n_l^{-k})$$

for some sequence $\{n_l\}$ of integers. ... However we were not able to build such a function."

In the present paper we will construct a function to satisfy the above requirements, that says, the conclusion of Theorem 1.2 cannot be strengthened to that of Theorem 1.1. It does give a partial answer to the question of Hasson.

We make a few historic remarks. The Hasson's results are nothing but local versions of a more general approach to lower estimates of the approximation error (see [5, 3]). Also, the functions of the form $\sum n^{-\alpha} T_n(x)$ (which we will use in the construction), where summation is taken over a subsequence $n = n_k$, are well known in analysis and approximation theory. In particular, Weierstrass introduced similar series as nowhere differentiable functions, and Bernstein discussed their approximation properties (see [1]).

In what follows, we always use C to indicate an absolute positive constant, while C_x to indicate a positive constant that may depend upon x, their values may be varied even in the same line. Sometimes, according to specific circumstances, we may use other symbols to indicate a constant, as specified.

2. A Constructive Example

Theorem 2.1. *Given $0 < \alpha \le 1$. There is a function $f \in C_{[-1,1]}$ satisfying $E_n(f) = O(n^{-\alpha})$, $n = 1, 2, \ldots,$ and*

$$D_+ f(0) \ne D^- f(0), \tag{1}$$

while

$$\limsup_{n\to\infty} n^\alpha \|f - P_n(f)\|_{[-r/n, r/n]} > 0$$

and

$$\liminf_{n\to\infty} n^\alpha \|f - P_n(f)\|_{[-r/n, r/n]} = 0$$

hold at the same time for some $r > 0$.

Proof. Let $T_n(x) = \cos(n \arccos x)$ be the Chebyshev polynomial of degree n. We know its zeros are $\cos\frac{(2k-1)\pi}{2n}$, $k = 1, \ldots, n$, and its extreme points are $\cos\frac{k\pi}{n}$, $k = 0, 1, \ldots, n$. Write

$$t_{4n} = \cos\frac{(4n-1)\pi}{8n}, \quad t^*_{4n} = \cos\frac{(4n+1)\pi}{8n},$$
$$r_{4n} = \cos\frac{(4n-1/2)\pi}{8n}, \quad r^*_{4n} = \cos\frac{(4n+1/2)\pi}{8n},$$

then clearly,

$$T_{4n}(0) = 1, \tag{2}$$

and

$$T_{4n}(t_{4n}) = T_{4n}(t^*_{4n}) = 0. \tag{3}$$

Furthermore, there are positive constants M_1, M_2 independent of n such that

$$\begin{aligned} M_1 n^{-1} &\le \min\{t_{4n},\ -t^*_{4n},\ r_{4n},\ -r^*_{4n}\} \\ &\le \max\{t_{4n},\ -t^*_{4n},\ r_{4n},\ -r^*_{4n}\} \le M_2 n^{-1}. \end{aligned} \tag{4}$$

Let $P_n(f, x)$ be the nth best polynomial approximant of f. We recall the well-known Freud theorem ([2]) which says that for $f_0 \in C_{[-1,1]}$, there is a constant $\lambda_{n,f_0} > 0$ depending upon n and f_0 only such for all $f \in C_{[-1,1]}$ that

$$\|P_n(f_0) - P_n(f)\| \le \lambda_{n,f_0} \|f_0 - f\|. \tag{5}$$

Select $\{n_j\} \subset \{4n\}$ by induction. Set $n_1 = 4$, after choose n_j, $j = 1, 2, \ldots, k$, and set

$$h_j(x) = n_j^{-\alpha} T_{n_j}(x), \quad j = 1, 2, \ldots, k,$$

take n_{k+1} satisfying

$$n_{k+1} \geq \left(2n_k^{2/\alpha}\left(\max_{n_{k-1}\leq n<n_k}\{\lambda_{n,h_k}\}+1\right)^{1/\alpha}\right), \tag{6}$$

and (note $T'_{n_j}(0)=0$)

$$\max_{1\leq j\leq k} n_j^{-\alpha}|T'_{n_j}(x)| \leq k^{-1}n_k^{-1} \tag{7}$$

for $x\in[-M_2n_{k+1}^{-1}, M_2n_{k+1}^{-1}]$, where $\lambda_{n,f}$ is the constant appearing in (5) and M_2 is the constant appearing in (4). Define

$$f(x)=\sum_{j=1}^{\infty} n_j^{-\alpha}T_{n_j}(x),$$

and write

$$f_k(x)=\sum_{j=k}^{\infty} n_j^{-\alpha}T_{n_j}(x)$$

for convenience. Clearly, from condition (6), $f\in C_{[-1,1]}$. We check that

$$E_n(f)\approx n_k^{-\alpha} \tag{8}$$

for $n_{k-1}\leq n<n_k$, $k=1,2,\ldots$, where $n_0=0$. The inequalities

$$E_n(f)=O\left(\sum_{j=k}^{\infty} n_j^{-\alpha}\right)=O(n_k^{-\alpha})$$

for $n_{k-1}\leq n<n_k$ come from (6). At the same time, by a simple well-known fact, for $n_{k-1}\leq n<n_k$ (note $P_n(h_k,x)=0$ for $n_{k-1}\leq n<n_k$ due to the oscillation property of the Chebyshev polynomials),

$$\begin{aligned} f(x)-P_n(f,x) &= f_k(x)-P_n(f_k,x) \\ &= h_k(x)-P_n(h_k,x)+f_{k+1}(x)-(P_n(f_k,x)-P_n(h_k,x)) \\ &= h_k(x)+f_{k+1}(x)+P_n(f_k,x)-P_n(h_k,x), \end{aligned}$$

so that by (5) and (6),

$$E_n(f)\geq n_k^{-\alpha}\|T_{n_k}\|-(1+\lambda_{n,h_k})\|f_{k+1}\| = n_k^{-\alpha}-(1+\lambda_{n,h_k})\sum_{j=k+1}^{\infty} n_j^{-\alpha}$$

$$\geq n_k^{-\alpha}-(1+\lambda_{n,h_k})n_{k+1}^{-\alpha}\geq n_k^{-\alpha}-Cn_k^{-2}. \tag{9}$$

That means, $E_n(f)\geq Cn_k^{-\alpha}$ for $n_{k-1}\leq n<n_k$ and sufficiently large k. Altogether, (8) holds. We easily note that

$$\|h_k\|_{[r^*_{n_k}, r_{n_k}]}\geq Cn_k^{-\alpha},$$

by the same technique as (9) we can reach that

$$\|f - P_n(f)\|_{[r^*_{n_k}, r_{n_k}]} \geq C n_k^{-\alpha}$$

for $n_{k-1} \leq n < n_k$, $k = 1, 2, \ldots$. This means that there exists an $r > 0$ such that (in view of (4))

$$\|f - P_{n_k - 1}(f)\|_{[-r/(n_k-1), r/(n_k-1)]} \geq C(n_k - 1)^{-\alpha},$$

or

$$\limsup_{n\to\infty} n^\alpha \|f - P_n(f)\|_{[-r/n, r/n]} > 0.$$

At the same time, it is obvious from (8) and (6) that

$$\|f - P_{n_{k-1}}(f)\|_{[-r/n_{k-1}, r/n_{k-1}]} \leq E_{n_{k-1}}(f) \leq C n_k^{-\alpha} \leq C n_{k-1}^{-2},$$

or

$$\liminf_{n\to\infty} n^\alpha \|f - P_n(f)\|_{[-r/n, r/n]} = 0.$$

Finally, we verify (1). Write

$$\begin{aligned} \frac{f(t_{n_k}) - f(0)}{t_{n_k}} \quad &= \quad \sum_{j=1}^{k-1} n_j^{-\alpha} T'_{n_j}(\xi_j) + n_k^{-\alpha} t_{n_k}^{-1} (T_{n_k}(t_{n_k}) - T_{n_k}(0)) \\ &\quad + \sum_{j=k+1}^{\infty} n_j^{-\alpha} t_{n_k}^{-1} (T_{n_j}(t_{n_k}) - T_{n_j}(0)) =: I_1 + I_2 + I_3, \end{aligned}$$

where $\xi_j \in (0, t_{n_k})$. From (4) and (6),

$$|I_3| \leq C t_{n_k}^{-1} n_{k+1}^{-\alpha} \leq C n_k^{-1},$$

while

$$|I_1| \leq \sum_{j=1}^{k-1} n_j^{-\alpha} \|T'_{n_j}\|_{[-M_2 n_k^{-1}, M_2 n_k^{-1}]} \leq n_{k-1}^{-1}$$

by (7). At the same time there is a constant $C_1 > 0$ such that

$$I_2 = -n_k^{-\alpha} t_{n_k}^{-1} \leq -C_1 n_k^{1-\alpha}$$

by (2), (3) and (4). Therefore,

$$\frac{f(t_{n_k}) - f(0)}{t_{n_k}} \leq -C_1$$

holds for sufficiently large k, as well as $t_{n_k} \searrow 0$ as $k \to \infty$. This indicates that $D_+ f(0) \leq -C_1$. The same argument can be applied to have a constant $C_2 > 0$ for which $D^- f(0) \geq C_2$. Combining these two inequalities, we evidently see that (1) holds. □

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