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PUPPE EXACT SEQUENCE AND ITS APPLICATION IN THE FIBREWISE CATEGORY MAP

Yoshifumi Konami and Takuo Miwa
Izumo-Nishi High School and Shimane University, Japan

Abstract. In this paper, we study Puppe exact sequence and its application in the fibrewise category **MAP**. The application shows that we can prove the generalized formula for the suspension of fibrewise product spaces. Further, introducing an intermediate fibrewise category $\mathbf{TOP}_B^H$, we give an another proof of the original formula in $\mathbf{TOP}_B$ using the concepts of $\mathbf{TOP}_B^H$.

1. Introduction

For a base space B, the category $\mathbf{TOP}_B$ is the fibrewise topology over B. For General Topology of Continuous Maps or Fibrewise General Topology, see B.A. Pasynkov [7],[8]. In [1], D. Buhagiar studied fibrewise topology in the category of all continuous maps, called **MAP** by him (as a way of thinking of a category, **MAP** can be seen in earlier works, see for example [10]). The study of fibrewise topology in **MAP** is a generalization of it in the category $\mathbf{TOP}_B$. In the previous paper [4], we studied fibrewise (pointed) cofibrations and fibrations in the category **MAP**. In this paper, we continue the previous work. In section 3, we prove that Puppe sequence is exact in **MAP**. In section 4, we study an application of Puppe exact sequence. In this study of section 4, we need not to consider any generalized concept of fibrewise non-degenerate spaces [5; section 22], and we can prove the generalized formula for the suspension of fibrewise product spaces. In section 5, we introduce an intermediate fibrewise category $\mathbf{TOP}_B^H$ which combines **MAP** with $\mathbf{TOP}_B$, and we prove an extended theorem in $\mathbf{TOP}_B^H$ of [5] Proposition 22.11 by using theorems in $\mathbf{TOP}_B^H$ (see Theorem 5.4 and Proposition 5.5). As a corollary

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of the extended theorem and the fact that a fibrewise non-degererate space is H-fibrewise well-pointed (see Proposition 5.5), we give an another proof of [5] Proposition 22.11.

The objects of **MAP** are continuous maps from any topological space into any topological space. For two objects $p : X \to B$ and $p' : X' \to B'$, a morphism from p into p' is a pair (ϕ, α) of continuous maps $\phi : X \to X', \alpha : B \to B'$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{\alpha} & B' \end{array}$$

is commutative. We note that this situation is a generalization of the category $\mathbf{TOP}_B$ since the category $\mathbf{TOP}_B$ is isomorphic to the particular case of **MAP** in which the spaces $B' = B$ and $\alpha = id_B$. We call an object $p : X \to B$ an **M**-*fibrewise space* and denote (X, p, B). Also, for two **M**-fibrewise spaces $(X, p, B), (X', p', B')$, we call the morphism (ϕ, α) from p into p' an **M**-*fibrewise map*, and denote $(\phi, \alpha) : (X, p, B) \to (X', p', B')$.

Furthermore, in this paper we often consider the case that an **M**-fibrewise space (X, p, B) has a section $s : B \to X$, we call it an **M**-*fibrewise pointed space* and denote (X, p, B, s). For two **M**-fibrewise pointed spaces (X, p, B, s), (X', p', B', s'), if an **M**-fibrewise map $(\phi, \alpha) : (X, p, B) \to (X', p', B')$ satisfies $\phi s = s'\alpha$, we call it an **M**-*fibrewise pointed map* and denote $(\phi, \alpha) : (X, p, B, s) \to (X', p', B', s')$.

In this paper, we assume that all spaces are topological spaces, all maps are continuous and id is the identity map of $I = [0, 1]$ into itself. Moreover, we use the following notation : For any $t \in I$, the maps $\sigma_t : X \to I \times X$ and $\delta_t : B \to I \times B$ are defined by

$$\sigma_t(x) = (t, x),\ \ \delta_t(b) = (t, b)\ (x \in X,\ b \in B).$$

For other undefined terminology, see [3], [4] and [5].

2. M-fibrewise pointed homotopy

In this section, first we shall define an **M**-fibrewise pointed homotopy, **M**-fibrewise pointed cofibration and **M**-fibrewise pointed cofibred pair which are introduced in [4]. Next, we shall introduce some concepts, for example, **M**-fibrewise pointed mapping cylinder, **M**-fibrewise pointed collapse, **M**-fibrewise pointed cone and **M**-fibrewise pointed nulhomotopic. Last, we shall prove some propositions. These concepts and propositions are used in latter section. We begin with the following definitions.

DEFINITION 2.1. (1) *([4; Definition 5.1]) Let*

$$(\phi, \alpha), (\theta, \beta) : (X, p, B, s) \to (X', p', B', s')$$

be $\mathbf{M}$*-fibrewise pointed maps. If there exists an* $\mathbf{M}$*-fibrewise pointed map* $(H,h) : (I \times X, id \times p, I \times B, id \times s) \to (X', p', B', s')$ *such that* (H,h) *is an* $\mathbf{M}$*-fibrewise homotopy of* (ϕ,α) *into* (θ,β) *(that is;* $H\sigma_0 = \phi,\ H\sigma_1 = \theta,\ h\delta_0 = \alpha,\ h\delta_1 = \beta$*), we call it an* $\mathbf{M}$-fibrewise pointed homotopy *of* (ϕ,α) *into* (θ,β)*. If there exists an* $\mathbf{M}$*-fibrewise pointed homotopy of* (ϕ,α) *into* (θ,β)*, we say* (ϕ,α) *is* $\mathbf{M}$-fibrewise pointed homotopic *to* (θ,β) *and write* $(\phi,\alpha) \simeq^{\mathbf{M}}_{(\mathbf{P})} (\theta,\beta)$.

(2) *([4; Definition 5.2]) An* $\mathbf{M}$*-fibrewise pointed map* $(\phi,\alpha) : (X,p,B,s) \to (X',p',B',s')$ *is called an* $\mathbf{M}$-fibrewise pointed homotopy equivalence *if there exists an* $\mathbf{M}$*-fibrewise pointed map* $(\theta,\beta) : (X',p',B',s') \to (X,p,B,s)$ *such that*

$$(\theta\phi,\beta\alpha) \simeq^{\mathbf{M}}_{(\mathbf{P})} (id_X, id_B),\ (\phi\theta,\alpha\beta) \simeq^{\mathbf{M}}_{(\mathbf{P})} (id_{X'}, id_{B'}).$$

Then we denote $(X,p,B,s) \cong^{\mathbf{M}}_{(\mathbf{P})} (X',p',B',s')$.

It is obvious that the relations $\simeq^{\mathbf{M}}_{(\mathbf{P})}$ and $\cong^{\mathbf{M}}_{(\mathbf{P})}$ are equivalence relations. Now, we define $\mathbf{M}$-fibrewise pointed cofibration and $\mathbf{M}$-fibrewise pointed cofibred pair as follows.

Definition 2.2. *([4; Definition 5.6]) An* $\mathbf{M}$*-fibrewise pointed map* $(u,\gamma) : (X_0,p_0,B_0,s_0) \to (X,p,B,s)$ *is an* $\mathbf{M}$-fibrewise pointed cofibration *if* (u,γ) *has the following* $\mathbf{M}$*-fibrewise homotopy extension property : Let* $(\phi,\alpha) : (X,p,B,s) \to (X',p',B',s')$ *be an* $\mathbf{M}$*-fibrewise pointed map and* $(H,h) : (I \times X_0, id \times p_0, I \times B_0, id \times s_0) \to (X',p',B',s')$ *an* $\mathbf{M}$*-fibrewise pointed homotopy such that the following two diagrams*

$$\begin{array}{ccc} X_0 & \xrightarrow{\sigma_0} & I \times X_0 \\ u\downarrow & & \downarrow H \\ X & \xrightarrow{\phi} & X' \end{array} \qquad \begin{array}{ccc} B_0 & \xrightarrow{\delta_0} & I \times B_0 \\ \gamma\downarrow & & \downarrow h \\ B & \xrightarrow{\alpha} & B' \end{array}$$

are commutative. Then there exists an $\mathbf{M}$*-fibrewise pointed homotopy* $(K,k) : (I \times X, id \times p, I \times B, id \times s) \to (X',p',B',s')$ *such that* $K\kappa_0 = \phi, K(id \times u) = H, k\rho_0 = \alpha, k(id \times \gamma) = h,$ *where* $\kappa_0 : X \to I \times X$ *and* $\rho_0 : B \to I \times B$ *are defined by* $\kappa_0(x) = (0,x)$ *and* $\rho_0(b) = (0,b)$ *for* $x \in X,\ b \in B$.

Definition 2.3 (*[4; Definition 2.3 and p.210]*). (1) *Let* (X,p,B) *be an* $\mathbf{M}$*-fibrewise space. If* $X_0 \subset X, B_0 \subset B$ *and* $p(X_0) \subset B_0$*, we call* $(X_0, p|X_0, B_0)$ *an* $\mathbf{M}$-fibrewise subspace of (X,p,B)*. We sometimes use the notation* (X_0,p_0,B_0) *instead of* $(X_0,p|X_0,B_0)$*. By the same way, we define an* $\mathbf{M}$-fibrewise pointed subspace.

(2) *For an* $\mathbf{M}$*-fibrewise pointed subspace* (X_0,p_0,B_0,s_0) *of* (X,p,B,s)*, the pair* $((X,p,B,s),(X_0,p_0,B_0,s_0))$ *is called by an* $\mathbf{M}$-fibrewise pointed pair. *If* X_0 *is closed in* X *and* B_0 *is closed in* B*, it is called a* closed $\mathbf{M}$-fibrewise pointed pair. *For an* $\mathbf{M}$*-fibrewise*

pointed pair $((X,p,B,s),(X_0,p_0,B_0,s_0))$, *if the inclusion map* $(u,\gamma):(X_0,p_0,B_0,s_0)\to(X,p,B,s)$ *is an* $\mathbf{M}$*-fibrewise pointed cofibration, we call the pair* $((X,p,B,s),(X_0,p_0,B_0,s_0))$ *an* $\mathbf{M}$-fibrewise pointed cofibred pair.

The following proposition, which will be used in section 4, can be proved by the same method of the proof in [4; Theorem 3.9].

PROPOSITION 2.4. *Let* $((X,p,B,s),(X_0,p_0,B_0,s_0))$ *and* $((X',p',B',s'),(X'_0,p'_0,B'_0,s'_0))$ *be two closed* $\mathbf{M}$*-fibrewise pointed cofibred pairs. Then*

$$((X\times X',p\times p',B\times B',s\times s'),(X_0\times X'\cup X\times X'_0,\overline{p},B_0\times B'\cup B\times B'_0,\overline{s}))$$

is also an $\mathbf{M}$*-fibrewise pointed cofibred pair, where*

$$\overline{p}=p\times p'|X_0\times X'\cup X\times X'_0, \overline{s}=s\times s'|B_0\times B'\cup B\times B'_0.$$

For cotriad, see [5]. We can also define the $\mathbf{M}$-fibrewise push-out of a cotriad as same as the fibrewise push-out in [5] as follows:

DEFINITION 2.5 (cf. [4; p.208–9]). *For an* $\mathbf{M}$*-fibrewise pointed map* $(u,\gamma):(X_0,p_0,B_0,s_0)\to(X_1,p_1,B_1,s_1)$, *we can construct the* $\mathbf{M}$*-fibrewise pointed push-out* (M,p,B,s) *of the cotraids*

$$(I\times X_0,\mathrm{id}\times p_0,I\times B_0,\mathrm{id}\times s_0)\xleftarrow{(\sigma_0,\delta_0)}(X_0,p_0,B_0,s_0)\xrightarrow{(u,\gamma)}(X_1,p_1,B_1,s_1)$$

where (σ_0,δ_0) *is an* $\mathbf{M}$*-fibrewise embedding to 0-level, as follows :* $M=(I\times X_0+X_1)/\sim$ *and* $B=(I\times B_0+B_1)/\approx$, *where* $(0,a)\sim u(a)$ *for* $a\in X_0$ *and* $(0,b)\approx\gamma(b)$ *for* $b\in B_0$, *and* $p:M\to B$ *and* $s:B\to M$ *are defined, respectively, by*

$$p(x)=\begin{cases}[\gamma p_0(a)] & \text{if } x=[u(a)], a\in X_0\\ [t,p_0(a)] & \text{if } x=[t,a], t\neq 0\\ [p_1(x)] & \text{if } x\in X_1-u(X_0),\end{cases}$$

$$s(b)=\begin{cases}[s_1\alpha(d)] & \text{if } b=[\alpha(d)], d\in B_0\\ [t,s_0(d)] & \text{if } b=[t,d], t\neq 0\\ [s_1(b)] & \text{if } b\in B_1-\alpha(B_0),\end{cases}$$

where [*] is the equivalence class. Then it is easily verified that p *and* s *are well-defined and continuous. We call the* $\mathbf{M}$*-fibrewise pointed push-out of the cotriad the* $\mathbf{M}$-fibrewise pointed mapping cylinder *of* (u,γ), *and denote by* $M(u,\gamma)$.

Now we shall consider the case in which (X_0,p_0,B_0,s_0) is an $\mathbf{M}$-fibrewise pointed subspace of (X_1,p_1,B_1,s_1) and $(u,\gamma):(X_0,p_0,B_0,s_0)\to$

(X_1, p_1, B_1, s_1) is the inclusion. We can define an **M**-fibrewise pointed map $(e, \epsilon) : (M, p, B, s) \to (0 \times X_1 \cup I \times X_0, id \times p_1, 0 \times B_1 \cup I \times B_0, id \times s_1)$ by

$$e(x) = \begin{cases} (0, a) & \text{if } x = [u(a)],\ a \in X_0 \\ (t, a) & \text{if } x = [t, a],\ t \neq 0 \\ (0, x) & \text{if } x \in X_1 - u(X_0), \end{cases}$$

$\epsilon(b)$ is defined by a similar way. Moreover if X_0 is closed in X_1 and B_0 is closed in B_1, the maps e and ϵ are homeomorphisms and we may identify (M, p, B, s) with $(0 \times X_1 \cup I \times X_0, id \times p_1, 0 \times B_1 \cup I \times B_0, id \times s)$.

For each **M**-fibrewise pointed map $(u, \gamma) : (X_0, p_0, B_0, s_0) \to (X_1, p_1, B_1, s_1)$, we can define an **M**-fibrewise pointed map $(k, \xi) : (M, p, B, s) \to (I \times X_1, id \times p_1, I \times B_1, id \times s_1)$ by

$$k(x) = \begin{cases} (0, u(a)) & \text{if } x = [u(a)],\ a \in X_0 \\ (t, u(a)) & \text{if } x = [t, a],\ t \neq 0 \\ (0, x) & \text{if } x \in X_1 - u(X_0), \end{cases}$$

$$\xi(b) = \begin{cases} (0, \gamma(d)) & \text{if } b = [\gamma(d)],\ d \in B_0 \\ (t, \gamma(d)) & \text{if } x = [t, d],\ t \neq 0 \\ (0, b) & \text{if } b \in B_1 - \gamma(B_0). \end{cases}$$

Then we can obtain the following proposition by the same method of the proof in [4; Theorem 3.1].

PROPOSITION 2.6. *The* **M**-*fibrewise pointed map* $(u, \gamma) : (X_0, p_0, B_0, s_0) \to (X_1, p_1, B_1, s_1)$ *is an* **M**-*fibrewise pointed cofibration if and only if there is an* **M**-*fibrewise pointed map* $(L, l) : (I \times X_1, id \times p_1, I \times B_1, id \times s_1) \to (M, p, B, s)$ *such that* $Lk = id_M, l\xi = id_B$, *where* (M, p, B, s) *is the same one in Definition 2.5 and* $(k, \xi) : (M, p, B, s) \to (I \times X_1, id \times p_1, I \times B_1, id \times s_1)$ *is the same one in the above.*

The following lemma is used in the next section.

LEMMA 2.7. *For an* **M**-*fibrewise pointed map* $(u, \gamma) : (X_0, p_0, B_0, s_0) \to (X_1, p_1, B_1, s_1)$, *let* $M(u, \gamma) = (M, p, B, s)$ *be the* **M**-*fibrewise pointed mapping cylinder of* (u, γ) *constructed in Definition 2.5. Then the* **M**-*fibrewise pointed map* $(\sigma_1, \delta_1) : (X_0, p_0, B_0, s_0) \to M(u, \gamma)$ *is an* **M**-*fibrewise pointed cofibration, where* (σ_1, δ_1) *is defined by* $\sigma_1(x) = (1, x), \delta_1(b) = (1, b)$.

PROOF. Note that we can define an **M**-fibrewise pointed map $(k, \xi) : M(\sigma_1, \delta_1) \to (I \times M, id \times p, I \times B, id \times s)$ by the same method in Proposition 2.6. We shall prove this lemma by using Proposition 2.6. For this purpose, we now define an **M**-fibrewise pointed function

$$\begin{aligned}(J,j):(I\times(X_1+I\times X_0),id\times(p_1+id\times p_0),\\ I\times(B_1+I\times B_0),id\times(s_1+id\times s_0)\\ \longrightarrow((0\times X_1+\overline{I\times I}\times X_0,0\times p_1+\overline{id\times id}\times p_0,\\ 0\times B_1+\overline{I\times I}\times B_0,0\times s_1+\overline{id\times id}\times s_0\end{aligned}$$

where $\overline{I\times I}=(0\times I)\cup(I\times 1)$ and $\overline{id\times id}=id\times id|\overline{I\times I}$.

Let $r:I\times I\to\overline{I\times I}$ be a retraction defined by the projection from the point (2,0). Then (J,j) is defined by

$$\begin{aligned}J(t,x)&=(0,x) && (x\in X_1)\\ J(t,(t',x_0))&=(r(t,t'),x_0) && (t,t'\in I,x_0\in X_0)\\ j(t,b)&=(0,b) && (b\in B_1)\\ j(t,(t',b_0))&=(r(t,t'),b_0) && (t,t'\in I,b_0\in B_0).\end{aligned}$$

Then it is easy to see that for $x_0\in X_0$

$$J(t,(0,x_0))=(r(t,0),x_0)=(0,(0,x_0))=(0,u(x_0))=J(t,u(x_0)).$$

similarly $j(t,(0,b_0))=j(t,\gamma(b_0))$ and (J,j) is an **M**-fibrewise pointed map. Therefore it is easily verified that (J,j) induces the **M**-fibrewise pointed retraction

$$(L,l):(I\times M,id\times p,I\times B,id\times s)\to M(\sigma_1,\delta_1)$$

such that $Lk=id_{\overline{M}},l\xi=id_{\overline{B}}$, where $\overline{M}$ and $\overline{B}$ are the total space and the base space of $M(\sigma_1,\delta_1)$, respectively. Thus by using Proposition 2.6, we complete the proof. □

We now define **M**-fibrewise pointed collapse, **M**-fibrewise pointed cone and **M**-fibrewise pointed nulhomotopic.

Definition 2.8. (1) *Let (X,p,B,s)be an* **M***-fibrewise pointed space and (X_0,p_0,B_0,s_0) a closed* **M***-fibrewise pointed subspace. Let $\tilde{X}$ be a set $\cup_{b\in B}X_b/X_{0b}$, where $X_b=p^{-1}(b)$ and $X_{0b}=p_0^{-1}(b)$ for $b\in B$ (or $b\in B_0$). We introduce the set $\tilde{X}$ the quotient topology of X and put $\tilde{B}=B$. If we define maps $\tilde{p}:\tilde{X}\to\tilde{B}$ and $\tilde{s}:\tilde{B}\to\tilde{X}$ indeced by p and s respectively, then $(\tilde{X},\tilde{p},\tilde{B},\tilde{s})$ is an* **M***-fibrewise pointed space. We call $(\tilde{X},\tilde{p},\tilde{B},\tilde{s})$ an* **M**-fibrewise pointed collapse *of (X,p,B,s) with respect to (X_0,p_0,B_0,s_0) and denoted by*

$$(X,p,B,s)/_{\mathbf{M}}(X_0,p_0,B_0,s_0).$$

(For fibrewise collapse, see [5; section 5].)

(2) *For an* **M***-fibrewise pointed space (X,p,B,s), we call the* **M***-fibrewise pointed collapse*

$$(I\times X,id\times p,I\times B,id\times s)/_{\mathbf{M}}(1\times X,id\times p|1\times X,1\times B,id\times s|1\times B)$$

the **M**-fibrewise pointed cone *of* (X, p, B, s) *and denote by* $\Gamma(X, p, B, s)$. *(We denote the total space of* $\Gamma(X, p, B, s)$ *by* CX.*)*

Definition 2.9. *Let* $(\phi, \alpha) : (X, p, B, s) \to (X', p', B', s')$ *be an* **M**-*fibrewise pointed map. Then we call* (ϕ, α) *to be* **M**-fibrewise pointed nulhomotopic *if there is an* **M**-*fibrewise pointed map* $(c, \alpha_c) : (X, p, B, s) \to (X', p', B', s')$ *such that* $c = s'\alpha_c p$ *and* $(\phi, \alpha) \simeq^{\mathbf{M}}_{(\mathbf{P})} (c, \alpha_c)$.

We now prove the following proposition which is used in next section.

Proposition 2.10. *Let* $(\phi, \alpha) : (X, p, B, s) \to (X', p', B', s')$ *be an* **M**-*fibrewise pointed map. Then* (ϕ, α) *is* **M**-*fibrewise pointed nulhomotopic if and only if* $(\phi, \alpha) \circ (i, \epsilon)^{-1}$ *can be extended to an* **M**-*fibrewise pointed map of* $\Gamma(X, p, B, s)$ *to* (X', p', B', s'), *where* $(i, \epsilon) : (X, p, B, s) \to \Gamma(X, p, B, s)$ *is the natural embedding to 0-level of* $\Gamma(X, p, B, s)$.

Proof. "Only if" part: Let (ϕ, α) be **M**-fibrewise pointed nulhomotopic. By Definition 2.9, there is an **M**-fibrewise pointed map $(c, \alpha_c) : (X, p, B, s) \to (X', p', B', s')$ such that $c = s'\alpha_c p$, and there is an **M**-fibrewise pointed homotopy

$$(H, h) : (I \times X, id \times p, I \times B, id \times s) \to (X', p', B', s')$$

such that $(H_0, h_0) = (\phi, \alpha)$ and $(H_1, h_1) = (c, \alpha_c)$. Now, we can define an **M**-fibrewise pointed map

$$(\tilde{\phi}, \tilde{\alpha}) : \Gamma(X, p, B, s) \to (X', p', B', s')$$

by

$$\tilde{\phi}([t, x]) = H(t, x), \quad \tilde{\alpha}(t, b) = h(t, b).$$

Because it is obvious that $(\tilde{\phi}, \tilde{\alpha})$ is an **M**-fibrewise pointed map by the facts

$$p'\tilde{\phi}([t, x]) = p'H(t, x) = h(t, p(x)) = \tilde{\alpha}(t, p(x)) = \overline{p}([t, x])$$

where $\tilde{p}$ is the projection from the total space of $\Gamma(X, p, B, s)$ to the base space $I \times B$, and

$$\tilde{\phi}(id \times s)(t, b) = \tilde{\phi}([t, s(b)]) = H(t, s(b)) = H(id \times s)(t, b) = s'h(t, b) = s'\tilde{\alpha}(t, b)$$

since (H, h) is **M**-fibrewise pointed. Furthermore, it is easy to see that $(\tilde{\phi}, \tilde{\alpha}) \circ (i, \epsilon) = (\phi, \alpha)$, so $(\phi, \alpha) \circ (i, \epsilon)^{-1}$ can be extended to an **M**-fibrewise pointed map of $\Gamma(X, p, B, s)$ to (X', p', B', s').

"If" part: Let an **M**-fibrewise pointed map

$$(\tilde{\phi}, \tilde{\alpha}) : \Gamma(X, p, B, s) \to (X', p', B', s')$$

be an extension of the **M**-fibrewise pointed map $(\phi, \alpha) \circ (i, \epsilon)^{-1}$. Now, we can define **M**-fibrewise pointed functions

$$(H, h) : (I \times X, id \times p, I \times B, id \times s) \to (X', p', B', s')$$
$$(c, \alpha_c) : (X, p, B, s) \to (X', p', B', s')$$

by

$$\begin{aligned} H(t,x) &= \tilde{\phi}([t,x]) \\ h(t,b) &= \tilde{\alpha}(t,b) \\ c &= H_1 \\ \alpha_c &= h_1. \end{aligned}$$

Then it is clear that $(H,h),(c,\alpha_c)$ are continuous and $H_0=\phi, h_0=\alpha$. Further, since $(\tilde{\phi},\tilde{\alpha})$ is an **M**-fibrewise pointed map, we have

$$H(id\times s)(t,b)=H(t,s(b))=\tilde{\phi}([t,s(b)])=\tilde{\phi}(id\times s)(t,b)=s'\tilde{\alpha}(t,b)=s'h(t,b).$$

Therefore (H,h) is an **M**-fibrewise pointed map. Furthermore by the fact

$$\begin{aligned} s'\alpha_c p(x) &= s'\alpha_c(p(x))=s'\tilde{\alpha}(1,p(x))=s'\tilde{\alpha}\overline{p}([1,x]) \\ &= \tilde{\phi}([1,x])=H_1(x)=c(x), \end{aligned}$$

it is easy to see that $s'\alpha_c p=c$. Thus, (ϕ,α) is **M**-fibrewise pointed nulhomotopic. □

3. Puppe exact sequence

The main purpose of this section is the proof of Puppe exact sequence in **MAP**. (For this sequence, see [9] in the category **TOP** and [5] in $\mathbf{TOP}_B$.) We begin with defining **M**-fibrewise pointed mapping cone, **M**-fibrewise pointed contractible, **M**-fibrewise pointed suspension and exactness of sequence of **M**-fibrewise pointed maps.

Definition 3.1. *For an* **M***-fibrewise pointed map* $(\phi,\alpha):(X,p,B,s)\to(X',p',B',s')$*, we call* $(CX\cup_\phi X',\tilde{p},(I\times B)\cup_\alpha B',\tilde{s})$ *the* **M**-fibrewise pointed mapping cone *of* (ϕ,α)*, and denote* $\Gamma(\phi,\alpha)$*, where* CX *is the space in Definition 2.8.*

In this definition, the maps

$$\tilde{p}:CX\cup_\phi X'\to(I\times B)\cup_\alpha B',\quad \tilde{s}:(I\times B)\cup_\alpha B'\to CX\cup_\phi X'$$

are defined as the **M**-fibrewise pointed maps induced by the following maps, respectively:

$$\overline{p}:CX+X'\to(I\times B)+B',\quad \overline{s}:(I\times B)+B'\to CX+X'$$

$$\begin{array}{ll} \overline{p}([t,x])=(t,p(x)) & ([t,x]\in CX) \\ \overline{p}(x')=p'(x') & (x'\in X') \\ \overline{s}(t,b)=[t,s(b)] & ((t,b)\in I\times B) \\ \overline{s}(b')=s'(b') & (b'\in B'), \end{array}$$

where $CX\cup_\phi X',(I\times B)\cup_\alpha B'$ are adjunction spaces, respectively, determined by

$$\phi:X=0\times X\to X',\quad \alpha:B=0\times B\to B'.$$

For fibrewise adjunction space, see [6].

For an **M**-fibrewise pointed map $(\phi,\alpha):(X,p,B,s)\to(X',p',B',s')$ and the **M**-fibrewise mapping cone $\Gamma(\phi,\alpha)$, it is easy to see that there is the natural embedding

$$(\phi',\alpha'):(X',p',B',s')\to\Gamma(\phi,\alpha).$$

For two **M**-fibrewise pointed spaces (X,p,B,s) and (X',p',B',s'), we consider the set of all **M**-fibrewise pointed homotopy classes of **M**-fibrewise pointed maps from (X,p,B,s) to (X',p',B',s'), and denote it by $\pi((X,p,B,s),(X',p',B',s'))$. Then for **M**-fibrewise pointed map $(\phi,\alpha):(X,p,B,s)\to(X',p',B',s')$ and any **M**-fibrewise pointed space (X'',p'',B'',s''), we can define an induced map

$$(\phi,\alpha)^*:\pi((X',p',B',s'),(X'',p'',B'',s''))\to\pi((X,p,B,s),(X'',p'',B'',s''))$$

of (ϕ,α) by $(\phi,\alpha)^*([\phi',\alpha'])=[\phi'\phi,\alpha'\alpha]$. It is easy to see that this map is well-defined. Now we shall define exactness of a sequence of **M**-fibrewise pointed maps.

Definition 3.2. *A sequence of* **M***-fibrewise pointed maps*

$$(X_1,p_1,B_1,s_1)\xrightarrow{(\phi_1,\alpha_1)}(X_2,p_2,B_2,s_2)\xrightarrow{(\phi_2,\alpha_2)}(X_3,p_3,B_3,s_3)\xrightarrow{(\phi_3,\alpha_3)}\cdots$$

is exact *if for any* **M***-fibrewise pointed space* (X',p',B',s') *the induced sequence from one in the above*

$$\begin{aligned}\pi((X_1,p_1,B_1,s_1),(X',p',B',s'))&\xleftarrow{(\phi_1,\alpha_1)^*}\pi((X_2,p_2,B_2,s_2),(X',p',B',s'))\\&\xleftarrow{(\phi_2,\alpha_2)^*}\pi((X_3,p_3,B_3,s_3),(X',p',B',s'))\\&\xleftarrow{(\phi_3,\alpha_3)^*}\cdots\end{aligned}$$

is exact.

Remark 3.3. Note that the latter is exact if

$$\ker(\phi_i,\alpha_i)^*=\operatorname{im}(\phi_{i+1},\alpha_{i+1})^*$$

where

$$\ker(\phi_i,\alpha_i)^*=\{[\psi_{i+1},\beta_{i+1}]|(\psi_{i+1}\phi_i,\beta_{i+1}\alpha_i)\text{ is }\mathbf{M}\text{-fibrewise pointed nulhomotopic}\}.$$

We have the following proposition for exactness.

Proposition 3.4. *For* **M***-fibrewise pointed map* $(\phi,\alpha):(X,p,B,s)\to(X',p',B',s')$ *and* **M***-fibrewise pointed space* (X'',p'',B'',s'')*, the sequence*

$$\begin{aligned}\pi((X,p,B,s),(X'',p'',B'',s''))&\xleftarrow{(\phi,\alpha)^*}\pi((X',p',B',s'),(X'',p'',B'',s''))\\&\xleftarrow{(\phi',\alpha')^*}\pi(\Gamma(\phi,\alpha),(X'',p'',B'',s''))\end{aligned}$$

is exact.

PROOF. $\mathrm{im}(\phi', \alpha')^* \subset \ker(\phi, \alpha)^*$: It is easy to see that the **M**-fibrewise pointed map

$$(\phi', \alpha') \circ (\phi, \alpha) : (X, p, B, s) \to \Gamma(\phi, \alpha)$$

is extended to an **M**-fibrewise pointed map $\Gamma(X, p, B, s) \to \Gamma(\phi, \alpha)$. Therefore, by Proposition 2.10 $(\phi', \alpha') \circ (\phi, \alpha)$ is **M**-fibrewise pointed nulhomotopic.

$\ker(\phi, \alpha)^* \subset \mathrm{im}(\phi', \alpha')^*$: Let

$$(\psi, \beta) : (X', p', B', s') \to (X'', p'', B'', s'')$$

be an **M**-fibrewise pointed map such that $(\psi, \beta) \circ (\phi, \alpha)$ is **M**-fibrewise pointed nulhomotopic. Let

$$(\tilde{H}, \tilde{h}) : (I \times X, \mathrm{id} \times p, I \times B, \mathrm{id} \times s) \to (X'', p'', B'', s'')$$

be the **M**-fibrewise pointed nulhomotopy. Then by using $\tilde{H}, \phi, \psi$ and $\tilde{h}, \alpha, \beta$, we can construct

$$(\psi', \beta') : \Gamma(\phi, \alpha) \to (X'', p'', B'', s'')$$

such that $(\psi', \beta') \circ (\phi', \alpha') = (\psi, \beta)$. □

We shall define **M**-fibrewise pointed contractible and prove two propositions connecting with this concept.

DEFINITION 3.5. *An* **M**-*fibrewise pointed space* (X, p, B, s) *is* **M**-fibrewise pointed contractible *if there is an* **M**-*fibrewise pointed space* (B', p', B', p'^{-1}) *(where* p' *is a homeomorphism) such that*

$$(X, p, B, s) \cong^{\mathbf{M}}_{(\mathbf{P})} (B', p', B', p'^{-1}).$$

PROPOSITION 3.6. *An* **M**-*fibrewise pointed space* (X, p, B, s) *is* **M**-*fibrewise pointed contractible if and only if* (id_X, id_B) *is* **M**-*fibrewise pointed nulhomotopic.*

PROOF. "Only if" part: Let an **M**-fibrewise pointed space (X, p, B, s) be **M**-fibrewise pointed contractible. By the definition, there are an **M**-fibrewise pointed space (B', p', B', p'^{-1}) and an **M**-fibrewise pointed homotopy equivalence

$$(\phi, \alpha) : (X, p, B, s) \to (B', p', B', p'^{-1})$$

satisfying $\phi = p'^{-1}\alpha p$. Let (ψ, β) be an **M**-fibrewise pointed homotopy inverse of (ϕ, α). Then by the fact

$$\begin{aligned} s(\beta\alpha)p &= (s\beta p')(p'^{-1}\alpha p) \\ &= \psi\phi \end{aligned}$$

and $(\psi\phi, \beta\alpha) \simeq^{\mathbf{M}}_{(\mathbf{P})} (id_X, id_B)$, (id_X, id_B) is **M**-fibrewise pointed nulhomotopic.

"If" part: Let (id_X, id_B) be **M**-fibrewise pointed nulhomotopic. There is an **M**-fibrewise pointed map

$$(c, \alpha_c) : (X, p, B, s) \to (X, p, B, s)$$

such that $c = s\alpha_c p$ and $(id_X, id_B) \simeq^{\mathbf{M}}_{(\mathbf{P})} (c, \alpha_c)$. Then we shall prove

$$(X, p, B, s) \cong^{\mathbf{M}}_{(\mathbf{P})} (B, id_B, B, id_B).$$

First, since

$$(pc, \alpha_c) : (X, p, B, s) \to (B, id_B, B, id_B)$$

is **M**-fibrewise pointed, we have

$$\begin{aligned} s(pc) &= sp(s\alpha_c p) \\ &= s\alpha_c p \\ &= c \\ id_B\alpha_c &= \alpha_c \end{aligned}$$

and

$$(s(pc), id_B\alpha_c) = (c, \alpha_c) \simeq^{\mathbf{M}}_{(\mathbf{P})} (id_X, id_B).$$

Next, since

$$\begin{aligned} (pc)s &= p(s\alpha_c p)s \\ &= \alpha_c \\ \alpha_c id_B &= \alpha_c, \end{aligned}$$

we have

$$((pc)s, \alpha_c id_B) = (\alpha_c, \alpha_c) \simeq^{\mathbf{M}}_{(\mathbf{P})} (id_B, id_B).$$

Thus (s, id_B) is an **M**-fibrewise pointed homotopy inverse of (pc, α_c). Therefore, (pc, α_c) is an **M**-fibrewise pointed homotopy equivalence. □

PROPOSITION 3.7. *For an* **M***-fibrewise pointed space* (X, p, B, s)*, the* **M***-fibrewise pointed cone* $\Gamma(X, p, B, s)$ *is* **M***-fibrewise pointed contractible.*

PROOF. We define an **M**-fibrewise homotopy

$$(H_\tau, h_\tau) : \Gamma(X, p, B, s) \to \Gamma(X, p, B, s)$$

by

$$\begin{aligned} H_\tau(t, x) &= (t + \tau(1 - t), x) \\ h_\tau(t, b) &= (t + \tau(1 - t), b). \end{aligned}$$

Then this (H_τ, h_τ) is an **M**-fibrewise pointed nulhomotopy of (id_X, id_B). Therefore we complete the proof. □

From now on, to prove Puppe exact sequence in **MAP**, we shall give some propositions.

Proposition 3.8. *Let* $((X,p,B,s),(X_0,p_0,B_0,s_0))$ *be a closed* **M**-*fibrewise pointed cofibred pair. Assume that* (X_0,p_0,B_0,s_0) *is* **M**-*fibrewise pointed contractible. Then the natural projection*

$$(\pi, id_B):(X,p,B,s)\to (X,p,B,s)/_{\mathbf{M}}(X_0,p_0,B_0,s_0)$$

is an **M**-*fibrewise pointed homotopy equivalence.*

Proof. Let

$$(F_t,f_t):(X_0,p_0,B_0,s_0)\to (X_0,p_0,B_0,s_0)$$

be an **M**-fibrewise pointed nulhomotopy of (id_{X_0}, id_{B_0}), where (F_1,f_1) is **M**-fibrewise pointed constant. By the assumption, for

$$(id_X, id_B):(X,p,B,s)\to (X,p,B,s)$$

(F_t,f_t) can be extended to an **M**-fibrewise pointed homotopy $(H_t,h_t):(X,p,B,s)\to(X,p,B,s)$ of (id_X,id_B). Let $(H_1',h_1')=(H_1,h_1)(\pi,id_B)^{-1}$. Then the **M**-fibrewise pointed map

$$(H_1',h_1'):(X,p,B,s)/_{\mathbf{M}}(X_0,p_0,B_0,s_0)\to (X,p,B,s)$$

is induced from (H_t,h_t). Further $(\pi,id_B)\circ(H_t,h_t)$ induces an **M**-fibrewise pointed homotopy

$$(H_t'',h_t''):(X,p,B,s)/_{\mathbf{M}}(X_0,p_0,B_0,s_0)\to (X,p,B,s)/_{\mathbf{M}}(X_0,p_0,B_0,s_0).$$

Note that $H_1''=\pi H_1', h_1''=h_1'$. Then since

$$(H_1',h_1')\circ(\pi,id_B)=(H_1,h_1)\simeq^{\mathbf{M}}_{(\mathbf{P})}(H_0,h_0)=(id_X,id_B),$$

$$(\pi,id_B)\circ(H_1',h_1')=(H_1'',h_1'')\simeq^{\mathbf{M}}_{(\mathbf{P})}(H_0'',h_0'')$$

and (H_0'',h_0'') is the identity of $(X,p,B,s)/_{\mathbf{M}}(X_0,p_0,B_0,s_0)$, (π,id_B) is an **M**-fibrewise pointed homotopy equivalence. This completes the proof. □

Proposition 3.9. *Let* $(\phi,\alpha):(X,p,B,s)\to(X',p',B',s')$ *be an* **M**-*fibrewise pointed cofibration. Then the natural projection*

$$(\pi,id):\Gamma(\phi,\alpha)\to\Gamma(\phi,\alpha)/_{\mathbf{M}}\Gamma(X,p,B,s)\cong^{\mathbf{M}}_{(\mathbf{P})}(X',p',B',s')/_{\mathbf{M}}(X,p,B,s)$$

is an **M**-*fibrewise pointed homotopy equivalence.*

Proof. We shall construct an **M**-fibrewise pointed homotopy of (id,id) of the **M**-fibrewise pointed mapping cone $\Gamma(\phi,\alpha)$ which deforms the **M**-fibrewise pointed cone $\Gamma(X,p,B,s)$ into its section. Since, if this is done, $\Gamma(X,p,B,s)$ is an **M**-fibrewise pointed contractible in $\Gamma(\phi,\alpha)$, we can obtain the result from Proposition 3.8.

We can now define an **M**-fibrewise pointed nulhomotopy $(H,h):I\times\Gamma(X,p,B,s)\to\Gamma(\phi,\alpha)$ of the inclusion $\Gamma(X,p,B,s)\to\Gamma(\phi,\alpha)$ by

$$\begin{aligned} H(t,[t',x]) &= [t'+t(1-t'),x] \\ h(t,t',b) &= (t'+t(1-t'),b) \end{aligned}$$

Since (ϕ,α) is an **M**-fibrewise pointed cofibration, (H,h) can be extended to an **M**-fibrewise pointed homotopy

$$(G,g): I\times(X',p',B',s')\to\Gamma(\phi,\alpha)$$

of the inclusion $(X',p',B',s')\to\Gamma(\phi,\alpha)$. Then by using (H,h) and (G,g) we can construct an **M**-fibrewise pointed homotopy of (id,id) of the **M**-fibrewise pointed mapping cone $\Gamma(\phi,\alpha)$ which deforms the **M**-fibrewise pointed cone $\Gamma(X,p,B,s)$ into its section. ∎

By using Lemma 2.7 we can prove the next proposition.

PROPOSITION 3.10. *For an* **M***-fibrewise pointed map* $(\phi,\alpha):(X,p,B,s)\to(X',p',B',s')$*, the inclusion* $(\phi',\alpha'):(X',p',B',s')\to\Gamma(\phi,\alpha)$ *is an* **M***-fibrewise pointed cofibration.*

PROOF. Let $M(p,id_B)$ be the **M**-fibrewise pointed mapping cylinder of $(p,id_B):(X,p,B,s)\to(B,id_B,B,id_B)$. Then it is easy to see that $\Gamma(X,p,B,s)$ and $M(p,\mathrm{id}_B)$ are **M**-fibrewise pointed equivalent by an **M**-fibrewise pointed map of $\Gamma(X,p,B,s)$ to $M(p,\mathrm{id}_B)$ defined by

$$[t,x]\mapsto[1-t,x],\qquad (t,b)\mapsto(1-t,b)$$

Note by Lemma 2.7 that the map $(\sigma_1,\delta_1):(X,p,B,s)\to\Gamma(X,p,B,s)$ which maps to 1-level is an **M**-fibrewise pointed cofibration.

For any **M**-fibrewise pointed homotopy $(H_t,h_t):(X',p',B',s')\to(X'',p'',B'',s'')$, $(H_t\circ\phi,h_t\circ\alpha):(X,p,B,s)\to(X'',p'',B'',s'')$ is an **M**-fibrewise pointed homotopy. Since (σ_1,δ_1) in the above is an **M**-fibrewise pointed cofibration, $(H_t\circ\phi,h_t\circ\alpha)$ can be extended to $\Gamma(X,p,B,s)$. Thus by patching the extended homotopy and (H_t,h_t) in $\Gamma(\phi,\alpha)$ we can construct an **M**-fibrewise pointed homotopy of $\Gamma(\phi,\alpha)$ to (X'',p'',B'',s''). This completes the proof. ∎

DEFINITION 3.11. *For an* **M***-fibrewise pointed space* (X,p,B,s)*, let* $\overline{X}=\{0,1\}\times X$*,* $\overline{p}=id\times p|\{0,1\}\times X$*,* $\overline{B}=\{0,1\}\times B$ *and* $\overline{s}=id\times s|\{0,1\}\times B$*. Then the* **M***-fibrewise pointed collapse*

$$(I\times X,id\times p,I\times B,id\times s)/_{\mathbf{M}}(\overline{X},\overline{p},\overline{B},\overline{s})$$

is called to be **M**-fibrewise pointed suspension*, and denoted by* $\Sigma(X,p,B,s)$*. (We denote the total space of* $\Sigma(X,p,B,s)$ *by* ΣX*, and the projection of* $\Sigma(X,p,B,s)$ *by* Σp*.)*

PROPOSITION 3.12. *Let* $(\phi,\alpha):(X,p,B,s)\to(X',p',B',s')$ *be an* **M***-fibrewise pointed map, where* α *is a bijection. Then for the inclusion* $(\phi',\alpha'):(X',p',B',s')\to\Gamma(\phi,\alpha)$*,* $\Gamma(\phi',\alpha')$ *is* **M***-fibrewise pointed equivalent to the* **M***-fibrewise pointed suspension* $\Sigma(X,p,B,s)$*.*

PROOF. We use the same notation of Proposition 3.10. Since (ϕ', α') is an **M**-fibrewise pointed cofibration from Proposition 3.10, using Proposition 3.9, $\Gamma(\phi', \alpha')$ is **M**-fibrewise pointed homotopy equivalent to

$$\Gamma(\phi', \alpha')/_{\mathbf{M}}\Gamma(X', p', B', s') = \Gamma(\phi, \alpha)/_{\mathbf{M}}(X', p', B', s') = \Sigma(X, p, B, s).$$

This competes the proof. ∎

In the process in the above, $((\phi')', (\alpha')')$ is transformed into an **M**-fibrewise pointed map

$$(\phi'', \alpha'') : \Gamma(\phi, \alpha) \to \Sigma(X, p, B, s).$$

Repeating this process we find that $\Gamma((\phi')', (\alpha')')$ is **M**-fibrewise pointed equivalent to **M**-fibrewise pointed suspension $\Sigma(X', p', B', s')$, and in the process $(((\phi')')', ((\alpha')')')$ is transformed into the **M**-fibrewise pointed suspension

$$(\Sigma\phi, id \times \alpha) : \Sigma(X, p, B, s) \to \Sigma(X', p', B', s'),$$

where $\Sigma\phi$ is the map from ΣX to $\Sigma X'$. Thus we obtain the main theorem of Puppe exact sequence in **MAP**.

THEOREM 3.13. *For an* **M**-*fibrewise pointed map* $(\phi, \alpha) : (X, p, B, s) \to (X', p', B', s')$ *where* α *is a bijection, the following sequence is exact.*

$$\begin{array}{rcl} (X, p, B, s) & \xrightarrow{(\phi,\alpha)} & (X', p', B', s') \\ & \xrightarrow{(\phi',\alpha')} & \Gamma(\phi, \alpha) \\ & \xrightarrow{(\phi'',\alpha'')} & \Sigma(X, p, B, s) \\ & \xrightarrow{(\phi''',\alpha''')} & \Sigma(X', p', B', s') \\ & \longrightarrow & \cdots \end{array}$$

4. AN APPLICATION OF PUPPE EXACT SEQUENCE

In this section, by applying Puppe exact sequence we shall prove the generalized formula for the suspension of **M**-fibrewise pointed product spaces, and the proof is simpler than the one in fibrewise version of [5; section 22]. (In our proof, we need not to consider any generalized concept of fibrewise non-degenerate spaces in [5].)

DEFINITION 4.1. (1) *For two* **M**-*fibrewise pointed spaces* (X_1, p_1, B_1, s_1) *and* (X_2, p_2, B_2, s_2) *let*

$$\begin{aligned} X_1 \vee^{\mathbf{M}} X_2 &= \cup_{(b,b') \in B_1 \times B_2}(X_{1,b} \times s_2(b') \cup s_1(b) \times X_{2,b'}), \\ p_1 \vee^{\mathbf{M}} p_2 &= p_1 \times p_2 | X_1 \vee^{\mathbf{M}} X_2. \end{aligned}$$

The **M**-*fibrewise pointed space* $(X_1 \vee^{\mathbf{M}} X_2, p_1 \vee^{\mathbf{M}} p_2, B_1 \times B_2, s_1 \times s_2)$ *is called the* **M**-fibrewise pointed coproduct *of* (X_1, p_1, B_1, s_1) *and*

(X_2, p_2, B_2, s_2), and denoted by $(X_1, p_1, B_1, s_1) \vee^{\mathbf{M}} (X_2, p_2, B_2, s_2)$ *or* $\bigvee^{\mathbf{M}}_{i\in\{1,2\}}(X_i, p_i, B_i, s_i)$. *The* **M**-*fibrewise pointed collapse*

$$(X_1, p_1, B_1, s_1) \times (X_2, p_2, B_2, s_2)/_{\mathbf{M}}(X_1, p_1, B_1, s_1) \vee^{\mathbf{M}} (X_2, p_2, B_2, s_2)$$

is called the **M**-fibrewise smash product *of* (X_1, p_1, B_1, s_1) *and* (X_2, p_2, B_2, s_2), *and denoted by*

$$(X_1, p_1, B_1, s_1) \wedge^{\mathbf{M}} (X_2, p_2, B_2, s_2) \ \textit{or} \ \bigwedge^{\mathbf{M}}_{i\in\{1,2\}}(X_i, p_i, B_i, s_i).$$

Note that $(X_1, p_1, B_1, s_1) \times (X_2, p_2, B_2, s_2) = (X_1 \times X_2, p_1 \times p_2, B_1 \times B_2, s_1 \times s_2)$.

(2) *For* $n \geq 3$ *and* **M**-*fibrewise pointed spaces* (X_i, p_i, B_i, s_i) $(i = 1, \cdots, n)$, *we shall define* $\bigvee^{\mathbf{M}}_{i\in\{1,\cdots,n\}}(X_i, p_i, B_i, s_i)$ *and* $\bigwedge^{\mathbf{M}}_{i\in\{1,\cdots,n\}}(X_i, p_i, B_i, s_i)$ *inductively, as follows:*

$$\bigvee^{\mathbf{M}}_{i\in\{1,\cdots,n\}}(X_i, p_i, B_i, s_i) = (\bigvee^{\mathbf{M}}_{i\in\{1,\cdots,n-1\}}(X_i, p_i, B_i, s_i)) \vee^{\mathbf{M}} (X_n, p_n, B_n, s_n)$$

$$\bigwedge^{\mathbf{M}}_{i\in\{1,\cdots,n\}}(X_i, p_i, B_i, s_i) = (\bigwedge^{\mathbf{M}}_{i\in\{1,\cdots,n-1\}}(X_i, p_i, B_i, s_i)) \wedge^{\mathbf{M}} (X_n, p_n, B_n, s_n).$$

In this paper, we set up the following Hypothesis.

HYPOTHESIS: By Definitions 3.11 and 4.1, the base space of

$$\Sigma\{(X_1, p_1, B_1, s_1) \vee^{\mathbf{M}} (X_2, p_2, B_2, s_2)\}$$

and

$$\Sigma\{(X_1, p_1, B_1, s_1) \wedge^{\mathbf{M}} (X_2, p_2, B_2, s_2)\}$$

is $I \times (B_1 \times B_2)$, but the base space of $\Sigma(X_1, p_1, B_1, s_1) \vee^{\mathbf{M}} \Sigma(X_2, p_2, B_2, s_2)$ and $\Sigma(X_1, p_1, B_1, s_1) \wedge^{\mathbf{M}} \Sigma(X_2, p_2, B_2, s_2)$ is $(I \times B_1) \times (I \times B_2)$. So, since $\Sigma\{(X_1, p_1, B_1, s_1) \vee^{\mathbf{M}} (X_2, p_2, B_2, s_2)\}$ and $\Sigma(X_1, p_1, B_1, s_1) \vee^{\mathbf{M}} \Sigma(X_2, p_2, B_2, s_2)$, or $\Sigma\{(X_1, p_1, B_1, s_1) \wedge^{\mathbf{M}} (X_2, p_2, B_2, s_2)\}$ and $\Sigma(X_1, p_1, B_1, s_1) \wedge^{\mathbf{M}} \Sigma(X_2, p_2, B_2, s_2)$ have different base spaces, those **M**-fibrewise pointed spaces are different, respectively. We want to identify those spaces as **M**-fibrewise pointed space, we set the following hypothesis: In $\Sigma(X_1, p_1, B_1, s_1) \vee^{\mathbf{M}} \Sigma(X_2, p_2, B_2, s_2)$ and $\Sigma(X_1, p_1, B_1, s_1) \wedge^{\mathbf{M}} \Sigma(X_2, p_2, B_2, s_2)$, we always restrict the base space $(I \times B_1) \times (I \times B_2)$ to $\cup\{(t \times B_1) \times (t \times B_2) | t \in I\}$. By this hypothesis, the following equalities always hold.

$$\Sigma\{(X_1, p_1, B_1, s_1) \vee^{\mathbf{M}} (X_2, p_2, B_2, s_2)\} = \Sigma(X_1, p_1, B_1, s_1) \vee^{\mathbf{M}} \Sigma(X_2, p_2, B_2, s_2), \tag{4.1}$$

$$\Sigma\{(X_1, p_1, B_1, s_1) \wedge^{\mathbf{M}} (X_2, p_2, B_2, s_2)\} = \Sigma(X_1, p_1, B_1, s_1) \wedge^{\mathbf{M}} \Sigma(X_2, p_2, B_2, s_2). \tag{4.2}$$

In the rest of this paper, consider automatically these identifications if necessary. The we can prove easily the following lemma by the canonical correspondence.

LEMMA 4.2. *For* $\mathbf{M}$*-fibrewise pointed spaces* (X_i, p_i, B_i, s_i) $(i = 1, 2, 3)$*,*

(1)

$$((X_1, p_1, B_1, s_1) \vee^{\mathbf{M}} (X_2, p_2, B_2, s_2)) \vee^{\mathbf{M}} (X_3, p_3, B_3, s_3) \\ \cong^{\mathbf{M}}_{(\mathbf{P})} (X_1, p_1, B_1, s_1) \vee^{\mathbf{M}} ((X_2, p_2, B_2, s_2) \vee^{\mathbf{M}} (X_3, p_3, B_3, s_3))$$

(2)

$$((X_1, p_1, B_1, s_1) \wedge^{\mathbf{M}} (X_2, p_2, B_2, s_2)) \wedge^{\mathbf{M}} (X_3, p_3, B_3, s_3) \\ \cong^{\mathbf{M}}_{(\mathbf{P})} (X_1, p_1, B_1, s_1) \wedge^{\mathbf{M}} ((X_2, p_2, B_2, s_2) \wedge^{\mathbf{M}} (X_3, p_3, B_3, s_3))$$

(3)

$$((X_1, p_1, B_1, s_1) \wedge^{\mathbf{M}} (X_2, p_2, B_2, s_2)) \vee^{\mathbf{M}} (X_3, p_3, B_3, s_3) \\ \cong^{\mathbf{M}}_{(\mathbf{P})} ((X_1, p_1, B_1, s_1) \vee^{\mathbf{M}} (X_3, p_3, B_3, s_3)) \\ \wedge^{\mathbf{M}}((X_2, p_2, B_2, s_2) \vee^{\mathbf{M}} (X_3, p_3, B_3, s_3))$$

(4)

$$((X_1, p_1, B_1, s_1) \vee^{\mathbf{M}} (X_2, p_2, B_2, s_2)) \wedge^{\mathbf{M}} (X_3, p_3, B_3, s_3) \\ \cong^{\mathbf{M}}_{(\mathbf{P})} ((X_1, p_1, B_1, s_1) \wedge^{\mathbf{M}} (X_3, p_3, B_3, s_3)) \\ \vee^{\mathbf{M}}((X_2, p_2, B_2, s_2) \wedge^{\mathbf{M}} (X_3, p_3, B_3, s_3)).$$

DEFINITION 4.3. *For two sequences of* $\mathbf{M}$*-fibrewise pointed maps*

$$\mathcal{F} : (X_1, p_1, B_1, s_1) \to (X_2, p_2, B_2, s_2) \to \cdots \to (X_n, p_n, B_n, s_n) \to \cdots$$

$$\mathcal{F}' : (X'_1, p'_1, B'_1, s'_1) \to (X'_2, p'_2, B'_2, s'_2) \to \cdots \to (X'_n, p'_n, B'_n, s'_n) \to \cdots$$

if there are $\mathbf{M}$*-fibrewise pointed homotopy equivalences*

$$(\phi_n, \alpha_n) : (X_n, p_n, B_n, s_n) \to (X'_n, p'_n, B'_n, s'_n)$$

such that all diagrams induced by these maps are commutative, $\mathcal{F}$ *and* $\mathcal{F}'$ have the same $\mathbf{M}$-fibrewise pointed homotopy type.

The following proposition is obvious from the construction of Puppe exact sequence.

PROPOSITION 4.4. *Let* $\alpha : B \to B'$ *be a bijection. The* $\mathbf{M}$*-fibrewise pointed homotopy type of Puppe exact sequence (in the sense of* $\mathbf{M}$*-fibrewise pointed) induced from an* $\mathbf{M}$*-fibrewise pointed map* $(\phi, \alpha) : (X, p, B, s) \to (X', p', B', s')$ *is only depend on the* $\mathbf{M}$*-fibrewise pointed homotopy class of* (ϕ, α)*.*

In particular, if (ϕ, α) *is* $\mathbf{M}$*-fibrewise pointed nulhomotopic, the* $\mathbf{M}$*-fibrewise pointed homotopy type of the sequence induced from* (ϕ, α) *has the* *same* **M**-*fibrewise pointed homotopy type of the sequence induced from an* **M**-*fibrewise constant map* (c, α_c)

$$\begin{aligned}(X,p,B,s) &\xrightarrow{(c,\alpha_c)} (X',p',B',s')\\ &\longrightarrow (X',p',B',s') \vee^{\mathbf{M}} \Sigma(X,p,B,s)\\ &\longrightarrow \Sigma(X,p,B,s)\\ &\longrightarrow \cdots\end{aligned}$$

As an application of this proposition, we prove the generalized formula for the **M**-fibrewise pointed suspension of **M**-fibrewise pointed product spaces,

Let

$$(u, id) : (X,p,B,s) \vee^{\mathbf{M}} (X',p',B',s') \to (X,p,B,s) \times (X',p',B',s')$$

be an **M**-fibrewise pointed embedding. We denote the **M**-fibrewise pointed mapping cone $\Gamma(u, id)$ of (u, id) by $(X,p,B,s) \bar{\wedge}^{\mathbf{M}} (X',p',B',s')$. The Puppe sequence (in the sense of **M**-fibrewise pointed) of (u, id) is as follows:

$$\begin{aligned}(X,p,B,s) \vee^{\mathbf{M}} (X',p',B',s') &\xrightarrow{(u,id)} (X,p,B,s) \times (X',p',B',s')\\ &\xrightarrow{(v,\alpha_v)} (X,p,B,s) \bar{\wedge}^{\mathbf{M}} (X',p',B',s')\\ &\xrightarrow{(w,id)} \Sigma(X,p,B,s) \vee^{\mathbf{M}} \Sigma(X',p',B',s')\\ &\longrightarrow \cdots\end{aligned}$$

Then we obtain the following.

Proposition 4.5. *(w, id) in the above is* **M**-*fibrewise pointed nulhomotopic.*

Proof. Generally, for the inclusion $(f, i) : (X_0, p_0, B_0, s_0) \to (X_1, p_1, B_1, s_1)$, we can consider the **M**-fibrewise pointed mapping cone $\Gamma(f, i)$ of (f, i) as the subspace of $\Gamma(X_1, p_1, B_1, s_1)$. Then the inclusion $(g, \alpha_g) : \Gamma(f, i) \to \Gamma(X_1, p_1, B_1, s_1)$ is continuous.

By applying this to the inclusion $(X,p,B,s) \vee^{\mathbf{M}} (X',p',B',s') \to (X,p,B,s) \times (X',p',B',s')$, we find that the inclusion

$$(j, id) : (X,p,B,s) \bar{\wedge}^{\mathbf{M}} (X',p',B',s') \to \Gamma\{(X,p,B,s) \times (X',p',B',s')\}$$

is continuous. The **M**-fibrewise pointed map

$$(w, id) : (X,p,B,s) \bar{\wedge}^{\mathbf{M}} (X',p',B',s') \to \Sigma(X,p,B,s) \vee^{\mathbf{M}} \Sigma(X',p',B',s')$$

can be defined by

$$\begin{aligned} w(t,x,s'(b')) &= ((t,x),(t,s'(b'))) && (x \in X_b,\ t \in I - \{0\})\\ w(t,s(b),x') &= ((t,s(b)),(t,x')) && (x' \in X'_{b'},\ t \in I - \{0\})\\ w(0,x,x') &= ((0,s(b)),(0,s'(b'))) && (x \in X_b, x' \in X'_{b'}).\end{aligned}$$

Further, let

$$\overline{\Sigma}(X,p,B,s) = \Sigma(X,p,B,s)/_{\mathbf{M}}\left\{(t,x)|x\in X, t\leq \frac{1}{2}\right\}$$
$$\underline{\Sigma}(X',p',B',s') = \Sigma(X',p',B',s')/_{\mathbf{M}}\left\{(t,x')|x'\in X', t\geq \frac{1}{2}\right\}.$$

Then, since $\{(t,x)|x\in X, t\leq \frac{1}{2}\}$ and $\{(t,x)|x\in X, t\geq \frac{1}{2}\}$ are **M**-fibrewise pointed subspaces of $\Sigma(X,p,B,s)$, those are (**M**-fibrewise pointed) homeomorphic to an **M**-fibrewise pointed cone, so those are **M**-fibrewise pointed contractible from Proposition 3.7. Therefore from Proposition 3.8 we see that the natural projections

$$\Sigma(X,p,B,s)\to\overline{\Sigma}(X,p,B,s),\quad \Sigma(X',p',B',s')\to\underline{\Sigma}(X',p',B',s')$$

are **M**-fibrewise pointed homotopy equivaleces. By patching the projections, the **M**-fibrewise pointed map

$$(\rho,id):\Sigma(X,p,B,s)\vee^{\mathbf{M}}\Sigma(X',p',B',s')\to\overline{\Sigma}(X,p,B,s)\vee^{\mathbf{M}}\underline{\Sigma}(X',p',B',s')$$

is also an **M**-fibrewise pointed homotopy equivalence.

Next, we can define an **M**-fibrewise pointed map

$$(q,\mathrm{id}):\Gamma\{(X,p,B,s)\times(X',p',B',s')\}\to\overline{\Sigma}(X,p,B,s)\vee^{\mathbf{M}}\underline{\Sigma}(X',p',B',s')$$

as follows:

$$q(t,x,x')=\begin{cases}[(t,s(p(x))),(t,s'(p'(x')))] & (t\in\{0,\frac{1}{2},1\})\\ ((t,s(p(x))),(t,x')) & (0<t<\frac{1}{2})\\ ((t,x),(t,s'(p'(x')))) & (\frac{1}{2}<t<1).\end{cases}$$

Then it is easy to see that $(q,id)\circ(j,id)=(\rho,id)\circ(w,id)$. On the other hand, since $\Gamma\{(X,p,B,s)\times(X',p',B',s')\}$ is **M**-fibrewise pointed contractible from Proposition 3.7, $(\rho,id)\circ(w,id)$ is **M**-fibrewise pointed nulhomotopic. Since (ρ,id) is an **M**-fibrewise pointed homotopy equivalence, (w,id) is also **M**-fibrewise pointed nulhomotopic, which completes the proof. ∎

From Propositions 4.4 and 4.5, we find that the **M**-fibrewise pointed mapping cone $\Gamma(w,id)$ of (w,id) has the same **M**-fibrewise pointed homotopy type of

$$\Sigma(X,p,B,s)\vee^{\mathbf{M}}\Sigma(X',p',B',s')\vee^{\mathbf{M}}\Sigma\{(X,p,B,s)\,\overline{\wedge}^{\mathbf{M}}\,(X',p',B',s')\}.$$

Since the **M**-fibrewise pointed mapping cone $\Gamma(w,\mathrm{id})$ has the same **M**-fibrewise pointed homotopy type of $\Sigma\{(X,p,B,s)\times(X',p',B',s')\}$, we have the following.

Proposition 4.6.

$$\Sigma\{(X,p,B,s)\times(X',p',B',s')\}\cong^{\mathbf{M}}_{(\mathbf{P})}$$
$$\Sigma(X,p,B,s)\vee^{\mathbf{M}}\Sigma(X',p',B',s')\vee^{\mathbf{M}}\Sigma\{(X,p,B,s)\,\overline{\wedge}^{\mathbf{M}}\,(X',p',B',s')\}.$$

We now define the following.

Definition 4.7. *An* **M**-*fibrewise pointed space* (X, p, B, s) *is called* **M**-fibrewise well-pointed *if* $(s, id_B) : (B, id_B, B, id_B) \to (X, p, B, s)$ *is an* **M**-*fibrewise pointed cofibration and* $s(B)$ *is closed in* X.

The next lemma is obvious from Proposition 2.4 in this paper.

Lemma 4.8. *Assume that* **M**-*fibrewise pointed spaces* (X, p, B, s) *and* (X', p', B', s') *are* **M**-*fibrewise well-pointed. Then the* **M**-*fibrewise pointed pair*

$$((X, p, B, s) \times (X', p', B', s'), (X, p, B, s) \vee^{\mathbf{M}} (X', p', B', s'))$$

is an **M**-*fibrewise pointed cofibred pair.*

The next proposition is easily verified from this lemma and Proposition 3.9.

Proposition 4.9. *Assume that two* **M**-*fibrewise pointed spaces* (X, p, B, s) *and* (X', p', B', s') *are* **M**-*fibrewise well-pointed. Then the natural projection*

$$\begin{aligned}
&(X, p, B, s) \mathbin{\bar{\wedge}}^{\mathbf{M}} (X', p', B', s') \\
&\to (X, p, B, s) \mathbin{\bar{\wedge}}^{\mathbf{M}} (X', p', B', s')/_{\mathbf{M}}\Gamma\{(X, p, B, s) \vee^{\mathbf{M}} (X', p', B', s')\} \\
&= (X, p, B, s) \times (X', p', B', s')/_{\mathbf{M}}(X, p, B, s) \vee^{\mathbf{M}} (X', p', B', s') \\
&= (X, p, B, s) \wedge^{\mathbf{M}} (X', p', B', s')
\end{aligned}$$

is an **M**-*fibrewise pointed homotopy equivalence.*

We have the following from Propositions 4.6 and 4.9.

Corollary 4.10. *Assume that two* **M**-*fibrewise pointed spaces* (X, p, B, s) *and* (X', p', B', s') *are* **M**-*fibrewise well-pointed. Then the next formula holds.*

$$\begin{aligned}
&\Sigma\{(X, p, B, s) \times (X', p', B', s')\} \cong^{\mathbf{M}}_{(\mathbf{P})} \\
&\Sigma(X, p, B, s) \vee^{\mathbf{M}} \Sigma(X', p', B', s') \vee^{\mathbf{M}} \Sigma\{(X, p, B, s) \wedge^{\mathbf{M}} (X', p', B', s')\}.
\end{aligned}$$

By repeatedly using this formula and (4.1),(4.2), we can obtain the following formula.

Theorem 4.11. *Assume that* **M**-*fibrewise pointed spaces* (X_i, p_i, B_i, s_i), $(i = 1, \cdots n)$ *are* **M**-*fibrewise well-pointed. Then the next formula holds.*

$$\sum \left\{ \prod_{i=1}^{n} (X_i, p_i, B_i, s_i) \right\} \cong^{\mathbf{M}}_{(\mathbf{P})} \bigvee_{N}{}^{\mathbf{M}} \sum (\bigwedge_{i \in N}{}^{\mathbf{M}} (X_i, p_i, B_i, s_i))$$

where N *runs through all nonempty subsets of* $\{1, \ldots, n\}$.

5. An intermediate fibrewise category $\mathbf{TOP}_B^H$

The category **MAP** is an extended one of the category $\mathbf{TOP}_B$. Further, the proofs of theorems in **MAP** in this paper are simpler than those in $\mathbf{TOP}_B$ in a sense; for example, we need not to consider any generalized concept of fibrewise non-degenerate spaces [5; section 22]. But, we have to examine closely whether our theorems and propositions in this paper give another proofs of corresponding theorems and propositions in [5; sections 21 and 22]. For these, we introduce an intermediate fibrewise category $\mathbf{TOP}_B^H$ which combines **MAP** with $\mathbf{TOP}_B$. It is easily verified that Theorem 5.4 in $\mathbf{TOP}_B^H$ is proved by the same methods in **MAP**. Further, we can prove in Proposition 5.5 that a fibrewise non-degenerate space is H-fibrewise well pointed. Finally, we can give an another proof of [5] Proposition 22.11 as a corollary of Theorem 5.4 using Proposition 5.5.

In this section, for a fixed topological space B we consider in the category $\mathbf{TOP}_B$. In case considering homotopies in $\mathbf{TOP}_B$, for a fibrewise pointed space (X, p, B, s) we consider the fibrewise pointed space $(I \times X, id \times p, I \times B, id \times s)$, and the following fibrewise pointed spaces have the base space $I \times B$: fibrewise mapping cylinder, fibrewise pointed cone, fibrewise pointed mapping cone, fibrewise pointed suspension. Further, we naturally identify the diagonal Δ_B(or Δ_{B^n}) with B. Therefore we denote this category by $\mathbf{TOP}_B^H$. Terminologies "fibrewise $\cdots$" are in $\mathbf{TOP}_B$, and "H-fibrewise $\cdots$" are in $\mathbf{TOP}_B^H$. But we use the both in $\mathbf{TOP}_B^H$. We begin with the following definitions.

Definition 5.1. (1) *(cf. Definition 2.1) Let*

$$(\phi, id_B), (\theta, id_B) : (X, p, B, s) \to (X', p', B, s')$$

be fibrewise pointed maps. If there exists an **M***-fibrewise pointed map* $(H, h) : (I \times X, id \times p, I \times B, id \times s) \to (X', p', B, s')$ *such that* (H, h) *is an* **M***-fibrewise homotopy of* (ϕ, id_B) *into* (θ, id_B) *with* $h\delta_t = id_B$ *for* $t \in I$*, we call it an* H-fibrewise pointed homotopy *of* (ϕ, id_B) *into* (θ, id_B)*. If there exists an* **M***-fibrewise pointed homotopy of* (ϕ, id_B) *into* (θ, id_B)*, we say* (ϕ, id_B) *is* H-fibrewise pointed homotopic *to* (θ, id_B) *and write* $(\phi, id_B) \simeq^H_{(\mathbf{P})} (\theta, id_B)$.

A fibrewise pointed map $(\phi, id_B) : (X, p, B, s) \to (X', p', B, s')$ *is called an* H-fibrewise pointed homotopy equivalence *if there exists a fibrewise pointed map* $(\theta, id_B) : (X', p', B, s') \to (X, p, B, s)$ *such that* $(\theta\phi, id_B id_B) \simeq^H_{(\mathbf{P})} (id_X, id_B)$, $(\phi\theta, id_B id_B) \simeq^H_{(\mathbf{P})} (id_{X'}, id_B)$*. Then we denote* $(X, p, B, s) \cong^H_{(\mathbf{P})} (X', p', B, s')$.

(It is obvious that the relations $\simeq^H_{(\mathbf{P})}$ *and* $\cong^H_{(\mathbf{P})}$ *are equivalence relations.)*

(2) *(cf. Definition 2.2) A fibrewise pointed map* $(u, id_B) : (X_0, p_0, B, s_0) \to (X, p, B, s)$ *is an* H-fibrewise pointed cofibration *if* (u, id_B) *has the following* H*-fibrewise homotopy extension property : Let* $(\phi, id_B) : (X, p, B, s) \to (X', p', B, s')$ *be a fibrewise pointed map and* $(H, h) : (I \times X_0, id \times p_0, I \times B, id \times s_0) \to (X', p', B, s')$ *an* H*-fibrewise pointed homotopy such that the following two diagrams*

$$\begin{array}{ccc} X_0 & \xrightarrow{\sigma_0} & I \times X_0 \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle H} \\ X & \xrightarrow{\phi} & X' \end{array} \qquad \begin{array}{ccc} B & \xrightarrow{\delta_0} & I \times B \\ {\scriptstyle id_B}\downarrow & & \downarrow{\scriptstyle h} \\ B & \xrightarrow{id_B} & B \end{array}$$

are commutative. Then there exists an H*-fibrewise pointed homotopy* $(K, k) : (I \times X, id \times p, I \times B, id \times s) \to (X', p', B, s')$ *such that* $K\kappa_0 = \phi, K(id \times u) = H, k\rho_0 = id_B, k(id \times id_B) = h,$ *where* $\kappa_0 : X \to I \times X$ *and* $\rho_0 : B \to I \times B$ *are defined by* $\kappa_0(x) = (0, x)$ *and* $\rho_0(b) = (0, b)$ *for* $x \in X,\ b \in B$.

(3) *(cf. Definition 2.3) For a fibrewise pointed subspace* (X_0, p_0, B, s_0) *of* (X, p, B, s)*, the pair* $((X, p, B, s), (X_0, p_0, B, s_0))$ *is called by a* closed fibrewise pointed pair *if* X_0 *is closed in* X*. For a fibrewise pointed pair* $((X, p, B, s), (X_0, p_0, B, s_0))$*, if the inclusion map* $(u, id_B) : (X_0, p_0, B, s_0) \to (X, p, B, s)$ *is an* H*-fibrewise pointed cofibration, we call the pair* $((X, p, B, s), (X_0, p_0, B, s_0))$ *an* H-fibrewise pointed cofibred pair.

(4) *(cf. Definition 2.5) For a fibrewise pointed map* $(u, id_B) : (X_0, p_0, B, s_0) \to (X_1, p_1, B, s_1)$*, we can construct the* H*-fibrewise pointed push-out* $(M, p, I \times B, s)$ *of the cotriad*

$$(I \times X_0, \mathrm{id} \times p_0, I \times B, \mathrm{id} \times s_0) \xleftarrow{(\sigma_0, \delta_0)} (X_0, p_0, B, s_0) \xrightarrow{(u, id_B)} (X_1, p_1, B, s_1)$$

where (σ_0, δ_0) *is an* $\mathbf{M}$*-fibrewise embedding to 0-level, as follows :* $M = (I \times X_0 + X_1)/\sim$ *, where* $(0, a) \sim u(a)$ *for* $a \in X_0$*, and* $p : M \to I \times B$ *and* $s : I \times B \to M$ *are defined, respectively, by*

$$p(x) = \begin{cases} [t, p_0(a)] & \text{if } x = [t, a], t \neq 0 \\ [p_0(a)] & \text{if } x = [u(a)], a \in X_0 \\ [p_1(x)] & \text{if } x \in X_1 - u(X_0), \end{cases}$$

$$s(t, b) = \begin{cases} (t, s_0(b)) & \text{if } t \neq 0 \\ (0, s_1(b)) & \text{if } t = 0. \end{cases}$$

where [*] is the equivalence class. Then it is easily verified that p *and* s *are well-defined and continuous. We call the* H*-fibrewise push-out of the cotriad the* H-fibrewise mapping cylinder *of* (u, id_B)*, and denote by* $M(u, id_B)$.

(5) *(cf. Definition 2.8) For a fibrewise pointed space* (X, p, B, s)*, we call the* $\mathbf{M}$*-fibrewise pointed collapse (the same space in Definition 2.8)*

$$(I \times X, id \times p, I \times B, id \times s)/_{\mathbf{M}}(1 \times X, id \times p|1 \times X, 1 \times B, id \times s|1 \times B)$$

the H-fibrewise pointed cone *of* (X, p, B, s) *and denote by* $\Gamma_H(X, p, B, s)$ *in* $\mathbf{TOP}_B^H$*. (We denote the total space of* $\Gamma_H(X, p, B, s)$ *by* $C_H X$*.)*

(6) *(cf. Definition 2.9) Let* $(\phi, id_B) : (X, p, B, s) \to (X', p', B, s')$ *be a fibrewise pointed map. Then we call* (ϕ, id_B) *to be* H-fibrewise pointed nulhomotopic *if there is a fibrewise pointed map* $(c, id_B) : (X, p, B, s) \to (X', p', B, s')$ *such that* $c = s'p$ *and* $(\phi, id_B) \simeq^H_{(\mathbf{P})} (c, id_B)$.

(7) *(cf. Definition 3.1) For a fibrewise pointed map* $(\phi, id_B) : (X, p, B, s) \to (X', p', B, s')$*, we call* $((C_H X) \cup_\phi X', \tilde{p}, (I \times B) \cup_{id_B} B, \tilde{s})$ *the* H-fibrewise pointed mapping cone *of* (ϕ, id_B)*, and denote* $\Gamma_H(\phi, id_B)$*, where* $C_H X$ *is the space in this definition (5).*

(8) *(cf. Definition 3.5) A fibrewise pointed space* (X, p, B, s) *is* H-fibrewise pointed contractible *if there is a fibrewise pointed space* (B, id_B, B, id_B) *such that*

$$(X, p, B, s) \cong^H_{(\mathbf{P})} (B, id_B, B, id_B).$$

(9) *(cf. Definition 3.11) We use the same notation of Definition 3.11. For a fibrewise pointed space* (X, p, B, s)*, we call the* $\mathbf{M}$*-fibrewise pointed collapse (the same space in Definition 3.10)*

$$(I \times X, id \times p, I \times B, id \times s)/_{\mathbf{M}}(\overline{X}, \overline{p}, \overline{B}, \overline{s})$$

H-fibrewise pointed suspension*, and denoted by* $\Sigma_H(X, p, B, s)$ *in* $\mathbf{TOP}_B^H$*. (We denote* $\Sigma_H(X, p, B, s) = (\Sigma_H X, \Sigma_H\, p, I \times B, \Sigma_H\, s)$*.)*

(10) *(cf. Definition 4.1) For two fibrewise pointed spaces* (X_1, p_1, B, s_1) *and* (X_2, p_2, B, s_2) *let*

$$\begin{aligned} X_1 \vee^H X_2 &= \cup_{b \in B}(X_{1,b} \times s_2(b) \cup s_1(b) \times X_{2,b}), \\ p_1 \vee^H p_2 &= p_1 \times p_2 | X_1 \vee^H X_2. \end{aligned}$$

The fibrewise pointed space $(X_1 \vee^H X_2, p_1 \vee^H p_2, \Delta_B, s_1 \times s_2|\Delta_B)$ *is called the* H-fibrewise pointed coproduct *of* (X_1, p_1, B, s_1) *and* (X_2, p_2, B, s_2)*, and denoted by* $(X_1, p_1, B, s_1) \vee^H (X_2, p_2, B, s_2)$ *or* $\bigvee^H_{i \in \{1,2\}} (X_i, p_i, B, s_i)$*. The fibrewise pointed collapse*

$$(X_1 \times_B X_2, p_1 \times_B p_2, \Delta_B, s_1 \times s_2|\Delta_B)/_B(X_1, p_1, B, s_1) \vee^H (X_2, p_2, B, s_2)$$

is called the H-fibrewise smash product *of* (X_1, p_1, B, s_1) *and* (X_2, p_2, B, s_2)*, and denoted by* $(X_1, p_1, B, s_1) \wedge^H (X_2, p_2, B, s_2)$ *or* $\bigwedge^H_{i \in \{1,2\}} (X_i, p_i, B, s_i)$*, where* $p_1 \times_B p_2 = p_1 \times p_2 | X_1 \times_B X_2$*. (Note that from* $\Delta_B \approx B$*, as sets we can put* $(X_1, p_1, B, s_1) \vee^H (X_2, p_2, B, s_2) = X_1 \vee_B X_2$ *and* $(X_1, p_1, B, s_1) \wedge^H (X_2, p_2, B, s_2) = X_1 \wedge_B X_2$*.)*

For $n \geq 3$ and fibrewise pointed spaces (X_i, p_i, B, s_i) $(i = 1, \cdots, n)$, we shall define $\bigvee^H_{i\in\{1,\cdots,n\}}(X_i, p_i, B, s_i)$ and $\bigwedge^H_{i\in\{1,\cdots,n\}}(X_i, p_i, B, s_i)$ inductively, as follows:

$$\bigvee^{H}_{i\in\{1,\cdots,n\}}(X_i, p_i, B, s_i) = \Big(\bigvee^{H}_{i\in\{1,\cdots,n-1\}}(X_i, p_i, B, s_i)\Big)\vee^H(X_n, p_n, B, s_n)$$

$$\bigwedge^{H}_{i\in\{1,\cdots,n\}}(X_i, p_i, B, s_i) = \Big(\bigwedge^{H}_{i\in\{1,\cdots,n-1\}}(X_i, p_i, B, s_i)\Big)\wedge^H(X_n, p_n, B, s_n).$$

(11) *(cf. Definition 4.3) For two sequences of fibrewise pointed maps*

$$\mathcal{F} : (X_1, p_1, B, s_1) \to (X_2, p_2, B, s_2) \to \cdots \to (X_n, p_n, B, s_n) \to \cdots$$

$$\mathcal{F}' : (X'_1, p'_1, B, s'_1) \to (X'_2, p'_2, B, s'_2) \to \cdots \to (X'_n, p'_n, B, s'_n) \to \cdots$$

if there are H-fibrewise pointed homotopy equivalences

$$(\phi_n, id_B) : (X_n, p_n, B, s_n) \to (X'_n, p'_n, B, s'_n)$$

such that all diagrams induced by these maps are commutative, $\mathcal{F}$ and $\mathcal{F}'$ have the same H-fibrewise pointed homotopy type.

(12) *(cf. Definition 4.7) A fibrewise pointed space (X, p, B, s) is called H-*fibrewise well-pointed *if $(s, id_B) : (B, id_B, B, id_B) \to (X, p, B, s)$ is an H-fibrewise pointed cofibration and $s(B)$ is closed in X. (Note that Definition 4.7 is in* **MAP** *and this definition is in $\mathbf{TOP}^H_B$.)*

Remark 5.2. By Definition 5.1 (9) and (10), the base space of $\Sigma_H\{(X_1, p_1, B, s_1) \vee^H (X_2, p_2, B, s_2)\}$ and $\Sigma_H\{(X_1, p_1, B, s_1) \wedge^H (X_2, p_2, B, s_2)\}$ is $I \times \Delta_B$, but the base space of $\Sigma_H(X_1, p_1, B, s_1) \vee^H \Sigma_H(X_2, p_2, B, s_2)$ and $\Sigma_H(X_1, p_1, B, s_1) \wedge^H \Sigma_H(X_2, p_2, B, s_2)$ is $\Delta_{I\times B}$. So, we naturally identify $I \times \Delta_B$ with $\Delta_{I\times B}$, and we can have the same formulas (4.1) and (4.2) in $\mathbf{TOP}^H_B$.

If we consider in $\mathbf{TOP}^H_B$ the theory of sections 2, 3 and 4 , it can be verified by slight modifications that we have the same type theorems and propositions in $\mathbf{TOP}^H_B$. Thus we have the following main theorems.

Theorem 5.3. *For an H-fibrewise pointed map $(\phi, id_B) : (X, p, B, s) \to (X', p', B, s')$, the following sequence is exact in $\mathbf{TOP}^H_B$.*

$$\begin{aligned}(X, p, B, s) &\xrightarrow{(\phi, id_B)} (X', p', B, s')\\ &\xrightarrow{(\phi', \delta_0)} \Gamma_H(\phi, id_B)\\ &\xrightarrow{(\phi'', id\times id_B)} \Sigma_H(X, p, B, s)\\ &\xrightarrow{(\phi''', id\times id_B)} \Sigma_H(X', p', B, s')\\ &\longrightarrow \cdots\end{aligned}$$

THEOREM 5.4. *Assume that fibrewise pointed spaces* (X_i, p_i, B, s_i) $(i = 1, \cdots n)$ *are* H*-fibrewise well-pointed. Then the next formula holds in* $\mathbf{TOP}_B^H$.

$$\sum{}_H \left(\prod_{i=1}^{n}{}_B X_i, \prod_{i=1}^{n}{}_B\, p_i, \Delta_{B^n}, \prod_{i=1}^{n} s_i|\Delta_{B^n} \right)$$
$$\cong^H_{(\mathbf{P})} \bigvee_N{}^H \sum{}_H \Big(\bigwedge_{i\in N}{}^H \ (X_i, p_i, B, s_i) \Big)$$

where N *runs through all nonempty subsets of* $\{1, \cdots, n\}$.

We restate the concept of fibrewise non-degenerate space ([5; Definition 22.2]). Let (X, p, B, s) be a fibrewise pointed space. We regard the fibrewise mapping cylinder $M_B(s)$ of s (cf. [5; section 18]) as a fibrewise pointed space with the section $\check{s} : B \to M_B(s)$ defined by $\check{s}(b) = (1, b)$, and denote it by $(\check{X}_B, \check{p}, B, \check{s})$. Then the fibrewise pointed space (X, p, B, s) is *fibrewise non-degenerate* if the natural projection $(\rho, id_B) : (\check{X}_B, \check{p}, B, \check{s}) \to (X, p, B, s)$ is a fibrewise pointed homotopy equivalence.

We now prove the following proposition.

PROPOSITION 5.5. *Let* (X, p, B, s) *be a fibrewise non-degenerate space. Then* (X, p, B, s) *is an* H*-fibrewise well-pointed space.*

PROOF. Since (X, p, B, s) is fibrewise non-degenerate, the natural projection $(\rho, id_B) : (\check{X}_B, \check{p}, B, \check{s}) \to (X, p, B, s)$ is a fibrewise homotopy equivalence. Therefore there is a fibrewise pointed map $(\eta, id_B) : (X, p, B, s) \to (\check{X}_B, \check{p}, B, \check{s})$ (with $\eta s = \check{s}$) and a fibrewise pointed homotopy $(G, id_B) : I \tilde{\times} \check{X}_B \to \check{X}_B$ such that $G_0 = id_{\check{X}_B}$ and $G_1 = \eta\rho$, where $I \tilde{\times} \check{X}_B$ is the reduced fibrewise cylinder of $I \times \check{X}_B$ ([5; section 19]).

To prove that (X, p, B, s) is H-fibrewise well pointed, let $(\phi, id_B) : (X, p, B, s) \to (X', p', B, s')$ be a fibrewise pointed map, and

$$(H, h) : (I \times B, id \times id_B, I \times B, id \times id_B) \to (X', p', B, s')$$

an H-fibrewise pointed homotopy such that $H(0, b) = \phi(s(b))$ and $h(t, b) = b$. Then we can construct an H-fibrewise pointed homotopy $(\tilde{H}, \tilde{h}) : (I \times X, id \times p, I \times B, id \times s) \to (X', p', B, s')$ as follows: For any $(t, x) \in I \times X$, $(t, b) \in I \times B$,

$$\begin{aligned} \tilde{H}(t, x) &= \begin{cases} \phi\rho G(t, (t, x)) & (t = 0) \\ H(t, p\rho G(t, (t, sp(x)))) & (t \neq 0), \end{cases} \\ \tilde{h}(t, b) &= b. \end{aligned}$$

It is easily verified that $(\tilde{H}, \tilde{h})$ is an H-fibrewise pointed homotopy such that $(\tilde{H}, \tilde{h}) \circ (id \times s, id \times id_B) = (H, h)$ and $(\tilde{H}, \tilde{h}) \circ (i_X, i_B) = (\phi, id_B)$ where $i_X : X \to I \times X$ defined by $i_X(x) = (0, x)$ and $i_B : B \to I \times B$ defined by $i_B(b) = (0, b)$. This completes the proof. $\square$

Using this proposition we can prove [5] Proposition 22.11 as a corollary of Theorem 5.4, which gives an another proof of [5] Proposition 22.11.

PROPOSITION 22.11([5]) *Assume that fibrewise pointed spaces* (X_i, p_i, B, s_i) $(i = 1, \cdots n)$ *are fibrewise non-degenerate spaces. Then the next formula holds in* $\mathbf{TOP}_B$.

$$\sum{}_B^B(X_1 \times_B \cdots \times_B X_n) \cong_B^B \bigvee_N {}_B \sum{}_B^B \bigwedge_{i \in N} {}_B X_i$$

where N *runs through all nonempty subsets of* $\{1, \cdots, n\}$.

PROOF. First, note from Proposition 5.5 that each fibrewise non-degenerate space (X_i, p_i, B, s_i) $(i = 1, \ldots, n)$ is H-fibrewise well pointed. Next, we can prove this proposition by the following steps.

(1) For a fibrewise pointed space (X, p, B, s), the total space (we denote $\Sigma_B^B X$) of the reduced fibrewise suspension $\Sigma_B^B(X)$ in $\mathbf{TOP}_B$ can be obtained from the total space $\Sigma_H X$ of $\Sigma_H(X, p, B, s)$ in $\mathbf{TOP}_B^H$ as follows: the space $\Sigma_B^B X$ is just equal to the fibrewise push-out of the cotriad

$$\Sigma_H X \xleftarrow{i} (id \times s)(I \times B) \xrightarrow{\pi} \{0\} \times s(B)$$

where i is the natural inclusion and π is the natural projection. Further the diagram

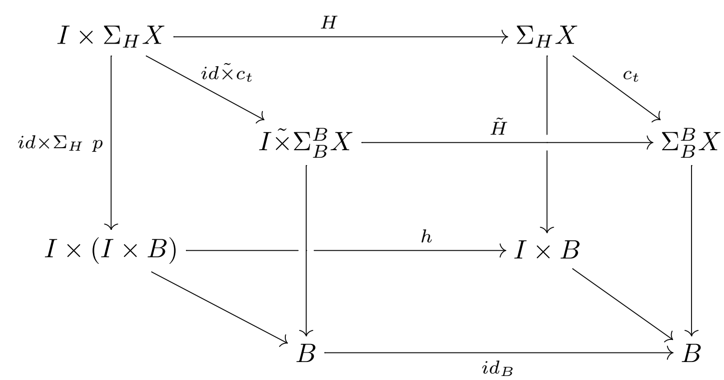
$$\begin{array}{ccc} \Sigma_H X & \xrightarrow{c_t} & \Sigma_B^B X \\ {\scriptstyle \Sigma_H\ p}\downarrow & & \downarrow{\scriptstyle \Sigma_B^B p} \\ I \times B & \xrightarrow{\pi_2} & B \end{array}$$

is commutative, where c_t is the (H-fibrewise pointed) compact map obtaining from the fibrewise push-out of the cotriad in the above, and $\Sigma_B^B p$ is the projection of $\Sigma_B^B(X)$.

(2) From $I \times \Sigma_B^B X$, we construct the reduced fibrewise cylinder $I \tilde{\times} \Sigma_B^B X$ ([5; section 19]), and obtain the natural map $id \tilde{\times} c_t : I \times \Sigma_H X \to I \tilde{\times} \Sigma_B^B X$ which is an (H-fibrewise pointed) compact map. Further, for an H-fibrewise pointed homotopy

$$(H, h) : I \times \Sigma_H(X, p, B, s) \to \Sigma_H(X, p, B, s),$$

we can define naturally the fibrewise pointed homotopy $(\tilde{H}, id_B) : I\tilde{\times}\Sigma_B^B(X) \to \Sigma_B^B(X)$ such that the following diagram is commutative.



(3) For the fibrewise pointed spaces (X_i, p_i, B, s_i) $(i = 1, \cdots, n)$, we can consider the formula in Theorem 5.4. Let

$$\begin{aligned} (X, p, I \times B, s) &= \sum{}_H \left(\prod_{i=1}^{n}{}_B X_i, \prod_{i=1}^{n}{}_B\ p_i, \Delta_{B^n}, \prod_{i=1}^{n} s_i | \Delta_{B^n} \right) \\ (Y, q, I \times B, t) &= \bigvee_N{}^H \sum{}_H (\bigwedge_{i\in N}{}^H\ (X_i, p_i, B, s_i)). \end{aligned}$$

Further, let $\tilde{X}$ and $\tilde{Y}$ be the total spaces of

$$\Sigma_B^B(X_1 \times_B \cdots \times_B X_n), \qquad \bigvee_N{}_B \Sigma_B^B(\bigwedge_{i\in N}{}_B\ X_i)$$

respectively. Note that Δ_B and Δ_{B^n} are identified with B, and $(X_1, p_1, B, s_1) \vee^H (X_2, p_2, B, s_2) = X_1 \vee_B X_2$, $(X_1, p_1, B, s_1) \wedge^H (X_2, p_2, B, s_2) = X_1 \wedge_B X_2$. By this note and (1) in this proof, there is a compact map $c_t' : Y \to \tilde{Y}$, and for an H-fibrewise pointed map $(f, id \times id_B) : (X, p, I \times B, s) \to (Y, q, I \times B, t)$ we can define a fibrewise pointed map $\tilde{f} : \tilde{X} \to \tilde{Y}$ such that the following diagram is commutative.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle c_t}\downarrow & & \downarrow{\scriptstyle c_t'} \\ \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \end{array}$$

(4) Let $(X, p, I \times B, s)$, $(Y, q, I \times B, t)$, $\tilde{X}$ and $\tilde{Y}$ be the same spaces as those in (3) of this proof. From Theorem 5.4, there are H-fibrewise pointed maps $(f, id \times id_B) : (X, p, I \times B, s) \to (Y, q, I \times B, t)$ and $(g, id \times id_B) : (Y, q, I \times B, t) \to (X, p, I \times B, s)$ such that $(gf, (id \times id_B)(id \times id_B)) \simeq^H_{(\mathbf{P})} (id_X, id \times id_B)$, $(fg, (id \times id_B)(id \times id_B)) \simeq^H_{(\mathbf{P})} (id_Y, id \times id_B)$. From these

H-homotopies and H-fibrewise pointed maps $(f, id \times id_B), (g, id \times id_B)$, we can construct, by the same methods of (2) and (3) in this proof, fibrewise pointed homotopies, fibrewise pointed maps $\tilde{f} : \tilde{X} \to \tilde{Y}$, $\tilde{g} : \tilde{Y} \to \tilde{X}$ such that $\tilde{g}\tilde{f} \simeq^B_B id_{\tilde{X}}, \tilde{f}\tilde{g} \simeq^B_B id_{\tilde{Y}}$. Thus, we complete the proof of [5] Proposition 22.11 by using Theorem 5.4. ∎

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Y. Konami
Izumo-Nishi High School
Izumo City 693-0032
Japan

T. Miwa
Department of Mathematics
Shimane University
Matsue 690-8504
Japan
E-mail: miwa@riko.shimane-u.ac.jp

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